On a class of nonlinear viscoelastic Kirchhoff plates: well-posedness and general decay rates

M. A. Jorge Silva *

Department of Mathematics, State University of Londrina,

86057-970 Londrina, PR, Brazil.

J. E. Muñoz Rivera †

National Laboratory of Scientific Computation, 25651-070 Petrópolis, RJ, Brazil,

and Institute of Mathematics, Federal University of Rio de Janeiro,

21941-909 Rio de Janeiro, PR, Brazil.

R. Racke ‡

Department of Mathematics and Statistics, University of Konstanz,

78457 Konstanz, Germany.

Abstract

This paper is concerned with well-posedness and energy decay rates to a class of nonlinear viscoelastic Kirchhoff plates. The problem corresponds to a class of fourth order viscoelastic equations with a non-locally Lipschitz perturbation of p-Laplacian type. The only damping effect is given by the memory component. We show that no additional damping is needed to obtain uniqueness in the presence of rotational forces. Then, we show that the general rates of energy decay are similar to ones given by the memory kernel, but generally not with the same speed, mainly when we consider the nonlinear problem with large initial data.

Keywords: Kirchhoff plates, well-posedness, p-Laplacian, general rates decay

MSC: 35B35, 35B40, 35L75, 74D99.

1 Introduction

This paper is motivated by models of Kirchhoff plates subject to a weak viscoelastic damping

$$u_{tt} - \sigma\Delta u_{tt} + \mu(0)\Delta^2 u + \int_{-\infty}^{t} \mu'(t-s)\Delta^2 u(s)\,ds = \mathcal{F}, \tag{1.1}$$

*Email: marcioajs@uel.br, corresponding author

†Email: rivera@lncc.br

‡Email: reinhard.racke@uni-konstanz.de

where $\sigma > 0$ is the uniform plate thickness, the kernel $\mu > 0$ corresponds to the viscoelastic flexural rigidity, and $\mathcal{F} = \mathcal{F}(x, t, u, u_t, \ldots)$ represents additional damping and forcing terms. The unknown function $u = u(x, t)$ represents the transverse displacement of a plate filament with prescribed history $u_0(x, t),\ t \leq 0$. The derivation of the linear mathematical model (1.1) with $\mathcal{F} = 0$ is given in Lagnese [16] and Lagnese and Lions [17], by assuming viscoelastic stress-strain laws on an isotropic material occupying a region of $\mathbb{R}^3$ and constant Poisson's ratio.

Lagnese [16, Chapter 6] studied the behavior of the energy associated to the linear model (1.1) in a bounded domain $\Omega \subset \mathbb{R}^2$, by introducing boundary feedback laws which induce further dissipation in the system, geometrical descriptions of $\partial\Omega$, and also

$$\mu \in C^2[0, \infty), \quad \mu(t) > 0, \quad \mu'(t) < 0, \quad \mu''(t) \geq 0, \quad \mu(\infty) > 0,$$

see also Lagnese [15]. Muñoz Rivera and Naso [26] considered an abstract model which encompasses equation (1.1) in the cases $\mathcal{F} = -u_t$ or else $\mathcal{F} = \Delta u_t$. They showed that the associated semigroup is not exponential stable in the first case (weak damping) whereas in the second case (strong damping) the corresponding semigroup is exponential stable. We note that in both cases $\mathcal{F}$ introduces an additional dissipation to the system.

More recently, Jorge Silva and Ma [13, 14] investigated the asymptotic behavior of a N-dimensional system like (1.1) with $\sigma = 0$ (without rotational inertia), by considering $\mathcal{F} = \Delta_p u - f(u) + h(x) + \Delta u_t$, where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u),\ p \geq 2$. Then (1.1) becomes to

$$u_{tt} + \mu(0)\Delta^2 u - \Delta_p u + f(u) + \int_{-\infty}^{t} \mu'(t-s)\Delta^2 u(s)\, ds - \Delta u_t = h(x),$$

In such case the strong damping plays an important role to obtain global well-posedness (mainly uniqueness) in higher dimensions $N \geq 3$ due to the presence of the p-Laplacian term $\Delta_p u$.

If we take $u_0 = 0$ for $t \leq 0$, $\mu(0) = 1$ and $g(t) = -\mu'(t)$, then (1.1) can be rewritten as follows,

$$u_{tt} - \sigma\Delta u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\, ds = \mathcal{F}. \tag{1.2}$$

Barreto et al. [3] investigated problem (1.2) in a bounded domain $\Omega \subset \mathbb{R}^2$ with mixed boundary condition, suitable geometrical hypotheses on $\partial\Omega$, and $\mathcal{F} = 0$. They established that the energy decays to zero with the same rate of the kernel g such as exponential and polynomial decay. To do so in the second case they made assumptions on g, g' and g'' which means that $g \approx (1+t)^{-p}$ for $p > 2$. Then they obtained the same decay rate for the energy. However, their approach can not be applied to prove similar results for $1 < p \leq 2$.

Concerning N-dimensional systems which cover the system (1.2) with $\sigma = 0$, both Cavalcanti et al. [5] and Andrade et al. [1] investigated the global existence, uniqueness and stabilization of energy. By taking a bounded or unbounded open set Ω and $\mathcal{F} = -M\left(\int_\Omega |\nabla u|\, dx\right) u_t$ as a kind of non-degenerated weak damping, where $M(s) > m_0 > 0$ for all $s \geq 0$, the authors showed in [5] that the energy goes to zero exponentially provided that g goes to zero at the same form. In [1] the authors studied the same concepts by considering a bounded domain and $\mathcal{F} = \Delta_p u - f(u) + \Delta u_t$, but replacing the fourth order memory term in (1.2) by a weaker memory of the form $\int_0^t g(t-s)\Delta u(s)\,ds$. It is worth noting that in both cases, respectively, the weak or strong damping constitutes an important role to obtain uniqueness and energy decay.

If we consider (1.2) with the Laplace operator instead of the bi-harmonic one we get the model

$$u_{tt} - \sigma\Delta u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds = \mathcal{F}, \tag{1.3}$$

which corresponds to a viscoelastic wave equation of second order. Equation (1.3) and related quasilinear problems with $|u_t|^\rho u_{tt}$ instead, $\rho > 0$, have been extensively studied by many researches with possible external forces $\mathcal{F}$ like source $f_1(u)$ and damping $f_2(u_t)$. See for instance [4, 6, 10, 11, 12, 19, 20, 22, 23, 24, 30, 31, 32] and the references therein.

In 2008 Messaoudi [22, 23] established a general decay of the energy solution to a viscoelastic equation corresponding to (1.3) with $\sigma = 0$, by taking $\mathcal{F} = 0$ and $\mathcal{F} = |u|^\gamma u,\ \gamma > 0$. More precisely, he considered the following decay condition on the memory kernel

$$g'(t) \ \leq\ -\xi(t)g(t), \quad \forall\ t > 0, \tag{1.4}$$

under proper conditions on $\xi(t) > 0$, and proved general decay of energy such as

$$E(t) \leq c_0\, e^{-c_1 \int_0^t \xi(s)\,ds}, \quad \forall\ t \geq 0, \tag{1.5}$$

for some $c_0, c_1 > 0$ depending on the weak initial data. Ever since several authors have used this condition to obtain arbitrary decay of energy for problems related to (1.3). See for instance the papers by Han and Wang [10, 11], Liu [19], Liu and Sun [21], Park and Park [30]. It is worth pointing that in all papers mentioned above when authors deal with nonlinear systems then c_1 is a proportional constant to $E(0)$ (denoted here by $c_1 \sim E(0)$), but it is not specified how this occurs. More recently, in [12], the author has illustrated that (1.5) provides decay rates which are faster than exponential one in the linear case. See also Messaoudi et al. [18, 25], Tatar [31] and Wu [32] for other kinds of interesting arbitrary decay rates in viscoelastic wave models related to (1.3).

There are also previous and recent works which encompass viscoelastic wave equations in a history framework and only employ memory dissipation to treat asymptotic behavior of solutions. We refer, for instance, the papers by Dafermos [7], Giorgi et al. [9], Muñoz Rivera and Salvatierra [27], and Pata [28, 29], Fabrizio et al. [8] and Araújo [2].

Our main goal in the present paper is to discuss the well-posedness and the asymptotic behavior of energy to the following nonlinear viscoelastic Kirchhoff plate equation

$$u_{tt} - \sigma\Delta u_{tt} + \Delta^2 u - \operatorname{div}F(\nabla u) - \int_0^t g(t-s)\Delta^2 u(s)\,ds = 0 \quad \text{in} \quad \Omega\times\mathbb{R}^+, \tag{1.6}$$

with simply supported boundary condition

$$u = \Delta u = 0 \quad \text{on} \quad \partial\Omega\times\mathbb{R}^+, \tag{1.7}$$

and initial conditions

$$u(\,\cdot\,,0) = u_0 \quad \text{and} \quad u_t(\,\cdot\,,0) = u_1 \quad \text{in} \quad \Omega, \tag{1.8}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\sigma \geq 0$, $F : \mathbb{R}^N \to \mathbb{R}^N$ is a vector field and $g : [0, \infty) \to \mathbb{R}^+$ is a real function. The hypotheses are given later.

Essentially, we consider the model (1.2) with $\mathcal{F} = \text{div}F(\nabla u)$ which constitutes a nonlinear and non-locally Lipschitz perturbation. Our main results are Theorems 2.3, 2.8 and 2.10. Making a comparison with the above related papers on the viscoelastic plate model (1.2) our main results yield the following improvements and contributions:

1. The only damping effect is caused by the memory term. Besides, our condition on the kernel g like (1.4) is less restrictive than those used in [3, 26, 5, 1]. Nevertheless, our general decay of energy (see (2.14) and (2.17)) generalizes all results on stability obtained in [1, 3, 26, 5]. We also specify how the decay rate depends on the initial data.

2. The p-Laplacian term $\Delta_p u$ is considered as a particular case. We show that in the presence of the rotational inertia term ($\sigma > 0$) the well-posedness of (1.6)-(1.8) is achieved without strong damping term. Moreover, for $\sigma \geq 0$ all results on stability are shown by exploiting only the memory dissipation. No additional weak or strong dissipation is necessary. Therefore our results improve those ones given in [1, 5, 13].

3. No further C^2-smoothness is imposed on the relaxation function g as regarded e.g. in [3, 5]. Moreover, the decay rate $(1+t)^{-\kappa}$ holds for every $\kappa > 0$ when g has a polynomial behavior. We note that the case $0 < \kappa \leq 2$ was not approached in [3].

4. The parameter σ is related to the uniform plate thickness and the results on stability hold by moving $\sigma \in [0, \infty)$ uniformly. Further, there is no result by now which treats the asymptotic behavior for plates with perturbation of p-Laplacian type just using dissipation from the memory. We also exemplify other interesting types of decay rates of energy beyond exponential and polynomial ones.

Remark 1.1. It is worth pointing out that the parameter σ changes the character of the system (1.6) depending whether it is null or not. In the case of choosing $\sigma > 0$, the term $-\sigma\Delta u_{tt}$ acts as a regularizing term by allowing us to consider stronger solutions and uniqueness (see Theorem 2.3). This is possible because the rotational inertia term gives a way to control the nonlinear perturbation $\text{div}F(\nabla u)$. On the other hand, if we consider $\sigma = 0$ in (1.6) then we can also check the existence of weak solutions (see Theorem 2.10 (i)) but uniqueness and stronger solutions are not provided once the term $\text{div}F(\nabla u)$ spoils the estimates along with lack of regularity for u_t. In spite of having two different systems according to parameter σ all results on stability hold in both cases (see Theorems 2.8 and 2.10 (ii)). In the second case (when $\sigma = 0$) the stability is obtained first for approximate solutions and then for weak solutions by taking $\liminf$ on the approximate energy.

The rest of the paper is organized as follows. In Section 2 we fix some notations and present our assumptions and main results. Section 3 is devoted to show that problem

(1.6)-(1.8) is well posed. Section 4 is dedicated to the proof of the energy decay. Finally, section 5 consists in an appendix where we first give some examples for different rates of decay. Then we provide some properties and examples for the vector field F.

2 Assumptions and main results

We begin by introducing the following Hilbert spaces

$$\mathcal{V}_0 = L^2(\Omega), \quad \mathcal{V}_1 = H_0^1(\Omega), \quad \mathcal{V}_2 = H^2(\Omega) \cap H_0^1(\Omega),$$

and

$$\mathcal{V}_3 = \{u \in H^3(\Omega) \cap H_0^1(\Omega);\ \Delta u \in H_0^1(\Omega)\},$$

with norms

$$\|u\|_{\mathcal{V}_0} = \|u\|_2, \quad \|u\|_{\mathcal{V}_1} = \|\nabla u\|_2, \quad \|u\|_{\mathcal{V}_2} = \|\Delta u\|_2, \quad \text{and} \quad \|u\|_{\mathcal{V}_3} = \|\nabla \Delta u\|_2,$$

respectively. As usual, $\|\cdot\|_p$ means the L^p-norm as well as $(\cdot,\cdot)$ denotes either the L^2-inner product or else a duality pairing between a Banach space V and its dual V'. The constants $\lambda_0, \lambda_1, \lambda_2, \lambda > 0$ represent the embedding constants

$$\lambda_0\|u\|_2^2 \le \|\nabla u\|_2^2, \quad \lambda_1\|u\|_2^2 \le \|\Delta u\|_2^2, \quad \lambda_2\|\nabla u\|_2^2 \le \|\Delta u\|_2^2, \quad \lambda = \frac{1}{\lambda_1} + \frac{1}{\lambda_2},$$

for $u \in \mathcal{V}_1$. We also consider the following phase spaces with their respective norms

$$\begin{aligned}
\mathcal{H} &= \mathcal{V}_2 \times \mathcal{V}_1 \quad \text{with} \quad ||(u,v)||_{\mathcal{H}}^2 = ||\Delta u||_2^2 + ||\nabla v||_2^2,\\
\mathcal{H}_1 &= \mathcal{V}_3 \times \mathcal{V}_2 \quad \text{with} \quad ||(u,v)||_{\mathcal{H}_1}^2 = ||\nabla \Delta u||_2^2 + ||\Delta v||_2^2,\\
\mathcal{W} &= \mathcal{V}_2 \times \mathcal{V}_0 \quad \text{with} \quad ||(u,v)||_{\mathcal{W}}^2 = ||\Delta u||_2^2 + ||v||_2^2.
\end{aligned}$$

2.1 The problem with rotational inertia

Let us first consider (1.6) with $\sigma > 0$. Without loss of generality we can take $\sigma = 1$. Setting $I = [0,T]$ with $T > 0$ arbitrary, weak solutions are defined as follows.

Given initial data $(u_0, u_1) \in \mathcal{H}$, we call a function $U := (u, u_t) \in C(I, \mathcal{H})$ a *weak solution* of the problem (1.6)-(1.8) on I if $U(0) = (u_0, u_1)$ and, for every $\omega \in \mathcal{V}_2$,

$$\begin{aligned}
&\frac{d}{dt}\Big[(u_t(t),\omega) + (\nabla u_t(t), \nabla\omega)\Big] + (\Delta u(t), \Delta\omega)\\
&\qquad + (F(\nabla u(t)), \nabla\omega) - \int_0^t g(t-s)(\Delta u(s), \Delta\omega)\, ds = 0 \ \text{ a.e. in } I.
\end{aligned}$$

The energy corresponding to the problem with rotational inertia is defined as

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u_t(t)\|_2^2 + \frac{h(t)}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}(g\Box\Delta u)(t) + \int_\Omega f(\nabla u(t))\, dx, \quad (2.1)$$

where $h(t)$ is given below in (2.6) and

$$(g\Box w)(t) := \int_0^t g(t-s)\|w(t)-w(s)\|_2^2 ds.$$

Now let us precise the hypotheses on g and F.

Assumption A1. *The C^1-function $g : [0,\infty) \to \mathbb{R}^+$ satisfies*

$$l := 1 - \int_0^\infty g(s)\, ds > 0 \quad \text{and} \quad g'(t) \le 0, \quad \forall\, t \ge 0. \tag{2.2}$$

Assumption A2. *$F : \mathbb{R}^N \to \mathbb{R}^N$ is a C^1-vector field given by $F = (F_1, \dots, F_N)$ such that*

$$|\nabla F_j(z)| \le k_j(1 + |z|^{(p_j-1)/2}), \quad \forall\, z \in \mathbb{R}^N, \tag{2.3}$$

where, for every $j = 1, \dots, N$, we consider $k_j > 0$ and p_j satisfying

$$p_j \ge 1 \quad \text{if} \quad N = 1, 2 \quad \text{and} \quad 1 \le p_j \le \frac{N+2}{N-2} \quad \text{if} \quad N \ge 3. \tag{2.4}$$

Moreover, F is a conservative vector field with $F = \nabla f$, where $f : \mathbb{R}^N \to \mathbb{R}$ is a real valued function satisfying

$$-\alpha_0 - \frac{\alpha l}{2}|z|^2 \le f(z) \le F(z)\cdot z + \frac{\alpha l}{2}|z|^2, \quad \forall\, z \in \mathbb{R}^N, \tag{2.5}$$

with $\alpha_0 \ge 0$ and $\alpha \in [0, \lambda_2)$.

Remark 2.1. From the choice of l and α we have

$$h(t) := 1 - \int_0^t g(s)\, ds \ge l, \ \ t \ge 0, \quad \text{and} \quad \beta := l\left(1 - \frac{\alpha}{\lambda_2}\right) > 0. \tag{2.6}$$

Also, applying (2.4) it follows from Sobolev embedding that

$$\mathcal{V}_2 \hookrightarrow W_0^{1,p_j+1}(\Omega), \quad \forall\, j = 1, \dots, N. \tag{2.7}$$

Thereby, the constants $\mu_{p_1}, \dots, \mu_{p_N} > 0$ represent the embedding constants for

$$\|\nabla u\|_{p_j+1} \le \mu_{p_j}\|\Delta u\|_2, \ \ j = 1, \dots, N.$$

Remark 2.2. Without loss of generality we can consider $F(0) = 0$. Indeed, if $F(0) = F_0 \neq 0$, then we define $G(z) = F(z) - F_0$ so that G satisfies $G(0) = 0$, $|\nabla G_j(z)| = |\nabla F_j(z)|$, $j = 1, \dots, N$, and $G(z) = \nabla \widetilde{f}(z)$, where $\widetilde{f}(z) = f(z) - F_0 \cdot z$. Also, it is easy to check that $\widetilde{f}$ and $G(z)$ satisfy (2.5) for some constants $\widetilde{\alpha}_0 \ge 0$, $\widetilde{\alpha} \in [0, \lambda_2)$. Therefore, G is a C^1-conservative vector field satisfying (2.3)-(2.5). In the Section 5 we give some examples of vector fields satisfying such properties.

Our first two main results establish the Hadamard well-posedness of (1.6)-(1.8) with respect to weak solutions, and a general decay rate of the energy.

Theorem 2.3 (Well-Posedness)**.** *Under* **Assumptions A1** *and* **A2** *we have:*

(*i*) *If* $(u_0, u_1) \in \mathcal{H}_1$*, then problem* (1.6)-(1.8) *has a stronger weak solution satisfying*

$$u \in L^\infty_{loc}(\mathbb{R}^+, \mathcal{V}_3), \quad u_t \in L^\infty_{loc}(\mathbb{R}^+, \mathcal{V}_2), \quad (I-\Delta)u_{tt} \in L^\infty_{loc}(\mathbb{R}^+, \mathcal{V}_1'). \tag{2.8}$$

(*ii*) *If* $(u_0, u_1) \in \mathcal{H}$*, then problem* (1.6)-(1.8) *has a weak solution satisfying*

$$u \in L^\infty_{loc}(\mathbb{R}^+, \mathcal{V}_2), \quad u_t \in L^\infty_{loc}(\mathbb{R}^+, \mathcal{V}_1), \quad (I-\Delta)u_{tt} \in L^\infty_{loc}(\mathbb{R}^+, \mathcal{V}_2'). \tag{2.9}$$

(*iii*) *In both cases we have continuous dependence on initial data in* $\mathcal{H}$*, that is, given* $U_0 = (u_0, u_1)$*,* $V_0 = (v_0, v_1) \in \mathcal{H}$*, let us consider the corresponding weak solutions* $U = (u, u_t)$*,* $V = (v, v_t)$ *of the problem* (1.6)-(1.8)*. Then*

$$\|U(t) - V(t)\|_{\mathcal{H}} \leq C_T \|U_0 - V_0\|_{\mathcal{H}}, \quad \forall\, t \in I, \tag{2.10}$$

for some constant $C_T = C(\|U_0\|_{\mathcal{H}}, \|V_0\|_{\mathcal{H}}, T) > 0$*. In particular, problem* (1.6)-(1.8) *has a unique weak solution.*

Remark 2.4. The proof of existence is given by the Faedo-Galerkin method. We first prove the existence of stronger (weak) solutions and then the existence of a weak solution is given by density arguments. The uniqueness follows as a consequence of the continuous dependence of stronger and weak solutions. The proofs are given in Section 3.

Lemma 2.5. *Under the assumptions of Theorem* 2.3 *the energy* $E(t)$ *satisfies*

$$\frac{d}{dt}E(t) = \frac{1}{2}(g'\Box\Delta u)(t) - \frac{g(t)}{2}\|\Delta u(t)\|_2^2, \quad \forall\, t > 0. \tag{2.11}$$

Remark 2.6. From conditions (2.2) and (2.11) it follows that $t \mapsto E(t)$ is nonincreasing. Since $g(t) \leq g(0)$, for each $t \geq 0$, if we take $g(0) = 0$ then $g,\, g' \equiv 0$ and Lemma 2.5 implies that $E(t)$ is constant That is, the system (1.6)-(1.8) is conservative. This motivates us to define the following decay condition on the memory kernel $g(t)$.

Assumption A3. $g(0) > 0$*, and there exist a constant* $\xi_0 \geq 0$ *and a* C^1*-function* $\xi : [0, \infty) \to \mathbb{R}^+$ *such that*

$$g'(t) \leq -\xi(t)g(t), \quad \forall\, t > 0, \tag{2.12}$$

and

$$\xi(t) > 0, \quad \xi'(t) \leq 0, \quad \left|\frac{\xi'(t)}{\xi(t)}\right| \leq \xi_0, \quad \forall\, t \geq 0. \tag{2.13}$$

Remark 2.7. The first two conditions in (2.13) allow us to conclude that

$$\xi(t) \leq \xi(0) := \xi_1 > 0, \quad \forall\, t \geq 0.$$

Also, condition (2.12) implies that the memory kernel has the uniform decay

$$g(t) \leq g(0)e^{-\int_0^t \xi(s)\,ds}, \quad \forall\, t \geq 0.$$

Then our second main result is given by the following

Theorem 2.8. *Under the assumptions of Theorem* 2.3*, let* (u, u_t) *be the weak solution of problem* (1.6)-(1.8) *with given initial data* $(u_0, u_1) \in \mathcal{H}$*. If we additionally assume* **Assumption A3** *and* $\alpha_0 = 0$ *in* (2.5)*, then*

$$E(t) \le c\, e^{-\gamma \int_0^t \xi(s)\, ds}, \quad \forall\, t \ge 0, \tag{2.14}$$

where $c = 3E(0)\, e^{\gamma \int_0^1 \xi(s)\, ds} > 0$*, and*

$$\gamma \sim \frac{k}{1 + [E(0)]^{\frac{p-1}{2}}} \quad \text{with} \quad k > 0 \quad \text{and} \quad p = \begin{cases} \max\limits_{j=1,\ldots,N} \{p_j\} & \text{if} \quad E(0) \ge 1, \\ \min\limits_{j=1,\ldots,N} \{p_j\} & \text{if} \quad E(0) < 1. \end{cases}$$

Remark 2.9. Theorem 2.8 is proved in Section 4. Concerning to estimate (2.14) it is worth point out two issues: (i) when every component of $F = (F_1, \ldots, F_N)$ is linear, namely, when $p_1, \ldots, p_N = 1$ in (2.3) and so $p = 1$, then the general estimate (2.14) is similar to the decay of the memory kernel g and provides us several kinds of decay according to the feature of $\xi(t)$ independently of the size of the initial data; (ii) otherwise, in the presence of the nonlinear perturbation F, then estimate (2.14) can be very slow for large initial data even if the memory kernel g decays quickly. In the Section 5 we consider some concrete examples for function $\xi(t)$.

2.2 The problem without rotational inertia

Let us now consider (1.6) with $\sigma = 0$. Let us also take $I = [0, T]$ with $T > 0$.

Given initial data $(u_0, u_1) \in \mathcal{W}$, we say a function $U := (u, u_t) \in C(I, \mathcal{W})$ is a *weak solution* of (1.6)-(1.8) on I if $U(0) = (u_0, u_1)$ and, for every $\omega \in \mathcal{V}_2$,

$$\frac{d}{dt}(u_t(t), \omega) + (\Delta u(t), \Delta \omega) + (F(\nabla u(t)), \nabla \omega) - \int_0^t g(t-s)(\Delta u(s), \Delta \omega)\, ds = 0 \quad \text{a.e. in } I.$$

Now the energy associated to the problem without rotational inertia is given by

$$\mathcal{E}(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{h(t)}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}(g \Box \Delta u)(t) + \int_\Omega f(\nabla u(t))\, dx. \tag{2.15}$$

Our third main result is the following theorem.

Theorem 2.10. *Under* **Assumptions A1** *and* **A2***, we have:*

(i) *If* $(u_0, u_1) \in \mathcal{W}$*, then problem* (1.6)-(1.8) *has a weak solution in the class*

$$u \in L^\infty_{loc}(\mathbb{R}^+, \mathcal{V}_2), \quad u_t \in L^\infty_{loc}(\mathbb{R}^+, \mathcal{V}_0), \quad u_{tt} \in L^\infty_{loc}(\mathbb{R}^+, \mathcal{V}_2'). \tag{2.16}$$

(ii) *Besides, if* **Assumption A3** *holds and* $\alpha_0 = 0$ *in* (2.5)*, then* $\mathcal{E}(t)$ *also satisfies*

$$\mathcal{E}(t) \le c\, e^{-\gamma \int_0^t \xi(s)\, ds}, \quad \forall\, t \ge 0, \tag{2.17}$$

where $c > 0$ *and* $\gamma > 0$ *are given in terms of* $\mathcal{E}(0)$ *as in Theorem* 2.8.

Remark 2.11. To prove Theorem 2.10 (i) one also uses the Faedo-Galerkin method. Only one a priori estimate is necessary to get a weak solution satisfying (2.16). Since we do not have regularity for u_t the estimate (2.17) is shown first for the approximate energy. Then the proof of Theorem 2.10 (ii) will hold true by passing the approximate energy to the limit. The details of the proof of Theorem 2.10 are very similar to those ones used in the proofs of Theorem 2.3 and Theorem 2.8. Thus we omit them here.

3 Well-posedness

In this section we prove Theorem 2.3. We start with the following approximate problem

$$(u_{tt}^n(t),\omega_j)+(\nabla u_{tt}^n(t),\nabla\omega_j)+(\Delta u^n(t),\Delta\omega_j) \qquad (3.1)$$
$$+(F(\nabla u^n(t)),\nabla\omega_j)-\int_0^t g(t-s)(\Delta u^n(s),\Delta\omega_j)\,ds=0,$$
$$u^n(0)=u_0^n \quad\text{and}\quad u_t^n(0)=u_1^n, \qquad (3.2)$$

for $j=1,\dots,n$, which has a local solution

$$u^n(t)=\sum_{j=1}^n y_{jn}(t)\omega_j \,\in\, [\omega_1,\dots,\omega_n],$$

on $[0,t_n),\ n\in\mathbb{N}$, given by ODE theory, where $(\omega_j)_{j\in\mathbb{N}}$ is an orthonormal basis in $\mathcal{V}_0$ given by eigenfunctions of Δ^2 with boundary condition (1.7). The a priori estimates below imply that the local solution can be extended to the interval $[0,T]$ and allow us to concluded the existence of a weak solution.

Proof of Theorem 2.3 (i)**.** Let us take regular initial data $(u_0,u_1)\in\mathcal{H}_1:=\mathcal{V}_3\times\mathcal{V}_2$. Then we consider the approximate problem (3.1)-(3.2) with

$$u_0^n\to u_0 \text{ in } \mathcal{V}_3 \qquad\text{and}\qquad u_1^n\to u_1 \text{ in } \mathcal{V}_2. \qquad (3.3)$$

A Priori Estimate I. Replacing w_j by $u_t^n(t)$ in (3.1) and since it hold

$$\int_0^t g(t-s)(\Delta u^n(s),\Delta u_t^n(t))\,ds = -\frac{1}{2}\frac{d}{dt}\left\{(g\Box\Delta u^n)(t)-\left(\int_0^t g(s)\,ds\right)\|\Delta u^n(t)\|_2^2\right\}$$
$$+\frac{1}{2}(g'\Box\Delta u^n)(t)-\frac{1}{2}g(t)\|\Delta u^n(t)\|_2^2, \qquad (3.4)$$

and

$$\int_\Omega F(\nabla u^n(t))\cdot\nabla u_t^n(t)\,dx = \int_\Omega \nabla f(\nabla u^n(t))\cdot\nabla u_t^n(t)\,dx$$
$$=\frac{d}{dt}\int_\Omega f(\nabla u^n(t))\,dx, \qquad (3.5)$$

it follows that

$$\frac{d}{dt}E^n(t) = \frac{1}{2}(g'\Box\Delta u^n)(t)-\frac{1}{2}g(t)\|\Delta u^n(t)\|_2^2,\quad t>0, \qquad (3.6)$$

where

$$E^n(t) = \frac{1}{2}\|u_t^n(t)\|_2^2 + \frac{1}{2}\|\nabla u_t^n(t)\|_2^2 + \frac{h(t)}{2}\|\Delta u^n(t)\|_2^2 + \frac{1}{2}(g\Box\Delta u^n)(t) + \int_\Omega f(\nabla u^n(t))\,dx.$$

Assumption (2.2) and (3.6) imply that $E^n(t) \le E^n(0)$. From (2.6) and the first condition in (2.5) we get

$$\frac{h(t)}{2}\|\Delta u^n(t)\|_2^2 + \int_\Omega f(\nabla u^n(t))\,dx \;\ge\; \frac{\beta}{2}\|\Delta u^n(t)\|_2^2 - \alpha_0|\Omega|,$$

and consequently,

$$\frac{1}{2}\|\nabla u_t^n(t)\|_2^2 + \frac{\beta}{2}\|\Delta u^n(t)\|_2^2 \le E^n(t) + \alpha_0|\Omega| \le E^n(0) + \alpha_0|\Omega|.$$

From (3.3), second condition in (2.5), (5.3) and Hölder's inequality we conclude

$$\|\nabla u_t^n(t)\|_2^2 + \|\Delta u^n(t)\|_2^2 \le M_1, \quad \forall\, t \in [0,T],\ \forall\, n \in \mathbb{N}, \tag{3.7}$$

where $M_1 = M_1(\|\nabla u_1\|_2, \|\Delta u_0\|_2, |\Omega|) > 0$.

A Priori Estimate II. Replacing w_j by $-\Delta u_t^n(t)$ in (3.1), since (3.4) holds with $\nabla\Delta$ in the place of Δ, and also

$$\int_\Omega F(\nabla u^n(t))\cdot\nabla\Delta u_t^n(t)\,dx \;=\; \frac{d}{dt}\int_\Omega F(\nabla u^n(t))\cdot\nabla\Delta u^n(t)\,dx + J_F,$$

with J_F given by

$$J_F \;=\; -\int_\Omega \big[\big(\nabla F_1(\nabla u^n(t))\cdot\nabla u_t^n(t),\ldots,\nabla F_N(\nabla u^n(t))\cdot\nabla u_t^n(t)\big)\big]\cdot\nabla\Delta u^n(t)\,dx,$$

then we infer

$$\frac{d}{dt}F^n(t) \;=\; \frac{1}{2}\left(g'\Box\nabla\Delta u^n\right)(t) - \frac{1}{2}g(t)\|\nabla\Delta u^n(t)\|_2^2 + J_F \;\le\; J_F, \quad t > 0, \tag{3.8}$$

where

$$F^n(t) \;=\; \frac{1}{2}\|\nabla u_t^n(t)\|_2^2 + \frac{1}{2}\|\Delta u_t^n(t)\|_2^2 + \frac{h(t)}{2}\|\nabla\Delta u^n(t)\|_2^2 + \frac{1}{2}\left(g\Box\nabla\Delta u^n\right)(t) - I_F$$

with

$$I_F = \int_\Omega F(\nabla u^n(t))\cdot\nabla\Delta u^n(t)\,dx.$$

Let us estimate the right hand side of (3.8). Using assumption (2.3), generalized Hölder inequality, and (2.7) we get

$$\begin{aligned}
|J_F| \;&\le\; \sum_{j=1}^N \int_\Omega |\nabla F_j(\nabla u^n(t))|\,|\nabla u_t^n(t)|\,|\nabla\Delta u^n(t)|\,dx \\
&\le\; \sum_{j=1}^N k_j \int_\Omega \left(1 + |\nabla u^n(t)|^{(p_j-1)/2}\right)\,|\nabla u_t^n(t)|\,|\nabla\Delta u^n(t)|\,dx \\
&\le\; \sum_{j=1}^N k_j\mu_{p_j}\left(|\Omega|^{\frac{p_j-1}{2(p_j+1)}} + \|\nabla u^n(t)\|_{p_j+1}^{\frac{p_j-1}{2}}\right)\|\Delta u_t^n(t)\|_2\|\nabla\Delta u^n(t)\|_2.
\end{aligned}$$

From estimate (3.7) and using again (2.7) we obtain

$$\sum_{j=1}^{N} k_j \mu_{p_j} \left(|\Omega|^{\frac{p_j-1}{2(p_j+1)}} + \|\nabla u^n(t)\|_{p_j+1}^{\frac{p_j-1}{2}} \right) \leq C < \infty.$$

From this and Young's inequality there exists a constant $C_1 = C_1 (||\nabla u_1||_2, ||\Delta u_0||_2) > 0$ such that

$$|J_F| \leq C_1 \left(\|\Delta u_t^n(t)\|_2^2 + \|\nabla \Delta u^n(t)\|_2^2 \right). \tag{3.9}$$

Inserting (3.9) in (3.8) and integrating from 0 to $t \leq T$, yields

$$F^n(t) \leq F^n(0) + C_1 \int_0^t \left(\|\Delta u_t^n(s)\|_2^2 + \|\nabla \Delta u^n(s)\|_2^2 \right) ds, \quad t \geq 0. \tag{3.10}$$

On the other hand, from (5.3) in the appendix with $F(0) = 0$ and Hölder's inequality, we have

$$\begin{aligned} |I_F| &\leq \int_\Omega |F(\nabla u^n(t))| \, |\nabla \Delta u^n(t)| \, dx \\ &\leq K \sum_{j=1}^{N} \int_\Omega \left(|\nabla u^n(t)| + |\nabla u^n(t)|^{(p_j+1)/2} \right) |\nabla \Delta u^n(t)| \, dx \\ &\leq K \left(N \|\nabla u^n(t)\|_2 + \sum_{j=1}^{N} \|\nabla u^n(t)\|_{p_j+1}^{\frac{p_j+1}{2}} \right) \|\nabla \Delta u^n(t)\|_2. \end{aligned}$$

Moreover, the estimates (3.7) and (2.7) imply

$$K \left(N \|\nabla u^n(t)\|_2 + \sum_{j=1}^{N} \|\nabla u^n(t)\|_{p_j+1}^{\frac{p_j+1}{2}} \right) \leq C < \infty.$$

Using Young's inequality there exists a constant $C_2 = C_2 (||\nabla u_1||_2, ||\Delta u_0||_2) > 0$ such that

$$|I_F| \leq C_2 + \frac{l}{4} \|\nabla \Delta u^n(t)\|_2^2.$$

Since $h(t) \geq l$, then

$$\frac{h(t)}{2} \|\nabla \Delta u^n(t)\|_2^2 - I_F \geq \frac{l}{4} \|\nabla \Delta u^n(t)\|_2^2 - C_2,$$

and, consequently,

$$\frac{l}{4} \|\Delta u_t^n(t)\|_2^2 + \frac{l}{4} \|\nabla \Delta u^n(t)\|_2^2 \leq F^n(t) + C_2. \tag{3.11}$$

Combining (3.10) and (3.11) we arrive at

$$\|\Delta u_t^n(t)\|_2^2 + \|\nabla \Delta u^n(t)\|_2^2 \leq \frac{4}{l} (C_2 + F^n(0)) + \frac{4C_1}{l} \int_0^t \left(\|\Delta u_t^n(s)\|_2^2 + \|\nabla \Delta u^n(s)\|_2^2 \right) ds.$$

Taking into account (3.3) and applying Gronwall's inequality, we finally conclude

$$\|\Delta u_t^n(t)\|_2^2 + \|\nabla\Delta u^n(t)\|_2^2 \le M_2, \quad \forall\, t \in [0,T],\ \forall\, n \in \mathbb{N}, \tag{3.12}$$

where $M_2 = M_2(||\Delta u_1||_2, ||\nabla\Delta u_0||_2, |\Omega|, T) > 0$.

The estimates (3.7) and (3.12) are sufficient to pass to the limit in the approximate problem (3.1)-(3.2) and to obtain a stronger weak solution

$$(u, u_t) \in C\left([0,T], \mathcal{H}\right) \cap L^\infty(0,T;\mathcal{H}_1), \quad T > 0, \tag{3.13}$$

satisfying

$$(I-\Delta)u_{tt} = -\Delta^2 u + \mathrm{div}F(\nabla u) + \int_0^t g(t-s)\Delta^2 u(s)\, ds \quad \text{in} \quad L^\infty\left(0,T;\mathcal{V}_1'\right). \tag{3.14}$$

This finishes the proof of the existence of regular weak solutions. ■

Remark 3.1. Unless for the term which involves $\mathrm{div}F(\nabla u)$, the limit on the other terms in the approximate system can be done in a usual way. With respect to this term we only need to apply estimates (3.7) and (5.2) along with the Aubin-Lions Lemma. Then it will hold later in the case of weak solutions, see for instance [1, 13]. ■

Proof of Theorem 2.3 *(iii)* **(stronger weak solutions).** We first show that solution in (3.13) satisfies the continuous dependence property (2.10).

Let us consider two stronger weak solutions $U = (u, u_t)$, $V = (v, v_t)$ of the problem (1.6)-(1.8) corresponding to initial data $U_0 = (u_0, u_1)$, $V_0 = (v_0, v_1) \in \mathcal{H}_1$, respectively. By setting $w = u - v$, then function $(w, w_t) = U - V$ solves the equation

$$w_{tt} - \Delta w_{tt} + \Delta^2 w - \int_0^t g(t-s)\Delta^2 w(s)\, ds = \mathrm{div}F(\nabla u) - \mathrm{div}F(\nabla v) \tag{3.15}$$

in $L^\infty\left(0,T;\mathcal{V}_1'\right)$, with initial data $(w(0), w_t(0)) = U_0 - V_0$.

Since $w_t(t) \in \mathcal{V}_2 \hookrightarrow \mathcal{V}_1$, then multiplying equation (3.15) by $w_t(t)$ and integrating over Ω, we get

$$\frac{d}{dt}W(t) \ = \ \frac{1}{2}\left(g'\Box\Delta w\right)(t) - \frac{1}{2}g(t)\|\Delta w(t)\|_2^2 + L_F \ \le \ L_F, \quad t > 0, \tag{3.16}$$

where

$$W(t) = \frac{1}{2}\|w_t(t)\|_2^2 + \frac{1}{2}\|\nabla w_t(t)\|_2^2 + \frac{h(t)}{2}\|\Delta w(t)\|_2^2 + \frac{1}{2}(g\Box\Delta w)(t), \quad t \ge 0,$$

and

$$L_F \ = \ -\int_\Omega [F(\nabla u(t)) - F(\nabla v(t))] \cdot \nabla w_t(t)\, dx.$$

Estimate (5.2) from the appendix, the generalized Hölder inequality, and (2.7) imply

$$\begin{aligned} |L_F| &\leq \int_\Omega |F(\nabla u(t)) - F(\nabla v(t))| \, |\nabla w_t(t)| \, dx \\ &\leq K \sum_{j=1}^{N} \int_\Omega \left(1 + |\nabla u(t)|^{(p_j-1)/2} + |\nabla v(t)|^{(p_j-1)/2}\right) |\nabla w(t)| \, |\nabla w_t(t)| \, dx \\ &\leq K \sum_{j=1}^{N} \mu_{p_j} \left(|\Omega|^{\frac{p_j-1}{2(p_j+1)}} + \|\nabla u(t)\|_{p_j+1}^{\frac{p_j-1}{2}} + \|\nabla v(t)\|_{p_j+1}^{\frac{p_j-1}{2}} \right) \|\Delta w(t)\|_2 \|\nabla w_t(t)\|_2. \end{aligned}$$

From (2.7) and (3.13) we obtain

$$K \sum_{j=1}^{N} \mu_{p_j} \left(|\Omega|^{\frac{p_j-1}{2(p_j+1)}} + \|\nabla u(t)\|_{p_j+1}^{\frac{p_j-1}{2}} + \|\nabla v(t)\|_{p_j+1}^{\frac{p_j-1}{2}} \right) \leq C < \infty,$$

and making use of Young inequality there exists a constant $C_3 = C_3\left(||\nabla u_1||_2, ||\Delta u_0||_2\right) > 0$ such that

$$|L_F| \leq C_3 \left(\|\Delta w(t)\|_2^2 + \|\nabla w_t(t)\|_2^2\right). \tag{3.17}$$

Inserting (3.17) into (3.16) and integrating from 0 to $t \leq T$, one has

$$W(t) \leq W(0) + C_3 \int_0^t \left(\|\Delta w(s)\|_2^2 + \|\nabla w_t(s)\|_2^2\right) ds, \quad t \geq 0. \tag{3.18}$$

On the other hand it is easy to check that

$$\|\Delta w(t)\|_2^2 + \|\nabla w_t(t)\|_2^2 \leq \frac{2}{l} W(t), \quad t \geq 0, \tag{3.19}$$

and

$$W(0) \leq \frac{1}{2}\left(1 + \frac{1}{\lambda_0}\right)\left(\|\Delta w(0)\|_2^2 + \|\nabla w_t(0)\|_2^2\right). \tag{3.20}$$

Combining (3.18)-(3.20) and applying Gronwall's inequality we conclude

$$\left(\|\Delta w(t)\|_2^2 + \|\nabla w_t(t)\|_2^2\right) \leq C_T^2 \left(\|\Delta w(0)\|_2^2 + \|\nabla w_t(0)\|_2^2\right), \quad \forall\, t \in [0, T], \tag{3.21}$$

for some constant $C_T = C(\|U_0\|_{\mathcal{H}}, \|V_0\|_{\mathcal{H}}, T) > 0$. This shows that the estimate (2.10) is guaranteed for regular solutions since we have $(w, w_t) = U - V$. ■

Proof of Theorem 2.3 *(ii)***.** Let us take initial data $(u_0, u_1) \in \mathcal{H}$. Then there exists a sequence $(u_0^n, u_1^n) \in \mathcal{H}_1$ such that

$$u_0^n \to u_0 \text{ in } \mathcal{V}_2 \quad \text{and} \quad u_1^n \to u_1 \text{ in } \mathcal{V}_1. \tag{3.22}$$

For each regular initial data (u_0^n, u_1^n), $n \in \mathbb{N}$, there exists a regular solution (u^n, u_t^n) satisfying (3.13)-(3.14). Taking the multiplier $u_t^n(t)$ in (3.14) and proceeding analogously as in (3.4)-(3.6) then estimate (3.7) holds true. This implies

$$(u^n, u_t^n) \overset{*}{\rightharpoonup} (u, u_t) \quad \text{in} \quad L^\infty(0, T; \mathcal{H}). \tag{3.23}$$

Besides, if we consider $m, n \in \mathbb{N}$, $m \geq n$, and $w = u^m - u^n$, then function (w, w_t) satisfies

$$w_{tt} - \Delta w_{tt} + \Delta^2 w - \int_0^t g(t-s)\Delta^2 w(s)\, ds = \mathrm{div}F(\nabla u^m) - \mathrm{div}F(\nabla u^n)$$

in $L^\infty(0, T; \mathcal{V}_1')$, with initial data $(w(0), w_t(0)) = (u_0^m - u_0^n, u_1^m - u_1^n)$. Taking the multiplier $w_t(t)$ and using analogous arguments as given in (3.16)-(3.20), the estimate (3.21) also holds. This means that

$$\|\Delta(u^m(t) - u^n(t))\|_2^2 + \|\nabla(u_t^m(t) - u_t^n(t))\|_2^2 \leq C\left(\|\Delta(u_0^m - u_0^n)\|_2^2 + \|\nabla(u_1^m - u_1^n)\|_2^2\right),$$

for any $t \in [0, T]$, and some constant $C = C(\|(u_0, u_1)\|_{\mathcal{H}}, T) > 0$. From (3.13) and (3.22), and since C is a constant depending only on the initial data in $\mathcal{H}$, we infer

$$(u^n, u_t^n) \;\; \to \;\; (u, u_t) \quad \text{in} \quad C([0, T], \mathcal{H}). \tag{3.24}$$

Finally, we note that the limits (3.23) and (3.24) are enough to pass to the limit in the approximate problem (3.1)-(3.2) and to obtain a weak solution $(u, u_t) \in C([0, T], \mathcal{H})$ satisfying (2.9) and

$$(I - \Delta)u_{tt} = -\Delta^2 u + \mathrm{div}F(\nabla u) + \int_0^t g(t-s)\Delta^2 u(s)\, ds \quad \text{in} \quad L^\infty(0, T; \mathcal{V}_2').$$

This concludes the proof on existence of weak solutions. ■

Remark 3.2. In a similar procedure we can check that condition (3.23) also holds in the problem without rotational inertia, namely,

$$(u^n, u_t^n) \;\; \stackrel{*}{\rightharpoonup} \;\; (u, u_t) \quad \text{in} \quad L^\infty(0, T; \mathcal{W}).$$

This is sufficient to pass the limit on the corresponding approximate problem to obtain

$$u_{tt} = -\Delta^2 u + \mathrm{div}F(\nabla u) + \int_0^t g(t-s)\Delta^2 u(s)\, ds \quad \text{in} \quad L^\infty(0, T; \mathcal{V}_2').$$

Therefore, we can also conclude the proof of Theorem 2.10 (i). ■

Proof of Theorem 2.3 (iii) **(weak solutions).** Given initial data $U_0 = (u_0, u_1)$, $V_0 = (v_0, v_1) \in \mathcal{H}$, let us consider the corresponding initial regular data $U_0^n = (u_0^n, u_1^n)$, $V_0^n = (v_0^n, v_1^n) \in \mathcal{H}_1$ such that

$$(U_0^n, V_0^n) \;\; \to \;\; (U_0, V_0) \quad \text{in} \quad \mathcal{H} \times \mathcal{H}, \tag{3.25}$$

and the respective regular solutions $U^n = (u^n, u_t^n)$, $V^n = (v^n, v_t^n)$ converging to the weak solutions $U = (u, u_t)$, $V = (v, v_t)$ as in (3.24), namely

$$(U^n, V^n) \;\; \to \;\; (U, V) \quad \text{in} \quad C([0, T], \mathcal{H} \times \mathcal{H}). \tag{3.26}$$

Since (2.10) holds for stronger weak solutions we have

$$||U^n(t) - V^n(t)||_{\mathcal{H}} \; \leq \; C_T||U_0^n - V_0^n||_{\mathcal{H}}, \quad t \in [0, T], \; n \in \mathbb{N}, \tag{3.27}$$

for some constant $C_T = C(\|U_0\|_{\mathcal{H}}, \|V_0\|_{\mathcal{H}}, T) > 0$.

Therefore, (2.10) is given for weak solutions after passing (3.27) to the limit when $n \to \infty$ and applying (3.25)-(3.26). In particular, we have uniqueness of solution in both cases. This completes the proof of Theorem 2.3. ■

4 Uniform decay of the energy

Our methods on stability are similar to (but not equal to) those for viscoelastic wave equations, see for instance [11, 19, 22, 23, 30, 31]. The proofs of Lemma 2.5 and Theorem 2.8 are given first for regular solutions. Then by standard density arguments the conclusion of Theorem 2.8 also holds for weak solutions.

Proof of Lemma 2.5. By taking the multiplier u_t with (1.6) and using the identities (3.4)-(3.5) for the solution, then the energy defined in (2.1) satisfies (2.11). Therefore, the proof of Lemma 2.5 follows readily. ∎

Before proving Theorem 2.8 we need to state some technical lemmas.

Lemma 4.1. *Under the assumptions of Theorem* 2.3 *we have*

$$\frac{d}{dt}E(t) \;\leq\; \frac{1}{2}(g'\Box\Delta u)(t) \;\leq\; 0, \quad \forall\; t > 0. \tag{4.1}$$

Proof. Inequality (4.1) is an immediate consequence of Lemma 2.5. ∎

Let us first define the functionals

$$\begin{aligned}
\phi_\chi(t) &= \int_\Omega \left(\int_0^t g(t-s)|\chi(t)-\chi(s)|\,ds \right)^2 dx, \\
\psi_\chi(t) &= \int_\Omega \left(-\int_0^t g'(t-s)|\chi(t)-\chi(s)|\,ds \right)^2 dx, \\
\zeta_\chi(t) &= \int_\Omega \left(\int_0^t g(t-s)|\chi(s)|\,ds \right)^2 dx.
\end{aligned}$$

Lemma 4.2. *Under the assumptions of Theorem* 2.3 *we have:*

(*a*) $\phi_u(t) \leq \dfrac{(1-l)}{\lambda_1}(g\Box\Delta u)(t), \quad \forall\; t \geq 0.$

(*b*) $\phi_{\nabla u}(t) \leq \dfrac{(1-l)}{\lambda_2}(g\Box\Delta u)(t), \quad \forall\; t \geq 0.$

(*c*) $\phi_{\Delta u}(t) \leq (1-l)(g\Box\Delta u)(t), \quad \forall\; t \geq 0.$

(*d*) $\psi_u(t) \leq \dfrac{g(0)}{\lambda_1}(-g'\Box\Delta u)(t), \quad \forall\; t \geq 0.$

(*e*) $\psi_{\nabla u}(t) \leq \dfrac{g(0)}{\lambda_2}(-g'\Box\Delta u)(t), \quad \forall\; t \geq 0.$

(*f*) $\zeta_{\Delta u}(t) \leq 2(1-l)(g\Box\Delta u)(t) + 2(1-l)^2\|\Delta u(t)\|_2^2, \quad \forall\; t \geq 0.$

Proof. To prove the items (a)-(e) it is enough to apply Hölder's inequality along with the embeddings $\mathcal{V}_2 \hookrightarrow \mathcal{V}_1 \hookrightarrow \mathcal{V}_0$ and the first condition in (2.6). Moreover, from (2.6) and item (c) of Lemma 4.2, we prove the item (f) as follows.

$$\begin{aligned}\zeta_{\Delta u}(t) &\leq \int_\Omega \left(\int_0^t g(t-s)\big(|\Delta u(t)-\Delta u(s)|+|\Delta u(t)|\big)\,ds\right)^2 dx\\ &\leq \int_\Omega \left(\int_0^t g(t-s)|\Delta u(t)-\Delta u(s)|\,ds + \int_0^t g(t-s)\,ds|\Delta u(t)|\right)^2 dx\\ &\leq 2\phi_{\Delta u}(t) + 2\left(\int_0^t g(s)\,ds\right)^2 \|\Delta u(t)\|_2^2\\ &\leq 2(1-l)(g\Box\Delta u)(t) + 2(1-l)^2\|\Delta u(t)\|_2^2.\end{aligned}$$

■

Let us now define the functionals

$$G(t) \;=\; E(t) + \epsilon_1\Phi(t) + \epsilon_2\Psi(t), \quad t\geq 0, \tag{4.2}$$

where ϵ_1, $\epsilon_2 > 0$ will be fixed later and

$$\Phi(t) \;=\; \xi(t)\int_\Omega \big(u_t(t)-\Delta u_t(t)\big)u(t)\,dx, \tag{4.3}$$

$$\Psi(t) \;=\; -\xi(t)\int_\Omega \big(u_t(t)-\Delta u_t(t)\big)\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)dx. \tag{4.4}$$

Lemma 4.3. *Under the assumptions of Theorem* 2.8 *there exists a constant* $c_0 > 0$ *such that* $\Phi(t)$ *given in* (4.3) *satisfies*

$$\begin{aligned}\frac{d}{dt}\Phi(t) &\leq c_0\xi(t)\Big[\|u_t(t)\|_2^2 + \|\nabla u_t(t)\|_2^2 + (g\Box\Delta u)(t)\Big]\\ &\quad -\xi(t)\left[\frac{\beta_1}{2}\|\Delta u(t)\|_2^2 + E(t)\right], \quad \forall\, t>0,\end{aligned} \tag{4.5}$$

where $\beta_1 = \beta/2 > 0$.

Proof. Differentiating $t\mapsto\Phi(t)$, using equation (1.6) and integrating by parts we get

$$\begin{aligned}\frac{d}{dt}\Phi(t) &= \xi(t)\Big[\|u_t(t)\|_2^2 + \|\nabla u_t(t)\|_2^2\Big] + \xi'(t)J_1 + \xi(t)J_2\\ &\quad -\xi(t)\Big[\|\Delta u(t)\|_2^2 + \int_\Omega F(\nabla u(t))\cdot\nabla u(t)\,dx\Big],\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}J_1 &= (u_t(t),u(t)) + (\nabla u_t(t),\nabla u(t)),\\ J_2 &= \int_0^t g(t-s)(\Delta u(s),\Delta u(t))\,ds.\end{aligned}$$

Now, applying Young's inequality with $\eta_1 > 0$ and $\eta_2 > 0$, it is easy to check that

$$\begin{aligned}|J_1| &\leq \eta_1\lambda\|\Delta u(t)\|_2^2 + \frac{1}{4\eta_1}\left(\|u_t(t)\|_2^2 + \|\nabla u_t(t)\|_2^2\right),\\ |J_2| &\leq \left(\int_0^t g(s)\,ds\right)\|\Delta u(t)\|_2^2 + \eta_2\|\Delta u(t)\|_2^2 + \frac{1}{4\eta_2}(g\Box\Delta u)(t).\end{aligned}$$

Inserting these two last estimates into (4.6), using the third condition in (2.13), Adding and subtracting $\xi(t)E(t)$, we obtain

$$\begin{aligned}\frac{d}{dt}\Phi(t) &\leq \xi(t)\left(\frac{3}{2} + \frac{\xi_0}{4\eta_1}\right)\left[\|u_t(t)\|_2^2 + \|\nabla u_t(t)\|_2^2\right] + \xi(t)\left(\frac{1}{2} + \frac{1}{4\eta_2}\right)(g\Box\Delta u)(t)\\ &\quad - \xi(t)\frac{h(t)}{2}\|\Delta u(t)\|_2^2 + \xi(t)\int_\Omega \left[f(\nabla u(t)) - F(\nabla u(t))\cdot\nabla u(t)\right]dx\\ &\quad + \xi(t)(\lambda\xi_0\eta_1 + \eta_2)\|\Delta u(t)\|_2^2 - \xi(t)E(t).\end{aligned}$$

Now applying assumption (2.5) and condition (2.6), we have

$$\begin{aligned}\frac{d}{dt}\Phi(t) &\leq \xi(t)\left(\frac{3}{2} + \frac{\xi_0}{4\eta_1}\right)\left[\|u_t(t)\|_2^2 + \|\nabla u_t(t)\|_2^2\right] + \xi(t)\left(\frac{1}{2} + \frac{1}{4\eta_2}\right)(g\Box\Delta u)(t)\\ &\quad - \xi(t)\left(\frac{\beta}{2} - \lambda\xi_0\eta_1 - \eta_2\right)\|\Delta u(t)\|_2^2 - \xi(t)E(t). \qquad (4.7)\end{aligned}$$

Since $\beta_1 = \frac{\beta}{2} > 0$, so choosing $\eta_2 = \eta_1 \leq \frac{\beta_1}{2(1+\lambda\xi_0)}$, and setting $c_0 = \max\left\{\frac{3}{2} + \frac{\xi_0}{4\eta_1}, \frac{1}{2} + \frac{1}{4\eta_1}\right\}$ in (4.7), we conclude that (4.5) holds true. This completes the proof of Lemma 4.3. ■

Lemma 4.4. *Under the assumptions of Theorem* 2.8*, and given any* $\delta > 0$*, then there exists a constant* $c_\delta > 0$ *such that* Ψ *defined in* (4.4) *satisfies*

$$\begin{aligned}\frac{d}{dt}\Psi(t) &\leq \left(\delta(1+\xi_0) - \int_0^t g(s)\,ds\right)\xi(t)\left[\|u_t(t)\|_2^2 + \|\nabla u_t(t)\|_2^2\right] + 4\delta\xi(t)\|\Delta u(t)\|_2^2\\ &\quad + c_\delta\left(1 + [E(0)]^{\frac{p-1}{2}}\right)\xi(t)(g\Box\Delta u)(t) + c_\delta(-g'\Box\Delta u)(t), \quad \forall\ t > 0, \qquad (4.8)\end{aligned}$$

where

$$p = \begin{cases}\max\{p_1,\dots,p_N\} & \text{if} \quad E(0) \geq 1,\\ \min\{p_1,\dots,p_N\} & \text{if} \quad E(0) < 1.\end{cases}$$

Proof. Differentiating Ψ, using equation (1.6) and integrating by parts we get

$$\frac{d}{dt}\Psi(t) = -\left(\int_0^t g(s)\,ds\right)\xi(t)\left[\|u_t(t)\|_2^2 + \|\nabla u_t(t)\|_2^2\right] + \sum_{j=1}^{6} I_j + I_F, \qquad (4.9)$$

where

$$
\begin{aligned}
I_1 &= -\xi'(t)\int_\Omega u_t(t)\left(\int_0^t g(t-s)(u(t)-u(s))\,ds\right)dx,\\
I_2 &= -\xi'(t)\int_\Omega \nabla u_t(t)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)dx,\\
I_3 &= \xi(t)\int_\Omega \Delta u(t)\left(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))\,ds\right)dx,\\
I_4 &= -\xi(t)\int_\Omega \left(\int_0^t g(t-s)\Delta u(s)\,ds\right)\left(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))\,ds\right)dx,\\
I_5 &= \xi(t)\int_\Omega u_t(t)\left(-\int_0^t g'(t-s)(u(t)-u(s))\,ds\right)dx,\\
I_6 &= \xi(t)\int_\Omega \nabla u_t(t)\left(-\int_0^t g'(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)dx,\\
I_F &= \xi(t)\int_\Omega F(\nabla u(t))\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)dx,
\end{aligned}
$$

Now let us estimate I_j, $j=1,\dots,6$, and I_F. From Young's inequality with $\delta>0$, item (a) of Lemma 4.2 and assumption (2.13) we obtain

$$
\begin{aligned}
|I_1| &\le \left|\frac{\xi'(t)}{\xi(t)}\right|\xi(t)\left(\delta\|u_t(t)\|_2^2+\frac{1}{4\delta}\phi_u(t)\right)\\
&\le \delta\xi_0\xi(t)\|u_t(t)\|_2^2+\frac{\xi_0}{4\delta\lambda_1}(1-l)\xi(t)(g\Box\Delta u)(t) \qquad (4.10)
\end{aligned}
$$

Analogously, but using items (b) and (c) of Lemma 4.2 instead of (a), we have

$$
\begin{aligned}
|I_2| &\le \delta\xi_0\xi(t)\|\nabla u_t(t)\|_2^2+\frac{\xi_0}{4\delta\lambda_2}(1-l)\xi(t)(g\Box\Delta u)(t), \qquad (4.11)\\
|I_3| &\le \delta\xi(t)\|\Delta u(t)\|_2^2+\frac{1}{4\delta}(1-l)\xi(t)(g\Box\Delta u)(t). \qquad (4.12)
\end{aligned}
$$

Again from Young's inequality with $\delta>0$, items (c), (f) and (d) of Lemma 4.2, we deduce

$$
\begin{aligned}
|I_4| &\le \delta\xi(t)\zeta_{\Delta u}(t)+\frac{1}{4\delta}\xi(t)\phi_{\Delta u}(t)\\
&\le 2\delta(1-l)^2\xi(t)\|\Delta u(t)\|_2^2+\left(2\delta+\frac{1}{4\delta}\right)(1-l)\xi(t)(g\Box\Delta u)(t), \qquad (4.13)
\end{aligned}
$$

and

$$
\begin{aligned}
|I_5| &\le \delta\xi(t)\|u_t(t)\|_2^2+\frac{1}{4\delta}\xi(t)\psi_u(t)\\
&\le \delta\xi(t)\|u_t(t)\|_2^2+g(0)\frac{\xi_1}{4\delta\lambda_1}(-g'\Box\Delta u)(t). \qquad (4.14)
\end{aligned}
$$

Similarly with item (e) in the place of (d) in Lemma 4.2, we also have

$$|I_6| \;\leq\; \delta\xi(t)\|\nabla u_t(t)\|_2^2 + g(0)\frac{\xi_1}{4\delta\lambda_2}(-g'\Box\Delta u)(t). \tag{4.15}$$

Now with respect to I_F we have

$$|I_F| \;\leq\; \xi(t)\int_0^t g(t-s)\underbrace{\left(\int_\Omega |F(\nabla u(t))||\nabla u(t)-\nabla u(s)|\,dx\right)}_{:=I_F^1} ds.$$

Applying (5.3) from the appendix with $F(0)=0$, Hölder's inequality, Young's inequality with $\delta>0$, and since $\mathcal{V}_2 \hookrightarrow W_0^{1,p_j+1}(\Omega) \hookrightarrow \mathcal{V}_1,\ j=1,\ldots,N$, we infer

$$\begin{aligned}
I_F^1 \;&\leq\; K\int_\Omega \left(\sum_{j=1}^N \left(1+|\nabla u(t)|^{(p_j-1)/2}\right)\right)|\nabla u(t)||\nabla u(t)-\nabla u(s)|\,dx \\
&\leq\; K\sum_{j=1}^N \left(|\Omega|^{\frac{p_j-1}{2(p_j+1)}} + \|\nabla u(t)\|_{p_j+1}^{\frac{p_j-1}{2}}\right)\|\nabla u(t)\|_{p_j+1}\|\nabla u(t)-\nabla u(s)\|_2 \\
&\leq\; \|\Delta u(t)\|_2\left[\frac{K}{\lambda_2}\sum_{j=1}^N \mu_{p_j}\left(|\Omega|^{\frac{p_j-1}{2(p_j+1)}} + \|\nabla u(t)\|_{p_j+1}^{\frac{p_j-1}{2}}\right)\right]\|\Delta u(t)-\Delta u(s)\|_2 \\
&\leq\; \delta\|\Delta u(t)\|_2^2 + \frac{1}{4\delta}\underbrace{\left[\frac{K}{\lambda_2}\sum_{j=1}^N \mu_{p_j}\left(|\Omega|^{\frac{p_j-1}{2(p_j+1)}} + \|\nabla u(t)\|_{p_j+1}^{\frac{p_j-1}{2}}\right)\right]^2}_{:=I_F^2}\|\Delta u(t)-\Delta u(s)\|_2^2.
\end{aligned}$$

Since $\|\nabla u(t)\|_{p_j+1} \leq \mu_{p_j}\|\Delta u(t)\|_2$ and $\frac{\beta}{2}\|\Delta u(t)\|_2^2 \leq E(t) \leq E(0)$ for any $t>0$, then

$$\begin{aligned}
I_F^2 \;&\leq\; \frac{2K^2}{\lambda_2^2}\left(\sum_{j=1}^N \mu_{p_j}|\Omega|^{\frac{p_j-1}{2(p_j+1)}}\right)^2 + \frac{2K^2}{\lambda_2^2}\left(\sum_{j=1}^N \mu_{p_j}^{\frac{p_j+1}{2}}\left(\frac{2}{\beta}\right)^{\frac{p_j-1}{4}}[E(0)]^{\frac{p_j-1}{4}}\right)^2 \\
&\leq\; \mu_1 + \mu_2[E(0)]^{\frac{p-1}{2}},
\end{aligned}$$

where we consider

$$p := \begin{cases} \max\{p_1,\ldots,p_N\} & \text{if} \quad E(0)\geq 1, \\ \min\{p_1,\ldots,p_N\} & \text{if} \quad E(0)<1, \end{cases}$$

$$\mu_1 := \frac{2K^2}{\lambda_2^2}\left(\sum_{j=1}^N \mu_{p_j}|\Omega|^{\frac{p_j-1}{2(p_j+1)}}\right)^2 \quad \text{and} \quad \mu_2 := \frac{2K^2}{\lambda_2^2}\left(\sum_{j=1}^N \mu_{p_j}^{\frac{p_j+1}{2}}\left(\frac{2}{\beta}\right)^{\frac{p_j-1}{4}}\right)^2.$$

Thus,

$$I_F^1 \leq \delta\|\Delta u(t)\|_2^2 + \frac{1}{4\delta}\left(\mu_1 + \mu_2[E(0)]^{\frac{p-1}{2}}\right)\|\Delta u(t)-\Delta u(s)\|_2^2$$

from where it follows that

$$|I_F| \;\leq\; \delta\xi(t)\|\Delta u(t)\|_2^2 + \frac{1}{4\delta}\left(\mu_1 + \mu_2[E(0)]^{\frac{p-1}{2}}\right)\xi(t)(g\Box\Delta u)(t). \tag{4.16}$$

Inserting (4.10)-(4.16) into (4.9), and since $1-l<1$, yields

$$\begin{aligned}\frac{d}{dt}\Psi(t) \;\leq\; & \left(\delta(1+\xi_0)-\int_0^t g(s)\,ds\right)\xi(t)\Big[\|u_t(t)\|_2^2+\|\nabla u_t(t)\|_2^2\Big] \\ & +4\delta\xi(t)\|\Delta u(t)\|_2^2+g(0)\frac{\xi_1\lambda}{4\delta}(-g'\Box\Delta u)(t) \\ & +\frac{1}{4\delta}\left(2+\xi_0\lambda+8\delta^2+\mu_1+\mu_2[E(0)]^{\frac{p-1}{2}}\right)\xi(t)(g\Box\Delta u)(t).\end{aligned}$$

Therefore, inequality (4.8) follows by taking $c_\delta=\frac{1}{4\delta}\max\{2+\xi_0\lambda+8\delta^2+\mu_1,\mu_2,g(0)\xi_1\lambda\}$. This concludes the proof of Lemma 4.4. ■

Lemma 4.5. *Under the assumptions of Theorem* 2.8 *and fixing any* $t_0>0$*, then*

$$\frac{d}{dt}G(t)\;\leq\;-\epsilon_1\xi(t)E(t),\quad \forall\; t\geq t_0, \tag{4.17}$$

for some positive constant $\epsilon_1\sim\frac{c_1}{1+[E(0)]^{\frac{p-1}{2}}}$*, with* $c_1>0$ *independent of the initial data.*

Proof. From definition of $G(t)$ in (4.2), and Lemmas 4.1, 4.3 and 4.4, we get

$$\begin{aligned}\frac{d}{dt}G(t) \;\leq\; & \left(\epsilon_1c_0+\epsilon_2\Big(\delta b_1-\int_0^t g(s)\,ds\Big)\right)\xi(t)\Big[\|u_t(t)\|_2^2+\|\nabla u_t(t)\|_2^2\Big] \\ & -\left(\epsilon_1\frac{\beta_1}{2}-4\delta\epsilon_2\right)\xi(t)\|\Delta u(t)\|_2^2-\epsilon_1\xi(t)E(t) \\ & +\left(\epsilon_1c_0+\epsilon_2c_\delta\left(1+[E(0)]^{\frac{p-1}{2}}\right)\right)\xi(t)(g\Box\Delta u)(t)+\left(\frac{1}{2}-\epsilon_2c_\delta\right)(g'\Box\Delta u)(t),\end{aligned} \tag{4.18}$$

for any $\delta,\epsilon_1,\epsilon_2>0$, where we denote $b_1=1+\xi_0>0$. By fixing any $t_0>0$ we note that

$$\int_0^t g(s)\,ds\geq\int_0^{t_0}g(s)\,ds:=g_0>0,\quad\forall\; t\geq t_0.$$

From this and condition (2.12) we can rewrite (4.18) as follows

$$\begin{aligned}\frac{d}{dt}G(t) \;\leq\; & -\big(\epsilon_2(g_0-\delta b_1)-\epsilon_1c_0\big)\xi(t)\Big[\|u_t(t)\|_2^2+\|\nabla u_t(t)\|_2^2\Big] \\ & -\left(\epsilon_1\frac{\beta_1}{2}-4\delta\epsilon_2\right)\xi(t)\|\Delta u(t)\|_2^2-\epsilon_1\xi(t)E(t) \\ & +\left(\frac{1}{2}-\epsilon_1c_0-2\epsilon_2c_\delta\left(1+[E(0)]^{\frac{p-1}{2}}\right)\right)(g'\Box\Delta u)(t),\end{aligned} \tag{4.19}$$

for every $t\geq t_0$. Now we first choose $0<\delta\leq\min\{\frac{g_0}{2b_1},\frac{g_0\beta_1}{32c_0}\}$. Thus

$$g_0-\delta b_1\geq\frac{g_0}{2}\quad\text{ and }\quad 4\delta\leq\frac{g_0\beta_1}{8c_0}. \tag{4.20}$$

Once fixed $\delta > 0$ we pick out $\epsilon_1 > 0$ and $\epsilon_2 > 0$ small enough such that

$$\frac{1}{2}\epsilon_2 \,<\, \frac{2c_0}{g_0}\epsilon_1 \,<\, \epsilon_2 \,<\, \min\left\{\frac{1}{2g_0}, \frac{1}{8c_\delta\left(1+[E(0)]^{\frac{p-1}{2}}\right)}\right\}. \tag{4.21}$$

Therewith (4.20)-(4.21) imply that

$$\epsilon_2(g_0-\delta b_1)-\epsilon_1 c_0 > 0, \quad \epsilon_1\frac{\beta_1}{2}-4\delta\epsilon_2 > 0, \quad \frac{1}{2}-\epsilon_1 c_0 - 2\epsilon_2 c_\delta\left(1+[E(0)]^{\frac{p-1}{2}}\right) > 0.$$

Therefore, since $g' \leq 0$, we obtain from (4.19) that estimate (4.17) holds true for some positive constant $\epsilon_1 \sim c_1/\left(1+[E(0)]^{\frac{p-1}{2}}\right)$, where $c_1 > 0$ is independent of the initial data. This completes the proof of Lemma 4.5. ■

With the lemmas above we have obtained the main ingredients to prove Theorem 2.8.

Proof of Theorem 2.8. First of all we note that

$$\frac{1}{2}E(t) \,\leq\, G(t) \,\leq\, \frac{3}{2}E(t), \quad \forall\, t \geq 0, \tag{4.22}$$

where we take ϵ_1 and ϵ_2 such that

$$0 < \epsilon_1,\, \epsilon_2 < \beta_1 \min\left\{\frac{1}{2\xi_1}, \frac{1}{\lambda\xi_1}\right\}. \tag{4.23}$$

Indeed, to prove (4.22) we use Hölder and Young inequalities, Lemma 4.2 (a)-(b), and (2.5) with $\alpha_0 = 0$.

Now let us fix $t_0 = 1$ and choose $\epsilon_1 > 0$ and $\epsilon_2 > 0$ so that (4.21) and (4.23) are satisfied. Then from Lemma 4.5 and (4.22) we have

$$\frac{d}{dt}G(t) \leq -\epsilon_1\xi(t)E(t) \leq -\frac{2\epsilon_1}{3}\xi(t)G(t), \quad \forall\, t \geq 1.$$

A straightforward computation implies that

$$G(t) \leq G(1)e^{-\gamma\int_1^t \xi(s)\,ds}, \quad \forall\, t \geq 1,$$

where $\gamma = \frac{2\epsilon_1}{3} \sim k/\left(1+[E(0)]^{\frac{p-1}{2}}\right)$, for some $k > 0$. Applying again (4.22) and since $E(t)$ is nonincreasing we obtain

$$E(t) \leq 3E(1)\,e^{-\gamma\int_1^t \xi(s)\,ds} \leq \left(3E(0)\,e^{\gamma\int_0^1 \xi(s)\,ds}\right)\, e^{-\gamma\int_0^t \xi(s)\,ds},$$

from where it follows that

$$E(t) \leq c\,e^{-\gamma\int_0^t \xi(s)\,ds}, \quad \forall\, t \geq 1, \tag{4.24}$$

where $c = 3E(0)\,e^{\gamma\int_0^1 \xi(s)\,ds}$, and $\gamma \sim k/\left(1+[E(0)]^{\frac{p-1}{2}}\right)$ for some positive constant k.

On the other hand, since $0 < \xi(t) \leq \xi_1 > 0$ for any $t \geq 0$, we see that

$$1 < e^{\gamma \int_t^1 \xi(s)\, ds} = \left(e^{\gamma \int_0^1 \xi(s)\, ds} \right) e^{-\gamma \int_0^t \xi(s)\, ds} < e^{\gamma \xi_1} < \infty, \quad \forall\, t \in [0,1].$$

Then, since $E(0)\, e^{\gamma \int_0^1 \xi(s)\, ds} < c$, we get

$$E(t) \leq E(0) \leq c\, e^{-\gamma \int_0^t \xi(s)\, ds}, \quad \forall\, t \in [0,1]. \tag{4.25}$$

Therefore, the uniform decay (2.14) is ensured by estimates (4.24)-(4.25). The proof of Theorem 2.8 is now complete. ■

Remark 4.6. The proof of Theorem 2.10 (ii) can be done with minor changes on the above calculations. In fact, we can formally check that the energy $\mathcal{E}(t)$ satisfies (2.17) by excluding the terms appearing due to the presence of rotational inertia term. ■

5 Appendix

5.1 Rates of energy decay

We emphasize below that (2.14) provides several rates of energy decay according to the function $\xi(t)$. In fact, the Examples 5.1-5.4 are motivated by [22, 23, 19, 31, 32, 25]. Besides, the Examples 5.5 and 5.6 were first shown in [12] and illustrate the wide variety of decay rates provided by (2.14) which are faster than pure exponential decay when we consider small initial data or else the linear case.

Example 5.1. Let us consider $\xi(t) = \kappa > 0$. Then condition (2.13) is fulfilled and from (2.14) we obtain the following exponential decay

$$E(t) \leq c\, e^{-\kappa\gamma t}, \quad t \geq 0.$$

Example 5.2. Let us also consider $\xi(t) = \kappa \ln(a+1)$, where $a > 0$ and $\kappa > 0$. Then, $\xi(t)$ satisfies (2.13) and applying (2.14) results

$$E(t) \leq c\,(a+1)^{-\kappa\gamma t}, \quad t \geq 0.$$

Example 5.3. Let us take a rational function $\xi(t) = \frac{\kappa}{t+1}$, with $\kappa > 0$. It is also easy to check that condition (2.13) holds true. From (2.14) we have the following polynomial-type decay

$$E(t) \leq \frac{c}{(t+1)^{\kappa\gamma}}, \quad t \geq 0.$$

The case $0 < \kappa\gamma \leq 2$ is contemplated if we consider κ small enough. However, the interesting case consists in considering polynomial decay of higher order which is possible by taking large values for κ.

Example 5.4. Let us now take $\xi(t) = \frac{\kappa}{(t+e)\ln(t+e)}$, with $\kappa > 0$, then

$$\xi'(t) = -\kappa \frac{\ln(t+e)+1}{[(t+e)\ln(t+e)]^2} < 0 \quad \text{and} \quad \left|\frac{\xi'(t)}{\xi(t)}\right| = \frac{1}{(t+e)\ln(t+e)} + \frac{1}{t+e} < \frac{2}{e},$$

for every $t \geq 0$. From (2.14) we get the logarithmic decay type

$$E(t) \leq \frac{c}{[\ln(t+e)]^{\kappa\gamma}}, \qquad t \geq 0.$$

Example 5.5. If $\xi(t) = \kappa \coth(t+\theta)$, $t \geq 0$, where $\kappa > 0$ and $\theta = \ln(1+\sqrt{2})$, then

$$\xi'(t) = -\kappa[\cosh(t+\theta)]^2 < 0 \quad \text{and} \quad \left|\frac{\xi'(t)}{\xi(t)}\right| = \frac{1}{\cosh(t+\theta)\sinh(t+\theta)} < 1,$$

for every $t \geq 0$. From (2.14) we obtain the following hyperbolic decay

$$E(t) \leq \frac{c}{[\sinh(t+\theta)]^{\kappa\gamma}}, \qquad t \geq 0.$$

Example 5.6. We also consider $\xi(t) = \frac{\kappa(1+2\ln(t+e^{1/2}))}{t+e^{1/2}}$, $t \geq 0$, where $\kappa > 0$. Thus $\xi(t) > 0$, and

$$\xi'(t) = \frac{\kappa(1-2\ln(t+e^{1/2}))}{(t+e^{1/2})^2} \leq 0 \quad \text{and} \quad \left|\frac{\xi'(t)}{\xi(t)}\right| \leq \frac{1}{t+e^{1/2}} < \frac{1}{e^{1/2}},$$

for every $t \geq 0$, see for instance [12]. So $\xi(t)$ also fulfills condition (2.13). From (2.14) we conclude that the energy has a transcendental decay like

$$E(t) \leq \frac{c_\kappa}{(t+e^{1/2})^{\kappa\gamma(1+\ln(t+e^{1/2}))}},$$

for all $t \geq 0$, where $c_\kappa = e^{3\kappa/4}c > 0$.

5.2 The vector field F

In this section we show an interesting property to vector fields which satisfy a condition like (2.3). This is given as an application of the Mean Value Inequality.

Lemma 5.1. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a C^1-vector field given by $F = (F_1, \ldots, F_N)$.*

(a) If there exist positive constants $k_1, \ldots, k_N$ and $q_1, \ldots, q_N$ such that

$$|\nabla F_j(z)| \leq k_j(1+|z|^{q_j}), \quad \forall\, z \in \mathbb{R}^N, \ \forall\, j = 1, \ldots, N. \tag{5.1}$$

Then, there exists a constant $K = K(k_j, q_j, N) > 0$, $j = 1, \ldots, N$, such that

$$|F(x) - F(y)| \ \leq \ K \sum_{j=1}^{N} (1+|x|^{q_j}+|y|^{q_j})\, |x-y|, \quad \forall\, x, y \in \mathbb{R}^N. \tag{5.2}$$

(*b*) *In particular, we have*

$$|F(x)| \le |F(0)| + K\sum_{j=1}^{N}(1+|x|^{q_j})\,|x|, \quad \forall\, x\in\mathbb{R}^N. \tag{5.3}$$

Proof. (*a*) From condition (5.1) it is not so difficult to check that $F' : \mathbb{R}^N \to \mathcal{L}(\mathbb{R}^N)$ satisfies

$$||F'(z)||_{\mathcal{L}(\mathbb{R}^N)} \le N\sum_{j=1}^{N} k_j(1+|z|^{q_j}), \quad \forall\, z\in\mathbb{R}^N.$$

Given $x, y \in \mathbb{R}^N$, we consider $z \in [x,y] \subset \mathbb{R}^N$ written as

$$z = (1-\theta)y + \theta x, \quad \theta = \theta(x,y) \in [0,1].$$

Thus,

$$|z|^{q_j} \le 2^{q_j}(|x|^{q_j} + |y|^{q_j}), \quad \forall\, j = 1,\dots,N,$$

from where it follows that

$$||F'(z)||_{\mathcal{L}(\mathbb{R}^N)} \le K\sum_{j=1}^{N}(1+|x|^{q_j}+|y|^{q_j}), \quad \forall\, z\in[x,y],$$

where $K = N \max\limits_{1\le j\le N}\{2^{q_j}k_j\}$. Applying the Mean Value Inequality we obtain (5.2).

(*b*) It is suffices to define $G(z) = F(z) - F(0)$. ■

5.3 Examples for F

We finally give examples of vector fields satisfying conditions like (2.3) and (2.5). More generally, we show below some applications of conservative C^1-vector fields $F = \nabla f$ such that (5.1) holds and also

$$-a_0 - a_1|z|^2 \le f(z) \le F(z)\cdot z + a_2|z|^2, \quad \forall\, z\in\mathbb{R}^N, \tag{5.4}$$

for some nonnegative constants $a_0, a_1, a_3 \ge 0$.

Example 5.7. Let us first consider

$$\begin{aligned} F : \mathbb{R}^N &\longrightarrow \mathbb{R}^N \\ z &\longmapsto F(z) = |z|^q z, \quad q \ge 0. \end{aligned}$$

Denoting by $F(z) = (F_1(z),\dots,F_N(z))$ and $z = (z_1,\dots,z_N) \in \mathbb{R}^N$, then

$$F_j(z) = |z|^q z_j, \quad j = 1,\dots,N.$$

If we consider $0 \ne z \in \mathbb{R}^N$ and $i, j = 1,\dots,N$, we have

$$\begin{aligned} \frac{\partial}{\partial z_i}F_j(z) &= q|z|^{q-2}z_iz_j && \text{for } i \ne j, \\ \frac{\partial}{\partial z_j}F_j(z) &= q|z|^{q-2}z_j^2 + |z|^q && \text{for } i = j. \end{aligned} \tag{5.5}$$

It is also easy to check by definition that

$$\frac{\partial}{\partial z_i} F_j(0) = 0 = \lim_{z \to 0} \frac{\partial}{\partial z_i} F_j(z), \quad i, j = 1, \dots, N.$$

Thus, the components $F_1, \dots, F_N$ are C^1-functions in $\mathbb{R}^N$. Besides, from (5.5) we get

$$\left| \frac{\partial}{\partial z_i} F_j(z) \right| \leq (1+q)|z|^q, \quad i, j = 1, \dots, N.$$

This is suffices to ensure that (5.1) holds true. Moreover, we note that $F = \nabla f$, where

$$\begin{aligned} f : \mathbb{R}^N & \longrightarrow \mathbb{R} \\ z & \longmapsto f(z) = \frac{1}{q+2} |z|^{q+2}, \end{aligned}$$

and then condition (5.4) is readily verified for any $a_0, a_1, a_2 \geq 0$.

Therefore, this vector field generates the following operator

$$\mathrm{div} F(\nabla u) = \mathrm{div} \left(|\nabla u|^q \nabla u \right),$$

which consists a p-Laplacian one by taking $p = 2q + 1$ satisfying (2.4).

Example 5.8. The above argument can be applied to the vector field $F = \nabla f$, where

$$f(z) = \frac{k}{q+2} |z|^{q+2} + \tau \cdot z, \quad z = (z_1, \dots, z_N) \in \mathbb{R}^N,$$

with $q \geq 0$, $k > 0$, and $\tau = (\tau_1, \dots, \tau_N) \in \mathbb{R}^N$. Thus, condition (5.1) follows analogously as above whereas (5.4) is fulfilled with $a_0 = \frac{|\tau|^2}{2}$ $a_1 = \frac{1}{2}$, and any $a_2 \geq 0$.

Example 5.9. Another case of p-Laplacian operator arises when we consider the vector field $F = (F_1, \dots, F_N)$ whose components F_j, $j = 1, \dots, N$, are given by

$$F_j(z) = |z_j|^{p-2} z_j, \quad \forall\, z = (z_1, \dots, z_N) \in \mathbb{R}^N,$$

where $p \geq 2$. In this case

$$\mathrm{div} F(\nabla u) = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left(\left| \frac{\partial u}{\partial x_j} \right|^{p-2} \frac{\partial u}{\partial x_j} \right).$$

Example 5.10. To illustrate another vector field, different one of p-Laplacian type, we consider $F = \nabla f$, where the potential function is given by

$$f(z) = \ln \left(\sqrt{|z|^2 + 1} \right), \quad z = (z_1, \dots, z_N) \in \mathbb{R}^N.$$

In such case we have

$$F(z) = \frac{z}{|z|^2 + 1}, \quad \forall\, z \in \mathbb{R}^N,$$

which vanishes when $z \to \infty$. It is easy to check that F and f satisfy (5.1) and (5.4).

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