Transmission problems in (thermo-)viscoelasticity with Kelvin-Voigt damping: non-exponential, strong and polynomial stability

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Abstract: We investigate transmission problems between a (thermo-)viscoelastic system with Kelvin-Voigt damping, and a purely elastic system. It is shown that neither the elastic damping by Kelvin-Voigt mechanisms nor the dissipative effect of the temperature in one material can assure the exponential stability of the total system when it is coupled through transmission to a purely elastic system. The approach shows the lack of exponential stability using Weyl's theorem on perturbations of the essential spectrum. Instead, strong stability can be shown using the principle of unique continuation. To prove polynomial stability we provide an extended version of the characterizations in [4]. Observations on the lack of compacity of the inverse of the arising semigroup generators are included too. The results apply to thermo-viscoelastic systems, to purely elastic systems as well as to the scalar case consisting of wave equations.

1 Introduction

We consider transmission problems for elastic materials in $d = 1, 2, 3$ dimensions, where one viscoelastic material experiences dissipation given by a Kelvin-Voigt damping mechanism and, in our most general case, also by heat conduction, while the second elastic material is purely elastic. The two systems have an interface where classical transmission

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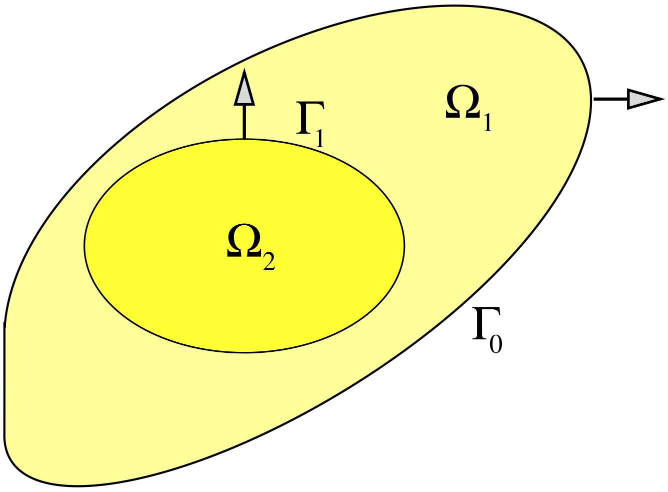
conditions are given. The configuration is described in the figure,



where the inner part Ω_2 represents the undamped purely elastic material, while Ω_1 represents the elastic material with a viscous damping of Kelvin-Voigt type and a damping through heat conduction. The interface is given by $\Gamma_1 = \partial\Omega_2$, where the usual continuity conditions on the displacement vectors and on the elastic normal derivatives prescribe the transmission condition below.

Let $u : \Omega_1 \times [0,\infty) \longrightarrow \mathbb{R}^d$ and $v : \Omega_2 \times [0,\infty) \longrightarrow \mathbb{R}^d$ denote the displacement vectors in the two different materials, let $\theta : \Omega_1 \times [0,\infty) \longrightarrow \mathbb{R}$ denote the temperature difference (absolute temperature minus a constant reference temperature) in Ω_1. Then u, v, θ satisfy the equations

$$\begin{aligned} \rho_1 u_{tt} + E_1 u + \beta E_1 u_t + \gamma \nabla\theta &= 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), & (1.1)\\ \rho_3\theta_t - \kappa\Delta\theta + \gamma \operatorname{div} u_t &= 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), & (1.2)\\ \rho_2 v_{tt} + E_2 v &= 0 \quad \text{in} \quad \Omega_2 \times (0,\infty), & (1.3) \end{aligned}$$

where

$$E_j := -\mu_j\Delta - (\mu_j + \delta_j)\nabla \operatorname{div}\,, \quad j = 1, 2,$$

denotes the formal elasticity operator for isotropic, homogeneous material with Lamé moduli μ_j, δ_j.

All constants $\rho_1, \beta, \dots, \mu_2$ are assumed to be positive. For the subsequent discussions we may assume w.l.o.g.

$$\rho_1 = \rho_2 = \rho_3 = \gamma = \kappa = 1.$$

We keep the parameter β as it is to point out the effect of the Kelvin-Voigt damping in different places in the following sections.

The transmission conditions on the interface Γ_1 are given by

$$u = v \quad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{1.4}$$

$$\partial_\nu^{E_1} u + \theta\,\nu + \beta \partial_\nu^{E_1} u_t = \partial_\nu^{E_2} v \quad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{1.5}$$

where

$$\partial_\nu^{E_j} = -\mu_j \partial_\nu - (\mu_j + \delta_j)\nu \operatorname{div},$$

and

$$\partial_\nu = \nu \nabla$$

(in each of the n components if it is applied to a vector) with the normal vector ν at the boundary as indicated in the picture above.

As remaining conditions on the smooth boundary we have

$$u = 0, \quad \theta = 0 \quad \text{on} \quad \Gamma_0 \times (0,\infty), \tag{1.6}$$

$$\partial_\nu \theta = 0 \quad \text{on} \quad \Gamma_1 \times (0,\infty). \tag{1.7}$$

The initial-boundary transmission problem is completed by initial conditions,

$$u(\cdot,0) = u^0, \quad u_t(\cdot,0) = u^1, \quad \theta(\cdot,0) = \theta^0 \quad \text{in} \quad \Omega_1, \tag{1.8}$$

$$v(\cdot,0) = v^0, \quad v_t(\cdot,0) = v^1 \quad \text{in} \quad \Omega_2. \tag{1.9}$$

We are interested in the asymptotic behavior of solutions. If there is just the dissipative problem in Ω_1 ($\Omega_2 = \emptyset$), then the Kelvin-Voigt damping is sufficient to exponentially stabilize the system. Therefore, it is interesting to see that even with an additional damping given by heat conduction, the transmission problem no longer shows exponential stability as we shall prove.

To prove the lack of exponential stability – often and in particular in one dimension –, one can use the well-known criterion for contraction semigroups, which states that the semigroup is exponentially stable if and only if the imaginary axis belongs to the resolvent set and the resolvent operator is uniformly bounded on the imaginary axis. Usually the non-uniform boundedness of the resolvent operator is shown by giving an explicit sequence of exact solutions of the system. For higher dimensions this is often not applicable due to the complexity of the resolvent operator. Here we compare the system with an undamped reference system and then demonstrate that the difference of the systems is of compact nature, and then apply Weyl's theorem on the perturbation of essential spectra by compact operators.

On the other hand, strong stability will be shown using the principle of unique continuation for the elastic operator in the isotropic case. That is, the damping material stabilizes through the interface the whole systems, oscillations will be damped to zero.

Moreover, we can prove a polynomial decay using an extended version of the characterization by Borichev and Tomilov [4], based on results of Latushkin and Shvydkov [10].

The special character of the generators appearing with Kelvin-Voigt damping is underlined by showing examples with non-compact inverses which is in contrast to standard situations.

Our considerations immediately extend to the system without temperature,

$$\rho_1 u_{tt} + E_1 u + \beta E_1 u_t = 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), \tag{1.10}$$

$$\rho_2 v_{tt} + E_2 v = 0 \quad \text{in} \quad \Omega_2 \times (0,\infty), \tag{1.11}$$

with transmission conditions

$$u = v \qquad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{1.12}$$

$$\partial_\nu^{E_1} u + \beta \partial_\nu^{E_1} u_t = \partial_\nu^{E_2} v \qquad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{1.13}$$

remaining boundary conditions

$$u = 0 \qquad \text{on} \quad \Gamma_0 \times (0,\infty), \tag{1.14}$$

and initial conditions

$$u(\cdot,0) = u^0, \quad u_t(\cdot,0) = u^1 \qquad \text{in} \quad \Omega_1, \tag{1.15}$$

$$v(\cdot,0) = v^0, \quad v_t(\cdot,0) = v^1 \qquad \text{in} \quad \Omega_2. \tag{1.16}$$

Moreover, we may consider instead of the elastic operator the scalar Laplacian looking at the corresponding transmission problem for wave equations for the scalar functions u, v,

$$\rho_1 u_{tt} - \kappa_1 \Delta u + \beta \kappa_1 \Delta u_t = 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), \tag{1.17}$$

$$\rho_2 v_{tt} - \kappa_2 \Delta v = 0 \quad \text{in} \quad \Omega_2 \times (0,\infty), \tag{1.18}$$

with positive constants κ_1, κ_2, and with transmission conditions

$$u = v \qquad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{1.19}$$

$$\kappa_1 \partial_\nu u + \beta \kappa_1 \partial_\nu u_t = \kappa_2 \partial_\nu v \qquad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{1.20}$$

remaining boundary conditions

$$u = 0 \qquad \text{on} \quad \Gamma_0 \times (0,\infty), \tag{1.21}$$

and initial conditions

$$u(\cdot,0) = u^0, \quad u_t(\cdot,0) = u^1 \qquad \text{in} \quad \Omega_1, \tag{1.22}$$

$$v(\cdot,0) = v^0, \quad v_t(\cdot,0) = v^1 \qquad \text{in} \quad \Omega_2. \tag{1.23}$$

For wave equations with localized *frictional* damping it is well-known that the system is exponentially stable when the damping is effective in a sufficient large neighborhood of the boundary, see for example [8, 12, 14, 15]. For the one-dimensional Euler-Bernoulli *beam*, also localized Kelvin-Voigt damping leads to exponential stability, as was shown by K. Liu and Z. Liu in [11].

On the other hand, as was proved in one dimension in [11], Kelvin-Voigt damping is not strong enough for the wave equation to give exponential stability in the transmission problem. Here we now prove this for $n \geq 2$.

The problem is related to the optimal design of material components, e.g. in damping mechanisms for bridges or in automotive industry, see [3, 16] or [13] and the references therein. As a consequence, one should consider various components with frictional damping if exponential stability is needed, for strong stability, where the oscillations at least tend to zero as time tends to infinity, adding material with Kelvin-Voigt damping properties (plus or without heat) is sufficient.

The paper is organized as follows. In Section 2 we shortly discuss the well-posedness of the thermo-viscoelastic transmission problem. In Section 3 we investigate smoothing properties of the related purely thermo-viscoelastic system. The main Section 4 provideds the proof of the lack of exponential stability. In Section 5, we prove the strong stability. The polynomial stability will be given in Section 6. Section 7 provides examples for Kelvin-Voigt operators yielding arguments for the non-compactness of the inverse of the generator of the arising semigropus. In Section 8 the results on the related purely elastic system and on problem for wave equations are given.

2 Well-posedness

Defining $W := (u, v, u_t, v_t, \theta)'$ ($'$ meaning transpose), we formally get from (1.1)–(1.3) the evolution equation

$$W_t(\cdot, t) = \mathcal{A}W(\cdot, t)$$

with the (yet formal) operator $\mathcal{A}$ acting on $\Phi = (u, v, U, V, \theta))'$ as

$$\mathcal{A}\Phi = \begin{pmatrix} U \\ V \\ -E_1 u - \beta E_1 U - \nabla\theta \\ -E_2 v \\ \Delta\theta - \operatorname{div} U \end{pmatrix}.$$

Introducing the spaces

$$\mathcal{H}^1 := \{\, (u, v) \in (H^1(\Omega_1))^d \times (H^1(\Omega_2))^d \mid \quad u = 0 \quad \text{on} \quad \Gamma_0, \quad u = v \quad \text{on} \quad \Gamma_1\},$$

$$\mathcal{L}^2 := (L^2(\Omega_1))^d \times (L^2(\Omega_1))^d \times L^2(\Omega_2),$$

with the classical Sobolev spaces $H^1(\dots),\ L^2(\dots)$, we choose as Hilbert space

$$\mathcal{H} := \mathcal{H}^1 \times \mathcal{L}^2,$$

with inner product

$$\begin{aligned}\langle \Phi_1, \Phi_2 \rangle_{\mathcal{H}} \ &:= \int_{\Omega_1} U_1\overline{U_2} + \mu_1 \nabla u_1 \overline{\nabla u_2} + (\mu_1+\delta_1)\operatorname{div} u_1 \overline{\operatorname{div} u_2} + \theta_1\overline{\theta_2}\, dx + \\ &\quad \int_{\Omega_2} V_1\overline{V_2} + \mu_2 \nabla v_1 \overline{\nabla v_2} + (\mu_2+\delta_2)\operatorname{div} v_1 \overline{\operatorname{div} v_2}\, dx, \qquad (2.1)\end{aligned}$$

where $\Phi_j = (u_j, v_j, U_j, V_j, \theta_j)'$.

The operator $\mathcal{A}$ is now defined as $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ by

$$\begin{aligned} D(\mathcal{A}) \ &:= \ \{\, \Phi \in \mathcal{H} \,|\quad (U,V) \in \mathcal{H}^1, \quad (u+\beta U, v) \in (H^2(\Omega_1))^d \times (H^2(\Omega_2))^d, \\ &\qquad \partial_\nu^{E_1}(u+\beta U) + \theta\,\nu = \partial_\nu^{E_2} v \quad \text{on} \quad \Gamma_1 \,\}. \end{aligned}$$

Then we have the dissipativity of the densely defined operator $\mathcal{A}$,

$$\operatorname{Re}\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = -\beta \int_{\Omega_1} \mu_1 |\nabla U|^2 + (\mu_1+\delta_1)|\operatorname{div} U|^2\, dx - \int_{\Omega_1} |\nabla\theta|^2\, dx. \qquad (2.2)$$

The equality (2.2) and the choice of the inner product resp. of $\mathcal{H}$ reflects the energy equality we have for (smooth) solutions of the transmission problem (1.1)–(1.9). That is, if

$$\begin{aligned} E(t) \equiv E(u,\theta,v\,;t) \ &:= \ \frac{1}{2}\Bigg(\int_{\Omega_1} |u_t|^2 + \mu_1|\nabla u|^2 + (\mu_1+\delta_1)|\operatorname{div} u|^2 + |\theta|^2\, dx + \\ &\qquad \int_{\Omega_2} |v_t|^2 + \mu_2 |\nabla v|^2 + (\mu_2+\delta_2)|\operatorname{div} v|^2\, dx\Bigg) \qquad (2.3)\end{aligned}$$

denotes the usual energy term associated to the equations, then

$$\frac{dE}{dt}(t) = -\beta \int_{\Omega_1} \mu_1 |\nabla u_t|^2 + (\mu_1+\delta_1)|\operatorname{div} u_t|^2\, dx - \int_{\Omega_1} |\nabla\theta|^2\, dx. \qquad (2.4)$$

Since the stationary transmission problem $A\Phi = F$ is uniquely solvable for any $F \in \mathcal{H}$ (cp. [2, 9, 5]) with continuous inverse operator, we have that $0 \in \varrho(\mathcal{A})$ (resolvent set). Together with the dissipativity (2.2), we conclude that $\mathcal{A}$ generates a contraction semigroup, hence we have

Theorem 2.1. *For any $W^0 \in D(\mathcal{A})$ there exists a unique solution W to*

$$W_t(t) = \mathcal{A}W(t), \quad W(0) = W^0,$$

satisfying

$$W \in C^1([0,\infty), \mathcal{H}) \cap C^0([0,\infty), D(\mathcal{A})).$$

3 Smoothing for pure thermo-viscoelasticity

Arguments needed to show the lack of exponential stability in Section 4 rely on the smoothing effect in pure, uncoupled thermo-viscoelasticity as we shall prove it now. For this purpose we consider the following thermo-viscoelastic initial value problem:

$$\widetilde{u}_{tt} + E_1\widetilde{u} + \beta E_1\widetilde{u}_t + \nabla\widetilde{\theta} = 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), \tag{3.1}$$

$$\widetilde{\theta}_t - \Delta\widetilde{\theta} + \operatorname{div}\widetilde{u}_t = 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), \tag{3.2}$$

with boundary conditions

$$\widetilde{u} = 0, \quad \widetilde{\theta} = 0 \qquad \text{on} \quad (\Gamma_0 \cup \Gamma_1) \times (0,\infty), \tag{3.3}$$

and initial conditions

$$\widetilde{u}(\cdot,0) = \widetilde{u}^0, \quad \widetilde{u}_t(\cdot,0) = \widetilde{u}^1, \quad \widetilde{\theta}(\cdot,0) = \widetilde{\theta}^0 \qquad \text{in} \quad \Omega_1. \tag{3.4}$$

Then we have the following version of a smoothing effect for $t > 0$:

Theorem 3.1. *If*

$$(\widetilde{u}^0, \widetilde{u}^1, \widetilde{\theta}^0) \in (H_0^1(\Omega_1))^d \times (L^2(\Omega_1))^d \times L^2(\Omega_1),$$

then the solution $(\widetilde{u}, \widetilde{\theta})$ *to (3.1)–(3.4) satisfies*

$$(\widetilde{u}, \widetilde{u}_t, \widetilde{u}_{tt}) \in C^0\left((0,\infty), (H_0^1(\Omega_1))^d \times (H_0^1(\Omega_1))^d \times (H^1(\Omega_1))^d\right),$$

$$(\widetilde{\theta}, \widetilde{\theta}_t,) \in C^0\left((0,\infty), H^2(\Omega_1) \times H^2(\Omega_1)\right).$$

Proof: We present energy estimates for assumed *smooth* solutions which will prove the theorem by density. Let

$$h(\cdot,t) := t^5\widetilde{u}(\cdot,t), \quad p(\cdot,t) := t^5\widetilde{\theta}(\cdot,t).$$

Then

$$h_{tt} + E_1 h + \beta E_1 h_t + \nabla p = 20t^3\widetilde{u} + 10t^4\widetilde{u}_t + 5\beta t^4 E_1\widetilde{u}, \tag{3.5}$$

$$p_t - \Delta p + \operatorname{div} h_t = 5t^4\widetilde{\theta} + 5t^4 \operatorname{div}\widetilde{u}_t. \tag{3.6}$$

Multiplication of (3.5) by h_t and (3.6) by p yields

$$\frac{d\mathcal{E}_1}{dt}(t) + \beta\int_{\Omega_1} \mu_1|\nabla h_t|^2 + (\mu_1+\delta_1)|\operatorname{div} h_t|^2\,dx + \int_{\Omega_1} |\nabla p|^2\,dx =$$

$$\int_{\Omega_1}\left(20t^3\widetilde{u} + 10t^4\widetilde{u}_t + 5\beta t^4 E_1\widetilde{u}\right)\overline{h_t}\,dx + \int_{\Omega_1}\left(5t^4\widetilde{\theta} + 5t^4\operatorname{div}\widetilde{u}_t\right)\overline{p}\,dx, \tag{3.7}$$

where

$$\mathcal{E}_1(t) := \mathcal{E}(h, p\,; t)$$

with

$$\mathcal{E}(h, p\,; t) := \frac{1}{2}\left(\int_{\Omega_1} |h_t|^2 + \mu_1|\nabla h|^2 + (\mu_1+\delta_1)|\operatorname{div} h|^2 + |p|^2\, dx\right).$$

Using

$$h(\cdot,0) = h_t(\cdot,0) = p(\cdot,0) = 0,$$

we obtain, after integration with respect to $t \in [0,T]$, for some fixed $T > 0$,

$$2\mathcal{E}_1(t) + \int_0^t \left(\beta\int_{\Omega_1} \mu_1|\nabla h_t|^2 + (\mu_1+\delta_1)|\operatorname{div} h_t|^2\, dx + \int_{\Omega_1} |\nabla p|^2\, dx\right)\, ds$$

$$\leq C\int_0^t\int_{\Omega_1} |\widetilde{u}_t|^2 + |\nabla\widetilde{u}|^2 + |\widetilde{\theta}|^2\, dx\, ds, \tag{3.8}$$

where C will denote a generic positive constant at most depending on T.

Differentiating in (3.5), (3.6) with respect to t, we have

$$\begin{aligned} h_{ttt} + E_1 h_t + \beta E_1 h_{tt} + \nabla p_t &= 60t^2\widetilde{u} + 60t^3\widetilde{u}_t + 10t^4\widetilde{u}_{tt} + \\ &\qquad 20\beta t^3 E_1\widetilde{u} + 5\beta t^4 E_1\widetilde{u}_t, \qquad (3.9)\\ p_{tt} - \Delta p_t + \operatorname{div} h_{tt} &= 20t^3\widetilde{\theta} + 5t^4\widetilde{\theta}_t + 20t^3\operatorname{div}\widetilde{u}_t + 5t^4\operatorname{div}\widetilde{u}_{tt}. \qquad (3.10)\end{aligned}$$

Similary as (3.8) we obtain

$$2\mathcal{E}_2(t) + \int_0^t \left(\beta\int_{\Omega_1} \mu_1|\nabla h_{tt}|^2 + (\mu_1+\delta_1)|\operatorname{div} h_{tt}|^2\, dx + \int_{\Omega_1} |\nabla p_t|^2\, dx\right)\, ds$$

$$\leq C\int_0^t\int_{\Omega_1} |\widetilde{u}_{tt}|^2 + |\nabla\widetilde{u}_t|^2 + |\nabla\widetilde{u}|^2 + |\widetilde{\theta}|^2 + |\nabla\widetilde{\theta}|^2\, dx\, ds, \tag{3.11}$$

where

$$\mathcal{E}_2(t) := \mathcal{E}(h_t, p_t\,; t).$$

Continuing this way in differentiating the differential equations two more times, we get similar estimates for the third- and fourth-order energy terms

$$\mathcal{E}_3(t) := \mathcal{E}(h_{tt}, p_{tt}\,; t), \qquad \mathcal{E}_4(t) := \mathcal{E}(h_{ttt}, p_{ttt}\,; t).$$

Since the first-order energy for $(\widetilde{u},\widetilde{\theta})$,

$$\widetilde{\mathcal{E}}_1(t) := \mathcal{E}(\widetilde{u},\widetilde{\theta}\,; t),$$

satisfies

$$\widetilde{\mathcal{E}}_1(t) + \beta \int_{\Omega_1} \mu_1 |\nabla \widetilde{u}_t|^2 + (\mu_1 + \delta_1)|\operatorname{div} \widetilde{u}_t|^2 \, dx + \int_{\Omega_1} |\nabla \widetilde{\theta}|^2 \, dx = \widetilde{\mathcal{E}}_1(0), \tag{3.12}$$

we conclude, using the ellipticity of the operators E_1 and Δ,

$$\int_{\Omega_1} |u_t|^2 + |\nabla u|^2 + |\nabla h_t|^2 + \nabla h_{tt}|^2 + |\nabla h_{ttt}|^2 \, dx + \|(p, p_t, p_{tt}\|^2_{H^2} \leq$$

$$C \left(\widetilde{\mathcal{E}}_1(0) + \int_0^t \int_{\Omega_1} |\nabla(\widetilde{u}, \widetilde{u}_t, \widetilde{u}_{tt})|^2 + |(\widetilde{\theta}, \widetilde{\theta}_t, \widetilde{\theta}_{tt})|^2 + |\nabla(\widetilde{\theta}, \widetilde{\theta}_t, \widetilde{\theta}_{tt})|^2 \, dx \, ds \right). \tag{3.13}$$

Letting $\eta > 0$ be arbitrarily small, but fixed, we have from

$$\nabla h_t = 5t^4 \nabla \widetilde{u} + t^5 \nabla \widetilde{u}_t$$

the estimate

$$|\nabla \widetilde{u}_t(\cdot, t)|^2 \leq \frac{C}{\eta^2} \left(|\nabla h_t(\cdot, t)|^2 + |\nabla \widetilde{u}(\cdot, t)|^2 \right),$$

and so on. This way, we obtain, with a constant C_η depending at most on T and on η, for $t \geq \eta$,

$$f(t) := \|(\widetilde{u}, \widetilde{u}_t, \widetilde{u}_{tt}, \widetilde{u}_{ttt})(\cdot, t)\|^2_{H^1} + \|(\widetilde{\theta}, \widetilde{\theta}_t, \widetilde{\theta}_{tt})(\cdot, t)\|^2_{H^2} \leq C_\delta \left(\widetilde{\mathcal{E}}(0) + \int_0^t f(s) \, ds \right).$$

By Gronwall, this implies

$$\widetilde{u} \in W^{3,2}\left([\eta, T], (H^1(\Omega_1)^d\right), \qquad \widetilde{\theta} \in W^{2,2}\left([\eta, T], H^2(\Omega_1)\right).$$

By embedding, we complete the proof of Theorem 3.1.

4 Lack of exponential stability

The proof that the semigroup is *not* exponentially stable will use the so-called Weyl theorem, saying that the essential spectrum of a bounded operator is invariant under compact perturbations (see [7]). The basic idea is to consider, in addition to the given semigroup S with $(S(t) = \mathrm{e}^{t\mathcal{A}})_{t\geq 0}$ describing our transmission problem above, another semigroup S_0 with $(S_0(t))_{t\geq 0}$, for which the essential type $\omega_{ess}(S_0)$ is known to be zero, e. g. for the unitary semigroup defined below, and then to show that the difference $S(t) - S_0(t)$ is a compact operator (for some resp. all $t > 0$). This implies, using Weyl's theorem, that the essential types of S and of S_0 are the same, hence we will have for the type $\omega_0(S)$ of the semigroup S that

$$\omega_0(S) \geq \omega_{ess}(S) = \omega_{ess}(S_0) = 0,$$

hence the semigroup S will not be exponentially stable.

Actually, the arguments will be more subtle, since we cannot argue with S and S_0 directly, due to regularity properties, but we have to exploit the smoothing effect proved in the previous section to argue in a modified setting, see the proof of Theorem 4.4 below.

We define the new semigroup S_0 by the following initial boundary value problem over $\Omega_1 \cup \Omega_2$:

$$\widetilde{u}_{tt} + E_1\widetilde{u} + \beta E_1 \widetilde{u}_t + \nabla\widetilde{\theta} = 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), \tag{4.1}$$

$$\widetilde{\theta}_t - \Delta\widetilde{\theta} + \operatorname{div} \widetilde{u}_t = 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), \tag{4.2}$$

with boundary conditions

$$\widetilde{u} = 0, \quad \widetilde{\theta} = 0 \qquad \text{on} \quad (\Gamma_0 \cup \Gamma_1) \times (0,\infty), \tag{4.3}$$

and initial conditions

$$\widetilde{u}(\cdot,0) = \widetilde{u}^0, \quad \widetilde{u}_t(\cdot,0) = \widetilde{u}^1, \quad \widetilde{\theta}(\cdot,0) = \widetilde{\theta}^0 \qquad \text{in} \quad \Omega_1, \tag{4.4}$$

as well as

$$\widetilde{v}_{tt} + E_2\widetilde{v} = 0 \quad \text{in} \quad \Omega_2 \times (0,\infty), \tag{4.5}$$

with boundary conditions

$$\widetilde{v} = 0 \qquad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{4.6}$$

and initial conditions

$$\widetilde{v}(\cdot,0) = \widetilde{v}^0, \quad \widetilde{v}_t(\cdot,0) = \widetilde{v}^1 \qquad \text{in} \quad \Omega_2. \tag{4.7}$$

The problem in Ω_1 is the purely thermo-viscoelastic problem studied in the previous section. The problem in Ω_2 is energy conserving, i. e.

$$\int_{\Omega_2} |\widetilde{v}_t|^2 + \mu_2|\nabla\widetilde{v}|^2 + (\mu_2+\delta_2)|\operatorname{div}\widetilde{v}|^2\,dx = \int_{\Omega_2} |\widetilde{v}^1|^2 + \mu_2|\nabla\widetilde{v}^0|^2 + (\mu_2+\delta_2)|\operatorname{div}\widetilde{v}^0|^2\,dx.$$

Hence, the contraction semigroup S_0 associated to (4.1)–(4.7) has type

$$\omega_0(S_0) = 0. \tag{4.8}$$

Lemma 4.1. *Let $((\widetilde{u}^0_n, \widetilde{u}^1_n, \widetilde{\theta}^0_n))_n$ be a sequence of initial data which is bounded in*

$$(H^1_0(\Omega_1))^d \times (H^1_0(\Omega_1))^d \times H^1_0(\Omega_1),$$

and let $((\widetilde{u}_n, \widetilde{\theta}_n))_n$ denote the associated solutions to the purely thermo-viscoelastic problem (4.1)–(4.4). Then there exists a subsequence, again denoted by $((\widetilde{u}_n, \widetilde{\theta}_n))_n$, such that

$$\widetilde{u}_n + \beta\widetilde{u}_{n,t} \to \widetilde{u} + \beta\widetilde{u}_t \quad \textit{strongly in } L^2\left((0,T), \left((H^1(\Omega_1)\right)^d\right),$$

and

$$\widetilde{\theta}_n \to \widetilde{\theta} \quad \textit{strongly in } L^2\left((0,T), H^1(\Omega_1)\right),$$

for some $(\widetilde{u}, \widetilde{\theta})$.

PROOF: Writing (mostly) in the proof for simplicity

$$u := \widetilde{u}_n, \qquad \theta := \widetilde{\theta}_n,$$

and multiplying the differential equations (4.1) and (4.2) by u_{tt} and θ_t, respectively, we obtain, after integration,

$$\begin{aligned}
&\int_0^t \int_{\Omega_1} |u_{tt}|^2 \, dx \, ds + \frac{1}{2} \int_{\Omega_1} \mu_1 \nabla u \overline{\nabla u_t} + (\mu_1 + \delta_1) \operatorname{div} u \, \overline{\operatorname{div} u_t} \, dx + \\
&\qquad \frac{\beta}{2} \int_{\Omega_1} \mu_1 |\nabla u_t|^2 + (\mu_1 + \delta_1) |\operatorname{div} u_t|^2 \, dx + \operatorname{Re} \int_0^t \int_{\Omega_1} \nabla \theta \, \overline{u_{tt}} \, dx \, ds \quad = \\
&\frac{1}{2} \int_{\Omega_1} \mu_1 \nabla u^0 \overline{\nabla u^1} + (\mu_1 + \delta_1) \operatorname{div} u^0 \, \overline{\operatorname{div} u^1} \, dx + \frac{\beta}{2} \int_{\Omega_1} \mu_1 |\nabla u^1|^2 + (\mu_1 + \delta_1) |\operatorname{div} u^1|^2 \, dx \\
&\qquad + \int_0^t \int_{\Omega_1} \mu_1 |\nabla u_t|^2 + (\mu_1 + \delta_1) |\operatorname{div} u_t|^2 \, dx \, ds,
\end{aligned}$$

and

$$\int_0^t \int_{\Omega_1} |\theta_t|^2 \, dx \, ds + \frac{1}{2} \int_{\Omega_1} |\nabla \theta|^2 \, dx = \int_{\Omega_1} |\nabla \theta^0|^2 \, dx - \operatorname{Re} \int_0^t \int_{\Omega_1} \operatorname{div} u_t \, \overline{\theta_t} \, dx \, ds. \tag{4.9}$$

Summing up, we obtain from the last two identities for $t \in [0,T]$, $T > 0$ arbitrary, but fixed,

$$\begin{aligned}
&\int_0^t \int_{\Omega_1} |u_{tt}|^2 \, dx \, ds + \beta \int_{\Omega_1} \mu_1 |\nabla u_t|^2 + (\mu_1 + \delta_1) |\operatorname{div} u_t|^2 \, dx + \\
&\quad \int_0^t \int_{\Omega_1} |\theta_t|^2 \, dx \, ds + \int_{\Omega_1} |\nabla \theta|^2 \, dx \le C \int_{\Omega_1} |\nabla u^0|^2 + |\nabla u^1|^2 + |u^1|^2 + |\nabla \theta^0|^2 \, dx,
\end{aligned} \tag{4.10}$$

where C a positive constant (at most depending on T). Since

$$E_1(u + \beta u_t) = -u_{tt} - \nabla \theta,$$

we conclude from (4.10) the boundedness of $(\widetilde{u}_n + \beta \widetilde{u}_{n,t})_n$ in $L^2\left((0,T), ((H^2(\Omega_1))^d\right)$. Moreover, we conclude the boundedness of $(\widetilde{u}_{n,t} + \beta \widetilde{u}_{n,tt})_n$ in $L^2\left((0,T), (L^2(\Omega_1))^d\right)$.

By Aubin's compactness theorem, we obtain that there exists a subsequence with

$$\widetilde{u}_n + \beta \widetilde{u}_{n,t} \to \widetilde{u} + \beta \widetilde{u}_t \quad \text{strongly in } L^2\left((0,T), \left(H^1(\Omega_1)\right)^d\right), \tag{4.11}$$

for some $\widetilde{u} \in L^2\left((0,T),\left(H^1(\Omega_1)\right)^d\right)$. Finally, we get from (4.10) and (4.2) that $(\widetilde{\theta}_n)_n$ is bounded in $L^2\left((0,T),H^2(\Omega_1)\right)$ and that $(\widetilde{\theta}_{n,t})_n$ is bounded in $L^2\left((0,T),L^2(\Omega_1)\right)$, which implies by Aubin's theorem that there is subsequence such that

$$\widetilde{\theta}_n \to \widetilde{\theta} \quad \text{strongly in } L^2\left((0,T),H^1(\Omega_1)\right), \tag{4.12}$$

for some $\widetilde{\theta} \in L^2\left((0,T),H^1(\Omega_1)\right)$. This completes the proof of Lemma 4.1.

Remark 4.2. *The convergence in (4.11) and (4.12), respectively, can be obtained, for any $\varepsilon > 0$, in the space*

$$L^2\left((0,T),\left(H^{2-\varepsilon}(\Omega_1)\right)^d\right)$$

and

$$L^2\left((0,T),H^{2-\varepsilon}(\Omega_1)\right),$$

respectively.

Lemma 4.3. *Let w be the solution (with sufficient regularity for the following integrals to exist) to*

$$w_{tt} + E_2 w = f \quad \text{in} \quad \Omega_2 \times (0,T),$$

with boundary condition

$$w = 0 \quad \text{on} \quad \Gamma_1 \times (0,T),$$

where $T > 0$ is arbitrary, but fixed. Then w satisfies

$$\begin{aligned}\int_0^T \int_{\Gamma_1} |\partial_\nu w|^2 + |\operatorname{div} w|^2 \, do \, ds \;\leq\; & C_T \left(\int_0^T \int_{\Omega_2} |w_t|^2 + |\nabla w|^2 + |f|^2 \, dx \, ds \, + \right. \\ & \left. \int_{\Omega_2} |w_t(\cdot,0)|^2 + |\nabla w(\cdot,0)|^2 \, dx \right),\end{aligned}$$

for some constant $C_T > 0$ depending at most on T.

Lemma 4.3 easily follows by multiplication of the differential equation with $q\,\nabla w$ in $\left(L^2(\Omega_1)\right)^d$, where q is a smooth vector field satisfying $q = \nu$ on Γ_1, cp. [6, Lemma 4.1] and the proof Lemma 6.3 below. Now we state the main theorem:

Theorem 4.4. *The semigroup S with $S(t) = \mathrm{e}^{\mathcal{A}t}$ is not exponentially stable.*

Proof: Having an application of Weyl's theorem in mind, we prove a compactness result. For this purpose, let $((u_n^0, v_n^0, u_n^1, v_n^1, \theta_n^0))_n$ be a bounded sequence of initial data in the space

$$\mathcal{H}_0 := \left(H_0^1(\Omega_1)\right)^d \times \left(H_0^1(\Omega_2)\right)^d \times \left(L^2(\Omega_1)\right)^d \times \left(L^2(\Omega_2)\right)^d \times H_0^1(\Omega_1).$$

Let (u_n, v_n, θ_n) be the corresponding solution to the transmission problem (1.1)–(1.9) – with associated semigroup $S(t) = \mathrm{e}^{\mathcal{A}t}$ –, and let $(\widetilde{u}_n, \widetilde{v}_n, \widetilde{\theta}_n)$ be the solution to the uncoupled problem (4.1)–(4.7) – with associated semigroup $S_0(t) \equiv \mathrm{e}^{\widetilde{\mathcal{A}}t}$. Let the difference, for which we wish to show convergence in $\mathcal{H}$ for some subsequence, be defined as (dropping n in some places)

$$w := u_n - \widetilde{u}_n, \quad z := v_n - \widetilde{v}_n, \quad \eta := \theta_n - \widetilde{\theta}_n.$$

Then (w, z, η) satisfies the differential equations

$$w_{tt} + E_1 w + \beta E_1 w_t + \nabla\eta = 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), \tag{4.13}$$

$$\eta_t - \Delta\eta + \operatorname{div} w_t = 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), \tag{4.14}$$

$$z_{tt} + E_2 z = 0 \quad \text{in} \quad \Omega_2 \times (0,\infty), \tag{4.15}$$

with zero initial data,

$$w(\cdot,0) = 0, \quad w_t(\cdot,0) = 0, \quad \eta(\cdot,0) = 0 \quad \text{in} \quad \Omega_1, \tag{4.16}$$

$$z(\cdot,0) = 0, \quad z_t(\cdot,0) = 0 \quad \text{in} \quad \Omega_2. \tag{4.17}$$

Moreover, the following boundary conditions are in particular satisfied:

$$w = 0, \quad \eta = 0 \quad \text{on} \quad \Gamma_0 \times (0,\infty), \tag{4.18}$$

$$w = u, \quad \eta = \theta \quad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{4.19}$$

$$z = v \quad \text{on} \quad \Gamma_1 \times (0,\infty). \tag{4.20}$$

As usual, we obtain for the associated energy (indicating the dependence on n), cp. (2.3),

$$\begin{aligned} E(w,z,\eta\,;t) &= \frac{1}{2}\Big(\int_{\Omega_1} |w_t|^2 + \mu_1|\nabla w|^2 + (\mu_1+\delta_1)|\operatorname{div} w|^2 + |\eta|^2\,dx + \\ &\quad \int_{\Omega_2} |z_t|^2 + \mu_2|\nabla z|^2 + (\mu_2+\delta_2)|\operatorname{div} z|^2\,dx\Big) =: E_n(t), \end{aligned} \tag{4.21}$$

the identity

$$\begin{aligned} \frac{d}{dt}E_n(t) + \beta\int_{\Omega_1} |\nabla w_t|^2 + |\nabla\eta|^2\,dx &= \int_{\Gamma_1} \left(-\partial_\nu^{E_1} w - \beta\partial_\nu w_t + \nu\,\eta\right)\overline{w_t}\,do \\ &\quad + \int_{\Gamma_1} \partial_\nu\eta\,\overline{\eta}\,do + \int_{\Gamma_1} \partial_\nu^{E_2} z\,\overline{z_t}\,do. \end{aligned} \tag{4.22}$$

This implies, using the boundary conditions (4.18)–(4.20) and the transmission conditions (1.4), (1.5),

$$\begin{aligned} E_n(t) &\le \int_0^t\int_{\Gamma_1} \left(\partial_\nu^{E_1}(\widetilde{u}_n + \beta\widetilde{u}_{n,t})\right)\overline{u_{n,t}}\,do\,ds - \int_0^t\int_{\Gamma_1} \partial_\nu^{E_2}\widetilde{v}_n\,\overline{u_{n,t}}\,do\,ds \\ &\quad + \int_0^t\int_{\Gamma_1} \partial_\nu\widetilde{\theta}_n\,\overline{\theta_n}\,do\,ds \equiv J_n^1 + J_n^2 + J_n^3. \end{aligned} \tag{4.23}$$

Since we get the same for differences $u_n - u_m, \cdots$, and since the energy term is equivalent to the norm in the underlying Hilbert space $\mathcal{H}$, it suffices to show that E_n converges.

Using Lemma 4.1, see also Remark 4.2, we know

$$\widetilde{u}_n + \beta\widetilde{u}_{n,t} \to \widetilde{u} + \beta\widetilde{u}_t \quad \text{strongly in } L^2\left((0,T), \left(H^1(\Omega_1)\right)^d\right),$$

implying

$$\partial_\nu^{E_1}\left(\widetilde{u}_n + \beta\widetilde{u}_{n,t}\right) \to \partial_\nu^{E_1}\left(\widetilde{u} + \beta\widetilde{u}_t\right) \quad \text{strongly in } L^2\left((0,T), \left(H^{-\frac{1}{2}}(\Gamma_1)\right)^d\right).$$

Since also

$$u_{n,t} \to u_t \quad \text{strongly in } L^2\left((0,T), \left(H^{\frac{1}{2}}(\Gamma_1)\right)^d\right),$$

we get the convergence of $(J_n^1)_n$. By Lemma 4.3 and using the differential equation (4.5), we obtain the weak convergence

$$\partial_\nu^{E_2}\widetilde{v}_n \rightharpoonup \partial_\nu^{E_2}\widetilde{v} \quad \text{weakly in } L^2\left((0,T), \left(L^2(\Gamma_1)\right)^d\right),$$

and with the boundedness of $(u_{n,t})_n$ in $L^2\left((0,T), ((L^2(\Gamma_1))^d\right)$, we conclude the convergence of $(J_n^2)_n$. The convergence of $(J_n^3)_n$ follows from Lemma 4.1 resp. Remark 4.2, because

$$\partial_\nu\widetilde{\theta}_n \to \partial_\nu\widetilde{\theta} \quad \text{strongly in } L^2\left((0,T), H^{-\frac{1}{2}}(\Gamma_1)\right),$$

and

$$\theta_n \to \theta \quad \text{strongly in } L^2\left((0,T), H^{\frac{1}{2}}(\Gamma_1)\right).$$

Hence we have proved that a bounded sequence of initial data $(\Phi_n)_n$ in $\mathcal{H}_0$ leads to a convergent subsequence of $\left((S(t) - S_0(t))\Phi_n\right)_n$ in $\mathcal{H}$, for any $t > 0$, a property which we call *compactness over* $\mathcal{H}_0$.

Let

$$\widehat{\mathcal{H}_0} := \left(H_0^1(\Omega_1)\right)^d \times \left(L^2(\Omega_2)\right)^d \times \left(L^2(\Omega_1)\right)^d \times \left(L^2(\Omega_2)\right)^d \times L^2(\Omega_1).$$

Then, fixing $\delta > 0$, we have

$$S_0(\delta)\widehat{\mathcal{H}_0} \subset \mathcal{H}_0,$$

because of the smoothing property proved in Section 3. We consider

$$\widetilde{\mathcal{H}} := \overline{\{S(r)S_0(\delta)\Phi^0 \mid \Phi^0 \in \widehat{\mathcal{H}_0},\ r \geq 0\}},$$

with closure in $\mathcal{H}$, allowing us to exploit the smoothing property. $\widetilde{\mathcal{H}}$ is an invariant subspace for the semigroup S, with $\widehat{\mathcal{H}_0}$ being a closed subset of $\widetilde{\mathcal{H}}$. If P denotes the orthogonal projection onto $\widehat{\mathcal{H}_0}$, then, for fixed $t > 0$,

$$S(t) - S_0(t)S_0(\delta)P : \widetilde{\mathcal{H}} \longrightarrow \widetilde{\mathcal{H}}$$

is compact by the compactness over $\mathcal{H}_0$ as proved above, since $S_0(\delta)\widehat{\mathcal{H}_0}$ is a dense subspace of $\widetilde{\mathcal{H}}$. Therefore, we can apply Weyl's theorem on compact perturbations of the essential spectrum. Since in $\widetilde{\mathcal{H}}$

$$\omega_{ess}(S_0 S(\delta) P) = 0,$$

we thus get there

$$\omega_{ess}(S) = 0,$$

implying that S is not exponentially stable also in $\mathcal{H}$. This completes the proof of Theorem 4.4.

5 Strong stability

Though not exponentially stable, the damping thermo-viscoelastic part in Ω_1 is damping for the whole system in the sense of strong stability. For the proof we use in particular the principle of unique continuation for the elastic operator for homogeneous isotropic media.

Theorem 5.1. *(1) $i\mathbb{R} \subset \varrho(\mathcal{A})$.*
(2) The semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable, i.e. we have for any $W^0 \in \mathcal{H}$:

$$e^{t\mathcal{A}} W^0 \to 0, \qquad \text{as} \quad t \to \infty.$$

Proof. (2) is a direct consequence of (1), hence it suffices to prove prove (1).

Remark 5.2. *Since $\mathcal{A}^{-1}$ is not expected to be compact – compare the discussion of Kelvin-Voigt operators in Section 7 –, it is not sufficient to just exclude imaginary eigenvalues of $\mathcal{A}$.*

Since $0 \in \varrho(\mathcal{A})$, there is $R_1 > 0$ such that $i[-R_1, R_1] \subset \varrho(\mathcal{A})$. Let

$$\infty \geq \lambda^* := \sup N,$$

where

$$N := \{\, R > 0 \,|\, i[-R, R] \subset \varrho(\mathcal{A}) \,\}.$$

Then $\lambda^* > 0$, since $R_1 \in N$. If $\lambda^* = \infty$ the proof is complete. So assume

$$0 < \lambda^* < \infty.$$

Then there exists a sequence $(\lambda_n)_n \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} \|(i\lambda_n - \mathcal{A})^{-1}\| = \infty.$$

This implies the existence of $(\bar{F}_n)_n \subset \mathcal{H}$ with

$$\|\tilde{F}_n\| = 1, \qquad \lim_{n\to\infty} \|(i\lambda_n - \mathcal{A})^{-1}\tilde{F}_n\| = \infty.$$

Denoting $\tilde{\Phi}_n := (i\lambda_n - \mathcal{A})^{-1}\tilde{F}_n$ and $\Phi_n := \tilde{\Phi}_n/\|\tilde{\Phi}_n\|$, as well as $F_n := \tilde{F}_n/\|\tilde{\Phi}_n\|$, we have

$$(i\lambda_n - \mathcal{A})\Phi_n = F_n$$

and

$$\|\Phi_n\| = 1, \qquad F_n \to 0 \quad \text{strongly in} \quad \mathcal{H}.$$

By the dissipativity (2.2) we then obtain

$$\begin{aligned}
\beta \int_{\Omega_1} \mu_1 |\nabla U_n|^2 + (\mu_1 + \delta_1) |\operatorname{div} U_n|^2 \, dx + \int_{\Omega_1} |\nabla \theta_n|^2 \, dx &= -\mathrm{Re}\, \langle \mathcal{A}\Phi_n, \Phi_n \rangle_{\mathcal{H}} \\
&= -\mathrm{Re}\, \langle F_n, \Phi_n \rangle_{\mathcal{H}} \to 0,
\end{aligned}$$

hence

$$U_n \to 0 \quad \text{strongly in} \quad \left(H^1(\Omega_1)\right)^d, \quad \theta_n \to 0 \quad \text{strongly in} \quad H^1(\Omega_1).$$

Denoting $F_n = (F_{n,1}, \ldots, F_{n,5})'$, and since

$$i\lambda_n u_n - U_n F_{1,n},$$

we have

$$u_n \to 0 \quad \text{strongly in} \quad \left(H^1(\Omega_1)\right)^d.$$

By the boundedness of $\|\Phi_n\|_{\mathcal{H}}$ we conclude that there exist subsequences such that

$$v_n \to v \quad \text{strongly in} \quad \left(L^2(\Omega_1)\right)^d,$$

and

$$v_n \to v \quad \text{weakly in} \quad \left(H^1(\Omega_1)\right)^d.$$

Moreover,

$$i\lambda_n V_n - \mu_2 \Delta v_n - (\mu_2 + \delta_2)\nabla \operatorname{div} v_n = F_{4,n},$$

implying

$$\begin{aligned}
\int_{\Omega_2} \mu_2 |\nabla v_n|^2 + (\mu_2 + \delta_2) |\operatorname{div} v_n|^2 \, dx &= \int_{\Gamma_1} \partial_\nu^{E_2} v_n \overline{v_n} \, do - i\lambda_n \int_{\Omega_2} V_n \overline{v_n} \, dx + \\
&\quad \int_{\Omega_2} F_{4,n} \overline{v_n} \, dx.
\end{aligned}$$

Since

$$\partial_\nu^{E_1} u_n + \theta\, \nu + \beta \partial_\nu^{E_1} U_n = \partial_\nu^{E_2} v_n,$$

we conclude that

$$\partial_\nu^{E_2} v_n \to 0 \quad \text{strongly in} \quad \left(H^{-\frac{1}{2}}(\Gamma_1)\right)^d.$$

With the boundedness of $(v_n)_n$ in $\left(H^{\frac{1}{2}}(\Gamma_1)\right)^d$, we have

$$\int_{\Gamma_1} \partial_\nu^{E_2} v_n \overline{v_n}\, do \to 0.$$

Thus, we conclude the strong convergence of $(v_n)_n$ in $(H^1(\Omega_2))^d$. Hence, Φ_n converges strongly to some $\Phi \in \mathcal{H}$ with $\|\Phi\|_{\mathcal{H}} = 1$. Since $\mathcal{A}\Phi_n = i\lambda_n \Phi_n - F_n$ now converges strongly to $i\lambda\phi$ (with $\lambda = \pm\lambda^*$), we obtain

$$\Phi \in D(\mathcal{A}), \qquad (i\lambda - \mathcal{A})\Phi = 0.$$

We successively conclude, using the dissipativity once more, $\theta = 0$, $U = 0$, $u = 0$, $v_{|\Gamma_1} = 0$, $\partial_\nu^{E_2} v_{|\Gamma_1} = 0$, and

$$\begin{aligned} i\lambda v - V &= 0, \\ i\lambda V + E_2 v &= 0. \end{aligned}$$

Hence we have for v

$$E_2 v = \lambda^2 v, \tag{5.1}$$

$$v_{|\Gamma_1} = 0, \quad \partial_\nu^{E_2} v_{|\Gamma_1} = 0. \tag{5.2}$$

By the unique continuation principle for solutions to (5.1), (5.2), i. e. for isotropic, homogeneous elasticity (see [18, 1, 17]), we get

$$v = 0.$$

Hence $\Phi = 0$ which is a contradiction to $\|\Phi\|_{\mathcal{H}} = 1$. This completes the proof of Theorem 5.1.

6 Polynomial stability

In addition to the strong stability, we shall prove the following polynomial decay result.

Theorem 6.1. *The semigroup* $\left(\mathrm{e}^{t\mathcal{A}}\right)_{t\geq 0}$ *decays polynomially of order at least* $\frac{1}{3}$*, i.e.*

$$\exists\, C > 0\, \exists\, t_0 > 0\, \forall\, t \geq t_0\, \forall\, \Phi^0 \in D(\mathcal{A}) : \ \|\mathrm{e}^{t\mathcal{A}}\Phi_0\|_{\mathcal{H}} \leq C\, t^{-\frac{1}{3}} \|\mathcal{A}\Phi^0\|_{\mathcal{H}}. \tag{6.1}$$

For the proof we shall use the following extension of a result of Borichev and Tomilov [4] resp. Latushkin and Shvydkoy [10] for a general contraction semigroup $(\mathcal{T}(t))_{t\geq 0} = \left(\mathrm{e}^{t\mathcal{B}}\right)_{t\geq 0}$ in a Hilbert space $\mathcal{H}_1$ with $i\mathbb{R} \subset \rho(\mathcal{B})$.

Lemma 6.2. *Let $(\mathcal{T}(t))_{t\geq 0} = \left(e^{t\mathcal{B}}\right)_{t\geq 0}$ be a contraction semigroup in a Hilbert space $\mathcal{H}_1$ with $i\mathbb{R} \subset \rho(\mathcal{B})$. Then the following characterizations (6.2) and (6.3) are equivalent, where the parameters $\alpha, \beta \geq 0$ are fixed:*

$$\exists\, C > 0\; \exists\, \lambda_0 > 0\; \forall\, \lambda \in \mathbb{R},\, |\lambda| \geq \lambda_0\; \forall\, F \in D(\mathcal{B}^\alpha):\; \|\,(i\lambda - \mathcal{B})^{-1}\, F\|_{\mathcal{H}} \leq C|\lambda|^\beta \|\mathcal{B}^\alpha F\|_{\mathcal{H}}, \tag{6.2}$$

$$\exists\, C > 0\; \exists\, t_0 > 0\; \forall\, t \geq t_0\; \forall\, \Phi^0 \in D(\mathcal{B}):\; \|e^{t\mathcal{B}}\Phi_0\|_{\mathcal{H}} \leq C\, t^{-\frac{1}{\alpha+\beta}} \|\mathcal{B}\Phi^0\|_{\mathcal{H}}. \tag{6.3}$$

Proof of Lemma 6.2: For $\alpha = 0$ it is part of the results in [4, Theorem 2.4]. To extend the proof there to the case $\alpha > 0$, one has to extend [10, Theorem 3.2] which is possible due to the following observation: [10, Theorem 3.2] proves the equivalence of

$$\sup_\lambda \left\{ \frac{\|\,(\lambda - \mathcal{B})^{-1}\,\|}{1 + |\lambda|^\alpha} \right\} < \infty \qquad \text{and} \qquad \sup_\lambda \left\{ \|\,(\lambda - \mathcal{B})^{-1}\,\mathcal{B}^{-\alpha}\| \right\} < \infty \tag{6.4}$$

for λ in a strip $a < \operatorname{Re}\lambda < b$. In the proof there, it is shown that

$$d_1 \frac{\|\,(\lambda - \mathcal{B})^{-1}\,x\|}{|\lambda|^\alpha} - K_\alpha \|x\| \leq \|\,(\lambda - \mathcal{B})^{-1}\,\mathcal{B}^{-\alpha} x\| \leq d_2 \frac{\|\,(\lambda - \mathcal{B})^{-1}\,x\|}{|\lambda|^\alpha} + K_\alpha \|x\| \tag{6.5}$$

holds for $x \in \mathcal{H}_1$, with positive constants d_1, d_2, K_α. But this immediately implies

$$\begin{aligned} \tilde{d}_1 \frac{\|\,(\lambda - \mathcal{B})^{-1}\,x\|}{1 + |\lambda|^{\alpha+\beta}} - K_\alpha \frac{\|x\|}{1 + |\lambda|^\beta} \quad &\leq \quad \frac{\|\,(\lambda - \mathcal{B})^{-1}\,\mathcal{B}^{-\alpha} x\|}{1 + |\lambda|^\beta} \\ &\leq \quad \tilde{d}_2 \frac{\|\,(\lambda - \mathcal{B})^{-1}\,x\|}{1 + |\lambda|^{\alpha+\beta}} + K_\alpha \frac{\|x\|}{1 + |\lambda|^\beta} \end{aligned} \tag{6.6}$$

with positive constants $\tilde{d}_1, \tilde{d}_2$. Hence we have the equivalence of

$$\sup_\lambda \left\{ \frac{\|\,(\lambda - \mathcal{B})^{-1}\,\|}{1 + |\lambda|^{\alpha+\beta}} \right\} < \infty \qquad \text{and} \qquad \sup_\lambda \left\{ \frac{\|\,(\lambda - \mathcal{B})^{-1}\,\mathcal{B}^{-\alpha}\|}{1 + |\lambda|^\beta} \right\} < \infty, \tag{6.7}$$

and [10, Theorem 3.2] is thus extended. Then one can use the results in [4], and this proves Lemma 6.2.

In the sequel we will prove (6.2) for $\alpha = 1$ and $\beta = 2$ which will then prove Theorem 6.1 by Lemma 6.2. For the proof we need the following Lemma, in particular in order to estimate tangential derivatives on Γ_1, since $q\nu$ will there be positive.

Lemma 6.3. *Let $x_0 \in \Omega_2$ and $q(x) := x - x_0$ for $x \in \Omega_2$. Let w, W and $\lambda \in \mathbb{R}$ satisfy*

$$\begin{aligned} i\lambda w - W \;&=\; g_2 \in \left(L^2(\Omega_2)\right)^d, \qquad (6.8) \\ i\lambda W + E_2 w \;&=\; g_4 \in \left(L^2(\Omega_2)\right)^d. \qquad (6.9) \end{aligned}$$

Then we have

$$\frac{1}{2}\int_{\Omega_2} |\lambda w|^2 + \mu_2|\nabla w|^2 + (\mu_2+\delta_2)|\operatorname{div} w|^2\,dx + \frac{\mu_2}{2}\int_{\Gamma_1}(q\nu)|\nabla_\tau w|^2\,do$$
$$-\frac{1}{2}\int_{\Gamma_1}(q\nu)\left(|\lambda w|^2 + \mu_2|\partial_\nu w|^2 + (\mu_2+\delta_2)|\operatorname{div} w|^2\right)\,do$$
$$-\mu_2\mathrm{Re}\int_{\Gamma_1}\partial_\nu w_j q\nabla_\tau \overline{w_j}\,do + (\mu_2+\delta_2)\mathrm{Re}\int_{\Gamma_1}(q\nu)\operatorname{div} w\left((\nabla_\tau\overline{w_j})_{/j} - \nu_j q_k(\nabla_\tau\overline{w_j})_{/k}\right)\,do$$
$$-\mu_2\frac{d-1}{2}\mathrm{Re}\int_{\Gamma_1}\partial_\nu w\overline{w}\,do - (\mu_2+\delta_2)\frac{d-1}{2}\mathrm{Re}\int_{\Gamma_1}(\nabla_\tau w_j)_{/j}\nu_k\overline{w_k} + (\partial_\nu w_j)_{/j}\nu_k\overline{w_k}\,do$$
$$= \mathrm{Re}\int_{\Omega_2}(i\lambda g_2 + g_4)(q_k\partial_k\overline{w} + \overline{w})\,dx, \qquad (6.10)$$

where ∇_τ denotes tangential derivatives.

Proof: We obtain from (6.8), (6.9)

$$-\lambda^2 w + E_2 w = i\lambda g_2 + g_4. \qquad (6.11)$$

Multiplying this by $q_k\partial_k w$ in $(L^2(\Omega_2))^n$ and performing a partial integration, we get

$$\mathrm{Re}\int_{\Omega_2}(i\lambda g_2+g_4)q_k\partial_k\overline{w}\,dx = \underbrace{-\frac{1}{2}\int_{\Gamma_1}(q\nu)|\lambda w|^2\,do + \frac{d}{2}\int_{\Omega_2}|\lambda w|^2\,dx}_{=:I_1}$$
$$-\mu_2\mathrm{Re}\int_{\Gamma_1}\nu\nabla w_j q_k\partial_k\overline{w_j}\,do + \mu_2\mathrm{Re}\int_{\Omega_2}\nabla w_j\nabla(q_k\partial_k\overline{w_j})\,dx$$
$$-(\mu_2+\delta_2)\mathrm{Re}\int_{\Gamma_1}\operatorname{div} w\nu_j q_k\partial_k\overline{w_j}\,do + (\mu_2+\delta_2)\mathrm{Re}\int_{\Omega_2}\operatorname{div} w\operatorname{div}(q_k\partial_k\overline{w})\,dx$$
$$\equiv I_1 + I_2 + I_3 + I_4 + I_5. \qquad (6.12)$$

Observing

$$\nabla w_j = \nabla_\tau w_j + \langle\nabla w_j,\nu\rangle_{\mathbb{C}^n}\nu, \qquad \operatorname{div} q = d,$$

we obtain

$$I_2 = -\mu_2\int_{\Gamma_1}(q\nu)|\partial_\nu w|^2\,do - \mu_2\mathrm{Re}\int_{\Gamma_1}\partial_\nu w_j\,q\,\overline{\nabla_\tau w_j}\,do, \qquad (6.13)$$

$$I_3 = \mu_2\left(1-\frac{d}{2}\right)\int_{\Omega_2}|\nabla w|^2\,dx + \frac{\mu_2}{2}\int_{\Gamma_1}(q\nu)|\nabla_\tau w|^2\,do + \frac{\mu_2}{2}\int_{\Gamma_1}(q\nu)|\partial_\nu w|^2\,do, \qquad (6.14)$$

$$I_4 = -(\mu_2+\delta_2)\int_{\Gamma_1}(q\nu)|\operatorname{div} w|^2\,dx +$$
$$(\mu_2+\delta_2)\mathrm{Re}\int_{\Gamma_1}(q\nu)\operatorname{div} w\left((\nabla_\tau\overline{w_j})_{/j} - \nu_j q_k(\nabla_\tau\overline{w_j})_{/k}\right)\,do, \qquad (6.15)$$

$$I_5 = (\mu_2+\delta_2)\left(1-\frac{d}{2}\right)\int_{\Omega_2}|\operatorname{div} w|^2\,dx + \frac{(\mu_2+\delta_2)}{2}\int_{\Gamma_1}(q\nu)|\operatorname{div} w|^2\,do. \qquad (6.16)$$

Summarizing (6.12)–(6.16) we conclude

$$
\begin{aligned}
-\frac{1}{2}\int_{\Gamma_1}(q\nu)\left(|\lambda w|^2+\mu_2|\partial_\nu w|^2+(\mu_2+\delta_2)|\operatorname{div} w|^2\right)\,do+\frac{d}{2}\int_{\Omega_2}|\lambda w|^2\,dx&+\\
\mu_2\left(1-\frac{d}{2}\right)\int_{\Omega_2}|\nabla w|^2\,dx+\frac{\mu_2}{2}\int_{\Gamma_1}(q\nu)|\partial_\nu w|^2\,do&+\\
(\mu_2+\delta_2)\left(1-\frac{d}{2}\right)\int_{\Omega_2}|\operatorname{div} w|^2\,dx-\mu_2\mathrm{Re}\int_{\Gamma_1}\partial_\nu w_j\, q\,\overline{\nabla_\tau w_j}\,do&+\\
(\mu_2+\delta_2)\mathrm{Re}\int_{\Gamma_1}(q\nu)\operatorname{div} w\left((\nabla_\tau\overline{w_j})_{/j}-\nu_j q_k(\nabla_\tau\overline{w_j})_{/k}\right)\,do&\\
= \mathrm{Re}\int_{\Omega_2}(i\lambda g_2+g_4)q_k\partial_k\overline{w}\,dx.& \qquad (6.17)
\end{aligned}
$$

Multiplying (6.11) by $\eta\, w$ for some $\eta>0$, we obtain

$$
\begin{aligned}
-\eta\int_{\Omega_2}|\lambda w|^2\,dx+\eta\mu_2\int_{\Omega_2}|\nabla w|^2\,dx+\eta(\mu_2+\delta_2)\int_{\Omega_2}|\operatorname{div} w|^2\,dx&\\
-\eta\mu_2\int_{\Gamma_1}\partial_\nu w\overline{w}\,do-\eta(\mu_2+\delta_2)\int_{\Gamma_1}(\nabla_\tau w_j)_{/j}\nu_k\overline{w_k}\,do&\\
-\eta(\mu_2+\delta_2)\int_{\Gamma_1}(\partial_\nu w_j)_{/j}\nu_k\overline{w_k}\,do \;=\; \eta\mathrm{Re}\int_{\Omega_2}(i\lambda g_2+g_4)\overline{w}\,dx.& \qquad (6.18)
\end{aligned}
$$

Choosing $\eta:=\frac{d-1}{2}$ and then adding (6.17) and (6.18), we obtain the claim of Lemma 6.3.

Now, we can present the Proof of Theorem 6.1 in proving (6.2) for $\alpha=1$ and $\beta=2$.

Let $\lambda\in\mathbb{R}$ (sufficiently large in case), $(i\lambda-\mathcal{A})\,\Phi=F=(f_1,f_2,f_3,f_4,f_5)'\in D(\mathcal{A})$. Then, denoting again $\Phi=(u,v,U,V,\theta)'$, we have

$$
\begin{aligned}
i\lambda u-U &= f_1, &(6.19)\\
i\lambda v-V &= f_2, &(6.20)\\
i\lambda U+E_1u+\beta E_1U+\nabla\theta &= f_3, &(6.21)\\
i\lambda V+E_2v &= f_4, &(6.22)\\
i\lambda\theta-\Delta\theta+\operatorname{div} U &= f_5. &(6.23)
\end{aligned}
$$

We obtain from the dissipativity (2.2)

$$
\int_{\Omega_1}\mu_1|\nabla U|^2+(\mu_1+\delta_1)|\operatorname{div} U|^2\,dx+\int_{\Omega_1}|\nabla\theta|^2\,dx\le\|F\|_{\mathcal{H}}\|\Phi\|_{\mathcal{H}}. \qquad (6.24)
$$

This implies

$$
\int_{\Omega_1}|U|^2+|\theta|^2\,dx\le C\|F\|_{\mathcal{H}}\|\Phi\|_{\mathcal{H}}, \qquad (6.25)
$$

where the letter C will be used in the sequel to denote positive constants not depending on λ, ϕ or F. Using (6.19) and (6.25) we obtain

$$\int_{\Omega_1} |u|^2 + |\nabla u|^2 \, dx \leq C \left(\|F\|_{\mathcal{H}} \|\Phi\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2 \right). \tag{6.26}$$

It remains to estimate $\int_{\Omega_2} |\nabla v|^2 + |V|^2 \, dx$ appropriately, which is the difficult part and where we need in particular powers of $|\lambda|$ on the right-hand sides as well as the property of F to belong to the domain of $\mathcal{A}$.

Let

$$z := u + \beta U, \qquad \vartheta := (1 + i\lambda\beta) v.$$

Then we get from (6.21)–(6.23)

$$\begin{aligned} (1 + i\lambda\beta) E_1 z &= -(1 + i\lambda\beta)\nabla\theta - (1 + i\lambda\beta) i\lambda U + (1 + i\lambda\beta) f_3, \\ E_2 \vartheta &= -(1 + \imath\lambda\beta) i\lambda V + (1 + i\lambda\beta) f_4, \\ -\Delta\theta &= -i\lambda \operatorname{div} U - i\lambda\theta + f_5, \end{aligned}$$

with transmission conditions on Γ_1,

$$z = \vartheta - \beta f_2, \qquad (1 + i\lambda\beta)\partial_\nu^{E_1}(z + \theta\,\nu) = \partial_\nu^{E_2}\vartheta,$$

and boundary conditions $z = 0,\ \theta = 0$ on Γ_0, and $\partial_\nu\theta = 0$ on Γ_1. By elliptic regularity theory we obtain, assuming $\lambda \geq 1$ w.l.o.g.,

$$\begin{aligned} &|1 + i\lambda\beta| \|z\|_{H^2(\Omega_1)} + \|\vartheta\|_{H^2(\Omega_2)} + \|\theta\|_{H^2(\Omega_1)} \leq \\ &\qquad C|\lambda|^2 \left(\|U\|_{L^2(\Omega_1)} + \|\operatorname{div} U\|_{L^2(\Omega_1} + \|V\|_{L^2(\Omega_2)} + \|\theta\|_{L^2(\Omega_1)} \right) \\ &\qquad\qquad + C|\lambda| \|F\|_{\mathcal{H}} + C\|f_2\|_{H^2(\Omega_2)} \leq \\ &\qquad\qquad\qquad C \left(|\lambda|^2 \|V\|_{L^2(\Omega_2)} + |\lambda|^2 \|F\|_{\mathcal{H}}^{\frac{1}{2}} \|\Phi\|_{\mathcal{H}}^{\frac{1}{2}} + |\lambda| \|\mathcal{A}F\|_{\mathcal{H}} \right), \end{aligned} \tag{6.27}$$

where we used (6.24) and

$$\|f_2\|_{H^{\frac{3}{2}}(\Gamma_1)} \leq C\|f_2\|_{H^2(\Omega_2)} \leq C\|\mathcal{A}F\|_{\mathcal{H}}. \tag{6.28}$$

By interpolation we obtain, using (6.24)–(6.26),

$$\begin{aligned} &\|z\|_{H^{3/2}(\Omega_1)} + \|\nabla z\|_{L^2(\Gamma_1)} \leq C\|z\|_{H^1(\Omega_1)}^{\frac{1}{2}} \|z\|_{H^2(\Omega_1)}^{\frac{1}{2}} \\ &\leq \frac{C}{|1 + i\lambda\beta|^{\frac{1}{2}}} \left(\|F\|_{\mathcal{H}}^{\frac{1}{4}} \|\Phi\|_{\mathcal{H}}^{\frac{1}{4}} + \|F\|_{\mathcal{H}}^{\frac{1}{2}} \right) \left(|\lambda| \|V\|_{L^2(\Omega_2)}^{\frac{1}{2}} + |\lambda| \|F\|_{\mathcal{H}}^{\frac{1}{4}} \|\Phi\|_{\mathcal{H}}^{\frac{1}{4}} + |\lambda|^{\frac{1}{2}} \|\mathcal{A}F\|_{\mathcal{H}}^{\frac{1}{2}} \right) \\ &\leq C|\lambda|^{\frac{1}{2}} \Big(\|\mathcal{A}F\|_{\mathcal{H}}^{\frac{1}{4}} \|\Phi\|_{\mathcal{H}}^{\frac{1}{4}} \|V\|_{L^2(\Omega_2)}^{\frac{1}{2}} + \|\mathcal{A}F\|_{\mathcal{H}}^{\frac{1}{2}} \|\Phi\|_{\mathcal{H}}^{\frac{1}{2}} + \|\mathcal{A}F\|_{\mathcal{H}}^{\frac{3}{4}} \|\Phi\|_{\mathcal{H}}^{\frac{1}{4}} + \\ &\quad \|\mathcal{A}F\|_{\mathcal{H}}^{\frac{1}{2}} \|V\|_{L^2(\Omega_2)}^{\frac{1}{2}} + \|\mathcal{A}F\|_{\mathcal{H}}^{\frac{3}{4}} \|\Phi\|_{\mathcal{H}}^{\frac{1}{4}} + \|\mathcal{A}F\|_{\mathcal{H}} \Big) \\ &\leq C|\lambda|^{\frac{1}{2}} \left(\left(\|\mathcal{A}F\|_{\mathcal{H}}^{\frac{1}{4}} \|\Phi\|_{\mathcal{H}}^{\frac{1}{4}} + \|\mathcal{A}F\|_{\mathcal{H}}^{\frac{1}{2}} \right) \|V\|_{L^2(\Omega_2)}^{\frac{1}{2}} + \|\mathcal{A}F\|_{\mathcal{H}}^{\frac{1}{2}} \|\Phi\|_{\mathcal{H}}^{\frac{1}{2}} + \right. \\ &\quad \left. \|\mathcal{A}F\|_{\mathcal{H}}^{\frac{3}{4}} \|\Phi\|_{\mathcal{H}}^{\frac{1}{4}} + \|\mathcal{A}F\|_{\mathcal{H}} \right). \end{aligned} \tag{6.29}$$

In order to estimate $\int_{\Omega_2} |\nabla v|^2 + |V|^2 \, dx$ we apply Lemma 6.3 for $w := v, \ W := V_2$ with $g_2 := f_2, \ g_4 := f_4$. Using the strict positivity of $q\nu$ on Γ_1, we obtain

$$\begin{aligned}
&\|V\|^2_{L^2(\Omega_2)} + \|\nabla v\|^2_{L^2(\Omega_2)} + \|\nabla_\tau v\|^2_{L^2(\Gamma_1)} \leq \\
&\qquad C\left(\|v\|^2_{L^2(\Gamma_1)} + \epsilon\|\nabla_\tau v\|^2_{L^2(\Gamma_1)} + C_\epsilon\left(\|\partial_\nu v\|^2_{L^2(\Gamma_1)} + \|\operatorname{div} v\|^2_{L^2(\Gamma_1)}\right)\right. \\
&\qquad \left. +|\lambda|^2\left(\|f_2\|^2_{L^2(\Omega_2)} + \|f_4\|^2_{L^2(\Omega_2)}\right) + \|f_2\|^2_{L^2(\Omega_2)}\right), \qquad (6.30)
\end{aligned}$$

where $C_\epsilon > 0$ depends on ϵ, and ε is chosen small enough to get rid of the term $\epsilon\|\nabla_\tau v\|^2_{L^2(\Gamma_1)}$ on the right-hand side of (6.30). The last term in (6.30) arises from the estimate

$$\|V\|^2_{L^2(\Omega_2)} \ \leq \ 2\|\lambda v\|^2_{L^2(\Omega_2)} + 2\|f_2\|^2_{L^2(\Omega_2)}.$$

To further estimate this term we need the following type of Poincaré inequality.

Lemma 6.4.

$$\exists\, K > 0 \ \forall\, F \in \mathcal{H} \, : \|f_2\|_{L^2(\Omega_2)} \leq K\left(\|\nabla f_2\|_{L^2(\Omega_2)} + \|\nabla f_1\|_{L^2(\Omega_1)}.\right)$$

PROOF: Assuming that the inequality does not hold we get a sequence $(F_n)_n$ in $\mathcal{H}$ with $F_n = (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n})'$ such that

$$\|\nabla f_{2,n}\|_{L^2(\Omega_2)} + \|\nabla f_{1,n}\|_{L^2(\Omega_1)} \leq \frac{1}{n}\|f_{2,n}\|_{L^2(\Omega_2)}.$$

Defining

$$g_n := \frac{f_{1,n}}{\|f_{2,n}\|_{L^2(\Omega_2)}}, \quad h_n := \frac{f_{2,n}}{\|f_{2,n}\|_{L^2(\Omega_2)}},$$

we conclude, using $g_n = 0$ on Γ_0, that $g_n \to g := 0$ in $H^1(\Omega_1)$ as well as $(g_n)_{|\Gamma_1} \to g_{|\Gamma_1} = 0$ in $L^2(\Gamma_1)$. Since $\nabla h_n \to 0$ in $L^2(\Omega_2)$, and $\|h_n\|_{L^2(\Omega_2)} = 1$ we obtain from Rellich's compactness theorem that $h_n \to h$ in $H^1(\Omega_2)$ and $h_n \to h$ in $L^2(\Gamma_1)$ for some h satisfying $\nabla h = 0$. This implies $h = k$ for some constant k.

On the other hand we know $g_n = h_n$ on Γ_1 (reflecting the transmission condition in $\mathcal{H}$) which implies $h = g = 0$ on Γ_1 and hence $k = 0$, giving $h = 0$ in Ω_2. This is a contradiction to $\|h\|_{L^2(\Omega)} = \lim_n \|h_n\|_{L^2(\Omega)} = 1$ and hence proves the Lemma.

With this Lemma we conclude from (6.30)

$$\begin{aligned}
\|V\|^2_{L^2(\Omega_2)} + \|\nabla v\|^2_{L^2(\Omega_2)} \ &\leq \ C\left(\|\mathcal{A}F\|_{\mathcal{H}}\|\Phi\|_{\mathcal{H}} + \|\mathcal{A}F\|^2_{\mathcal{H}} + \right. \\
&\left. \|\partial_\nu v\|^2_{L^2(\Gamma_1)} + \|\operatorname{div} v\|^2_{L^2(\Gamma_1)} + |\lambda|^2\|\mathcal{A}F\|^2_{\mathcal{H}}\right). \qquad (6.31)
\end{aligned}$$

The following series of estimates now concerns the term $\|\partial_\nu v\|^2_{L^2(\Gamma_1)} + \|\operatorname{div} v\|^2_{L^2(\Gamma_1)}$ where the transmission conditions will be exploited again. We observe

$$(1 + i\lambda\beta)\partial_\nu v = \partial_\nu \vartheta, \quad (1 + i\lambda\beta)\operatorname{div} v = \operatorname{div} \vartheta.$$

Hence, with $c_1 := 2\mu_2(\mu_2+\delta_2)$,

$$\begin{aligned}
C|1+i\lambda\beta|^2\left(\|\partial_\nu v\|^2_{L^2(\Gamma_1)}+\|\operatorname{div} v\|^2_{L^2(\Gamma_1)}\right) &\le |\mu_2|^2\|\partial_\nu\vartheta\|^2_{L^2(\Gamma_1)}+|\mu_2+\delta_2|^2\|\operatorname{div}\vartheta\|^2_{L^2(\Gamma_1)}\\
&= \|\partial_\nu^{E_2}\vartheta\|^2_{L^2(\Gamma_1)}-c_1\int_{\Gamma_1}\partial_\nu\vartheta\operatorname{div}\bar\vartheta\,\nu\,do\\
&= \|\partial_\nu^{E_2}\vartheta\|^2_{L^2(\Gamma_1)}-c_1\int_{\Gamma_1}\partial_\nu\vartheta\operatorname{div}(\bar\vartheta-(\bar z+\bar f_2))\,\nu\,do\\
&\quad -c_1\int_{\Gamma_1}\partial_\nu\vartheta\operatorname{div}(\bar z+\bar f_2)\,\nu\,do\\
&\equiv (|1+i\lambda\beta|^2)(I+II+III). \qquad (6.32)
\end{aligned}$$

Observing

$$(1+i\lambda\beta)\partial_\nu^{E_1}(z+\theta\,\nu)=\partial_\nu^{E_2}\vartheta$$

we have

$$\begin{aligned}
I &\le 2\|\partial_\nu^{E_1}z\|^2_{L^2(\Gamma_1)}+2\|\theta\|^2_{L^2(\Gamma_1)}\\
&\le 2\|z\|^2_{H^{3/2}(\Omega_1)}+C\|\mathcal{A}F\|_{\mathcal{H}}\|\Phi\|_{\mathcal{H}}.
\end{aligned}$$

by (6.24). Next, using (6.28),

$$\begin{aligned}
|III| &\le \varepsilon\|\partial_\nu\vartheta\|^2_{L^2(\Gamma_1)}+C_\varepsilon\left(\|z\|^2_{H^{3/2}(\Omega_1)}+\|f_2\|^2_{H^1(\Gamma_1)}\right)\\
&\le \varepsilon\|\partial_\nu\vartheta\|^2_{L^2(\Gamma_1)}+C_\varepsilon\left(\|z\|^2_{H^{3/2}(\Omega_1)}+\|\mathcal{A}F\|^2_{\mathcal{H}}\right). \qquad (6.33)
\end{aligned}$$

Using $\vartheta = z+\beta f_2$ on Γ_1, we have

$$\operatorname{div}\,(\vartheta-(z+f_2))=\nu\,\partial_\nu\,(\vartheta-(z+f_2))\,.$$

This implies

$$\begin{aligned}
II &= -c_1\|\partial_\nu\vartheta\|^2_{L^2(\Gamma_1)}+c_1\int_{\Gamma_1}(\partial_\nu\vartheta\,\nu)\left(\partial_\nu\bar z+\partial_\nu\bar f_2\right)\nu\,do\\
&\equiv -c_1\|\partial_\nu\vartheta\|^2_{L^2(\Gamma_1)}+\tilde{II}.
\end{aligned}$$

Since we may neglect the negative term $-c_1\|\partial_\nu\vartheta\|^2_{L^2(\Gamma_1)}$ later on, and since

$$\begin{aligned}
|\tilde{II}| &\le \varepsilon\|\partial_\nu\vartheta\|^2_{L^2(\Gamma_1)}+C_\varepsilon\left(\|\partial_\nu z\|^2_{L^2(\Gamma_1)}+\|\partial_\nu f_2\|^2_{L^2(\Gamma_1)}\right)\\
&\le \varepsilon\|\partial_\nu\vartheta\|^2_{L^2(\Gamma_1)}+C_\varepsilon\left(\|z\|^2_{H^{3/2}(\Omega_1)}+\|\mathcal{A}F\|^2_{\mathcal{H}}\right), \qquad (6.34)
\end{aligned}$$

we get from (6.32)–(6.34), choosing ε small enough,

$$\|\partial_\nu v\|^2_{L^2(\Gamma_1)}+\|\operatorname{div} v\|^2_{L^2(\Gamma_1)} \le C\left(\|z\|^2_{H^{3/2}(\Omega_1)}+\|\mathcal{A}F\|^2_{\mathcal{H}}+\|\mathcal{A}F\|_{\mathcal{H}}\|\Phi\|_{\mathcal{H}}\right). \quad (6.35)$$

Combining (6.29) and (6.35) we obtain

$$\begin{aligned}\|\partial_\nu v\|^2_{L^2(\Gamma_1)} + \|\operatorname{div} v\|^2_{L^2(\Gamma_1)} &\leq C|\lambda| \left(\left(\|\mathcal{A}F\|^{\frac{1}{2}}_{\mathcal{H}} \|\Phi\|^{\frac{1}{2}}_{\mathcal{H}} + \|\mathcal{A}F\|_{\mathcal{H}} \right) \|V\|_{L^2(\Omega_2)} + \right. \\ &\quad \left. \|\mathcal{A}F\|_{\mathcal{H}} \|\Phi\|_{\mathcal{H}} + \|\mathcal{A}F\|^{\frac{3}{2}}_{\mathcal{H}} \|\Phi\|^{\frac{1}{2}}_{\mathcal{H}} + \|\mathcal{A}F\|^2_{\mathcal{H}} \right) \\ &\leq c|\lambda|^2 \left(\|\mathcal{A}F\|_{\mathcal{H}} \|\Phi\|_{\mathcal{H}} + \|\mathcal{A}F\|^{\frac{3}{2}}_{\mathcal{H}} \|\Phi\|^{\frac{1}{2}}_{\mathcal{H}} \right. \\ &\quad \left. + \|\mathcal{A}F\|^2_{\mathcal{H}} \right) + \frac{1}{2} \|V\|^2_{L^2(\Omega_2)}. \end{aligned} \tag{6.36}$$

A combination of (6.31) and (6.36) yields

$$\|V\|^2_{L^2(\Omega_2)} + \|\nabla v\|^2_{L^2(\Omega_2)} \leq C|\lambda|^2 \left(\|\mathcal{A}F\|_{\mathcal{H}} \|\Phi\|_{\mathcal{H}} + \|\mathcal{A}F\|^2_{\mathcal{H}} + \|\mathcal{A}F\|^{\frac{3}{2}}_{\mathcal{H}} \|\Phi\|^{\frac{1}{2}}_{\mathcal{H}} \right). \tag{6.37}$$

Finally combining (6.24)–(6.26) and (6.37) we have proved the estimate

$$\|\Phi\|^2_{\mathcal{H}} \leq C|\lambda|^4 \|\mathcal{A}F\|^2_{\mathcal{H}} + \frac{1}{2} \|\Phi\|^2_{\mathcal{H}}$$

implying

$$\| (i\lambda - \mathcal{A})^{-1} F \|_{\mathcal{H}} \leq C|\lambda|^2 \|\mathcal{A}F\|_{\mathcal{H}}$$

which completes the proof of Theorem 6.1.

7 On the lack of compactness of the inverse

We present examples to underline that the compactness of the inverse $\mathcal{A}^{-1}$ cannot be expected (cp. Remark 5.2).

7.1 Wave equation or Kirchhoff plate equation

We look at the classical Kelvin-Voigt damping – without transmission, $\Omega_2 = \emptyset$ – for the wave equation or the Kirchhoff plate equation, respectively, i.e. for

$$v_{tt} - \Delta(v + \beta v_t) = 0, \quad v = 0 \quad \text{on} \quad \Gamma, \tag{7.1}$$

or

$$v_{tt} + \Delta^2(v + \beta v_t) = 0, \quad v = \Delta v = 0 \quad \text{on} \quad \Gamma, \tag{7.2}$$

where

$$\Gamma := \Gamma_0 \cup \Gamma_1 = \partial\Omega_1.$$

If A denotes either $-\Delta$ or Δ^2 (with corresponding boundary conditions in its domain), then the usual transformation to a first-order system yields the stationary operator

$$\mathcal{B} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ -A(f + \beta g) \end{pmatrix}.$$

We look at the eigenvalue problem

$$\mathcal{B}\begin{pmatrix} f \\ g \end{pmatrix} = \lambda \begin{pmatrix} f \\ g \end{pmatrix}.$$

This turns into the equation

$$-(1+\beta\lambda)Af = \lambda^2 f.$$

Let

$$\lambda_0 := -\frac{1}{\beta}.$$

For $\lambda = \lambda_0$ we conclude $f = g = 0$, hence λ_0 is no eigenvalue.

Since A has (only) positive real eigenvalues we conclude that real $\lambda < \lambda_0$ exist as eigenvalues of $\mathcal{B}$, as there are for $n \in \mathbb{N}$ the λ_n satisfying

$$-\frac{\lambda_n^2}{1+\beta\lambda_n} = \xi_n,$$

where $(\xi_n)_n$ are the eigenvalues of A satisfying $\xi_n \to \infty$ as $n \to \infty$. We obtain

$$\lambda_n^{\pm} = \frac{1}{2}\left(-\beta\xi_n \pm \sqrt{\beta^2\xi_n^2 - 4\xi_n}\right).$$

Since

$$\lambda_n^+ \to -\frac{1}{\beta} = \lambda_0, \qquad \text{as} \quad n \to \infty,$$

we conclude that $\lambda_0 \in \sigma_{ess}(\mathcal{B})$ (essential spectrum of $\mathcal{B}$). As a consequence, $\mathcal{B}^{-1}$ cannot be compact.

Since the resolvent equation

$$(\lambda_0 - \mathcal{B})(f,g)' = (F,G)'$$

is equivalent to

$$g = \lambda_0 f - F, \qquad \lambda_0^2 f = \lambda_0 F + \beta AF + G,$$

it is solvable for $(F,G)' \in C_0^\infty(\Omega_1)$, hence λ_0 belongs to the continuous spectrum $\sigma_c(\mathcal{B})$ (cp. [19] for the Euler-Bernoulli beam in one dimension with variable coefficients).

7.2 Thermoelastic plate equation

As second example, we consider a thermoelastic plate equation with Kelvin-Voigt damping. The differential equations are, cp. (7.2),

$$v_{tt} + \Delta^2(v + \beta v_t) + \Delta\theta = 0, \tag{7.3}$$

$$\theta_t - \Delta\theta - \Delta u_t = 0, \tag{7.4}$$

with additional boundary conditions for θ of Dirichlet type: $\theta = 0$ on Γ.

The stationary operator for the corresponding first-order system is given by

$$\mathcal{B}\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} g \\ -\Delta^2(f+\beta g) - \Delta h \\ \Delta h + \Delta g \end{pmatrix}.$$

The eigenvalue problem $\mathcal{B}\Phi = \lambda\Phi$ leads, for $\Phi = (f, g, h)'$, to

$$g = \lambda f, \quad -\Delta^2(f+\beta g) - \Delta h = \lambda g, \quad \Delta h + \Delta g = \lambda h.$$

Eliminating f and g, we have to solve

$$-(1+\beta\lambda)\Delta^3 h - \lambda(\beta\lambda+2)\Delta^2 h + \lambda^2\Delta h - \lambda^3 h = 0, \tag{7.5}$$

with boundary conditions

$$h = \Delta h = \Delta^2 h = 0 \qquad (\text{on} \quad \Gamma).$$

Solving for h and λ, and defining g by ($\Delta g = \Delta h - \lambda h$, $g = 0$ on the boundary) and then $f := g/\lambda$ ($\lambda = 0$ is not possible), we obtain with this λ an eigenvalue with eigenvector (f, g, h) for $\mathcal{B}$.

Let again

$$\lambda_0 := -\frac{1}{\beta}.$$

Making for (7.5) the ansatz

$$h = h_n(x) = \chi_n(x),$$

where $(\chi_n)_n$ are the eigenvectors to the Dirichlet-Laplace operator with eigenvalues $(\zeta_n)_n$,

$$-\Delta\chi_n = \zeta_n\chi_n, \qquad \chi_n = 0 \quad \text{on} \quad \Gamma, \tag{7.6}$$

satisfying

$$0 < \zeta_n \to \infty, \qquad (\text{as} \quad n \to \infty),$$

we have to solve

$$P(\lambda) := (1+\beta\lambda)\zeta_n^3 + (\lambda(\beta\lambda+2)\zeta_n^2 + \lambda^2\zeta_n + \lambda^3 = 0. \tag{7.7}$$

We look for a sequence of real $(\lambda_n)_n$ satisfying $P(\lambda_n) = 0$ with $\lambda_n \to \lambda_0$.

Claim: For any $\varepsilon > 0$ there is a real zero λ_n of P, satisfying $\lambda_n \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ provided $n \geq n_0(\varepsilon)$.

Proof of the claim: Since

$$P(\lambda_0 \pm \varepsilon) \left\{ \begin{matrix} > \\ < \end{matrix} \right\} 0$$

if n is large enough depending on ε, the claim follows (here we used $\zeta_n \to \infty$).

Letting ε tend to zero, we can find a sequence of eigenvalues of $\mathcal{B}$ tending to λ_0. Hence

$$\lambda_0 \in \sigma(\mathcal{B}).$$

The question if λ_0 can be an eigenvalue, is answered with: no. For λ_0 to be an eigenvalue we conclude from (7.5) the solvability of the special problem

$$(\Delta^2 + \frac{1}{\beta}\Delta)h = -\frac{1}{\beta^2}h. \tag{7.8}$$

The operator $A_\beta := \Delta^2 + \frac{1}{\beta}\Delta$ corresponding to the boundary conditions $h = \Delta h = 0$ is a self-adjoint operator in $L^2(\Omega_1)$ which is bounded from below but not necessarily positive. Nevertheless, (7.8) does not have a solution since all eigenfunctions of A_β are given by the complete set of orthonormal eigenfunctions χ_n of the Laplace operator given in (7.6). Assuming that (7.8) has a solution we arrive at the necessary condition

$$\zeta_n^2 - \frac{1}{\beta}\zeta_n = -\frac{1}{\beta^2},$$

for some $n \in \mathbb{N}$, or, equivalently,

$$\left(\beta - \frac{1}{2\zeta_n}\right)^2 = -\frac{3}{4\zeta_n^2},$$

which is a contradiction. Conclusion:

$$\lambda_0 \in \sigma_{ess}(\mathcal{B}), \qquad \mathcal{B}^{-1} \text{ is not compact.}$$

The resolvent equation

$$(\lambda_0 - \mathcal{B})(f, g, h)' = (F, G, H)'$$

is, for given $(F, G, H)' \in C_0^\infty(\Omega_1)$, equivalent to

$$g = \lambda_0 f - F, \quad \lambda_0^2 f = \lambda_0^2 F + \Delta^2 F + G - \Delta h, \quad B_\beta h := (\Delta^2 + \frac{1}{\beta}\Delta + \frac{1}{\beta^2})h = Q,$$

were $Q = Q(F, G, H) \in C_0^\infty(\Omega_1)$. The problem $B_\beta h = Q$ can be solved for h, with arbitrarily smooth h, (with boundary conditions $h = \Delta h = 0$), since B_β is a positive, self-adjoint (elliptic) operator, the positivity following from

$$\zeta_n^2 - \frac{1}{\beta}\zeta_n + \frac{1}{\beta^2} = (\zeta_n - \frac{1}{\beta})^2 + \frac{3}{4\beta^2} > 0.$$

Hence λ_0 belongs to the continuous spectrum $\sigma_c(\mathcal{B})$.

With these examples in mind it is reasonable to assume that the inverse of the operator(s) $\mathcal{A}$ arising in Sections 1–6 is not compact.

8 Variations

All the results in the previous sections carry over to the situation without temperature, i. e. to the (viscoelasticity plus Kelvin-Voigt) $\longleftrightarrow$ (pure viscoelasticity) transmission problem (1.10)–(1.16), we have

Theorem 8.1. *The semigroup $\left(\mathrm{e}^{t\mathcal{A}}\right)_{t\geq 0}$ associated to the transmission problem (1.10)–(1.16) is not exponentially but strongly stable and satisfies the polynomial decay estimate*

$$\exists\, C>0\;\exists\, t_0>0\;\forall\, t\geq t_0\;\forall\,\Phi^0\in D(\mathcal{A}):\;\|\mathrm{e}^{t\mathcal{A}}\Phi_0\|_{\mathcal{H}}\leq C\,t^{-\frac{1}{3}}\|\mathcal{A}\Phi^0\|_{\mathcal{H}}. \tag{8.1}$$

On the other hand, the case of considering an additional temperature in Ω_2 (thermoelastic problem) is open and depending on the dimension, cp. the complex discussion of pure thermoelasticity in [6].

But we obtain the corresponding results for the Kelvin-Voigt transmission problem for scalar wave equations (1.17)–(1.23), we have

Theorem 8.2. *The semigroup $\left(\mathrm{e}^{t\mathcal{A}}\right)_{t\geq 0}$ associated to the transmission problem (1.17)–(1.23) is not exponentially but strongly stable and satisfies the polynomial decay estimate*

$$\exists\, C>0\;\exists\, t_0>0\;\forall\, t\geq t_0\;\forall\,\Phi^0\in D(\mathcal{A}):\;\|\mathrm{e}^{t\mathcal{A}}\Phi_0\|_{\mathcal{H}}\leq C\,t^{-\frac{1}{3}}\|\mathcal{A}\Phi^0\|_{\mathcal{H}}. \tag{8.2}$$

We just remark that Lemma 6.3 now, in the scalar case with the Laplace operator replacing the elastic operator, takes the (simpler) form, which we state for future references.

Lemma 8.3. *Let $x_0\in\Omega_2$ and $q(x):=x-x_0$ for $x\in\Omega_2$. Let w,W and $\lambda\in\mathbb{R}$ satisfy*

$$i\lambda w - W = g_2\in L^2(\Omega_2), \tag{8.3}$$

$$i\lambda W-\kappa_0\Delta w = g_4\in L^2(\Omega_2). \tag{8.4}$$

Then we have

$$\begin{aligned}\frac{1}{2}\int_{\Omega_2}|\lambda w|^2+\kappa_0|\nabla w|^2+\frac{\kappa_0}{2}\int_{\Gamma_1}(q\nu)|\nabla_\tau w|^2\,do-\frac{1}{2}\int_{\Gamma_1}(q\nu)\left(|\lambda w|^2+\kappa_0|\partial_\nu w|^2\right)\,do&\\ -\kappa_0\mathrm{Re}\int_{\Gamma_1}\partial_\nu w_j q\nabla_\tau\overline{w_j}\,do-\kappa_0\frac{d-1}{2}\mathrm{Re}\int_{\Gamma_1}\partial_\nu w\overline{w}\,do&\\ =\mathrm{Re}\int_{\Omega_2}(i\lambda g_2+g_4)(q_k\partial_k\overline{w}+\overline{w})\,dx.&\end{aligned} \tag{8.5}$$

We finally remark that for the results on polynomial stability we cannot exchange the roles of the domains Ω_1 and Ω_2, since the positivity of $q\nu$ (cp. Lemma 6.3 or Lemma 8.3) would then no longer be given, but was essentially used in the proofs. On the other hand, all the the results on smoothing, on the lack of exponential stability and on the strong stability in Theorem 3.1, Theorem 4.4, Theorem 5.1, Theorem 8.1, and Theorem 8.2 carry over to the situation where the purely elastic part is the surrounding one (Ω_1), and the (thermo-)viscoelastic material is the inner part (Ω_2).

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