DISSIPATIVE STRUCTURES FOR THERMOELASTIC PLATE EQUATIONS IN $\mathbb{R}^n$

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Abstract. We consider the Cauchy problem in $\mathbb{R}^n$ for linear thermoelastic plate equations where heat conduction is modeled by either the Cattaneo law or by the Fourier law – described by the relaxation parameter τ, where $\tau > 0$ corresponds to Cattaneo's law and $\tau = 0$ corresponds to Fourier's law. Additionally, we take into account possible inertial effects characterized by a parameter $\mu \geq 0$, where $\mu = 0$ corresponds to the situation without inertial terms.

For the Catteneo system without inertial term, being a coupling of a Schrödinger type equation (the elastic plate equation) with a hyperbolic system for the temperature and the heat flux, we shall show that a regularity-loss phenomenon appears in the asymptotic behavior as time tends to infinity, while this is not given in the standard model where the Cattaneo law is replaced by the standard Fourier law. This kind of effect of changing the qualitative behavior when moving from Fourier to Cattaneo reflects the effect known for bounded domains, where the system with Fourier law is exponentially stable while this property is lost when going to the Cattaneo law. In particular, we shall describe in detail the singular limit as $\tau \to 0$.

For the system with inertial term we demonstrate that it is of standard type, not of regularity loss type. The corresponding limit of a vanishing inertial term is also described.

All constants appearing in the main results are given explicitly, allowing for quantitative estimates. The optimality of the estimates is also proved.

Keywords: thermoelastic plate, decay structure, regularity-loss, Cauchy problem, inertial term, singular limit

MSC 2010: 35B35, 35B40, 35Q79, 74F05

1. Introduction

In this paper, we consider the Cauchy problem for the following linear thermoplastic plate equation, where heat conduction is modeled by Cattaneo's (Maxwell's, Vernotte's) law ($\tau > 0$) or by Fourier's law ($\tau = 0$), and where an inertial term may be present ($\mu > 0$) or not ($\mu = 0$):

$$\begin{aligned} u_{tt} + \Delta^2 u - \mu \Delta u_{tt} + \Delta\theta &= 0, \\ \theta_t + \operatorname{div} q - \Delta u_t &= 0, \\ \tau q_t + q + \nabla\theta &= 0. \end{aligned} \tag{1.1}$$

Here, u describes the elongation of a plate, while θ and q denote the temperature (difference to a fixed temperature) resp. the heat flux. For the Cattaneo law the relaxation parameter τ is a positive constant. The constant μ is a non-negative parameter in front of the inertial term. Taking $\mu = \tau = 0$, we obtain the standard thermoelastic plate equation:

$$\begin{aligned} u_{tt} + \Delta^2 u + \Delta\theta &= 0, \\ \theta_t - \Delta\theta - \Delta u_t &= 0, \end{aligned} \tag{1.2}$$

where the Cattaneo law

$$\tau q_t + q + \nabla\theta = 0 \tag{1.3}$$

has turned into the Fourier law

$$q + \nabla\theta = 0, \tag{1.4}$$

leading to the classical parabolic heat equation appearing in (1.2).

We mention that we set several constants (given from physics and usually appearing in (1.1) or (1.2)) equal to one without loss of generality for the discussion in the next sections.

One main purpose of this note is to analyze for $\mu = 0$ the dissipative structure for the system (1.1) and to compare the properties of (1.1) and (1.2) in terms of the relaxation parameter τ. We shall show that there shows up a so-called regularity-loss when moving from (1.2) to (1.1). In other words, the singular limit $\tau \to 0$ is reflected in removing this regularity-loss. Said-Houari [23] recently also proved the regularity-loss result in one space-dimension. Here, we consider the multi-dimensional case and, as main contribution, explicitly provide the dependence on τ for a better understanding of the singular limit. Moreover, we consider the eigenvalue problems for (1.1) and (1.2) in Subsection 3.2. This way we demonstrate the optimality of decay estimates. Some (non-sharp) decay estimates were also given in [10].

It is interesting to notice that a kind of essentially changing the qualitative behavior can also be observed for bounded domains (instead of the Cauchy problem in $\mathbb{R}^n$), where the corresponding initial boundary value problem typically shows exponential stability for $\tau = 0$, while it looses this property for $\tau > 0$, see the papers of Quintanilla & Racke [19, 20]. For bounded domains and $\tau = 0$, there are many results in particular on exponential stability, see for example [1, 11, 12, 13, 14, 15, 16, 17, 18] For results for the Cauchy problem or in general exterior domains see for example [2, 3, 4, 17, 18]. For $\mu > 0$ the exponential stability is always given [5].

Similar effects are known for the thermoelastic Timoshenko system in one space dimensions. Here, we also have that the system with the Fourier model for heat conduction may show exponential stability in bounded domains (in the case of equal wave speeds of the two wave equations involved), while this property is lost with the Cattaneo model, see Fernańdez Sare & Racke [6]. Moreover, for the Cauchy problem in $\mathbb{R}^1$, one has the same effect, i.e., a regularity-loss phenomenon when changing from Fourier's to Cattaneo's law, see Ide & Kawashima [8], Ide & Haramoto & Kawashima [7], Ueda & Duan & Kawashima [26], Said-Houari & Kasimov [24].

One should also know that there are thermoelastic systems – with second-order elasticity - which behave very much the same, no matter if $\tau = 0$ or if $\tau > 0$, see [21, 22]; this is true even on a quantitative level, see [9].

We shall also investigate the case $\mu > 0$ reflection an inertial term and will demonstrate that there is no regularity loss but the standard type present. This corresponds to the fact that in bounded domains exponential stability its given. The limit $\mu \to 0$ is investigated too. This way, we obtain the global picture for the limits $\tau, \mu \to 0$ in a detailed manner.

The estimates in the main theorems will be given with explicit constants, thus allowing for even quantitative estimates[1].

System (1.1) with $\mu = 0$ is a coupling of an equation of Schrödinger type (for u) and a hyperbolic system (for θ, q), while system (1.2) couples the former one to a parabolic equation (for θ). For *hyperbolic* systems with partial symmetric relaxation or with partial non-symmetric relaxation (like in Timoshenko systems) there exist results and general conditions like the Shizuta-Kawashima condition to guarantee stability resp. to show a regularity-loss. We mention here [27, 25] and Ueda & Duan & Kawashima [26]. These results are not applicable to the system under investigation.

Thus, another contribution of our paper therefore shall be to initiate investigations on couplings as those given in (1.1) or (1.2) looking for characterizations of (no) regularity-loss.

The paper is organized as follows. In Section 2 we consider the system without inertial term ($\mu = 0$) and with the Fourier model ($\tau = 0$), giving the optimal decay estimates with explicit constants. Section 3 provides the main theorems for the corresponding Cattaneo model ($\tau > 0$). More precisely, in Subsection 3.1 we show the decay with the characterization of the dependence on τ. Subsection 3.2 provides an expansion of the eigenvalues of the characteristic equation giving the dependence on τ and showing optimality of decay rates. Furthermore, in Subsection 3.3 we compare the solutions for the system with $\tau = 0$ to those of the system with $\tau > 0$ and obtain a difference of order τ^2 on the level of the energy terms. Finally, in Section 4 we consider system (1.1) with inertial term ($\mu > 0$) and demonstrate that there is no regularity loss phenomenon.

Throughout the paper, we always assume

$$0 \le \tau \le 1,$$

and we use standard notation, in particular the Sobolev spaces $L^p = L^p(\mathbb{R}^n)$, $p \ge 1$, and $H^s = W^{s,2}(\mathbb{R}^n)$, $s \in \mathbb{N}_0$, with their associated norms $\|\cdot\|_{L^p}$ resp. $\|\cdot\|_{H^s}$.

2. The standard thermoelastic plate equation

Before studying the general case (1.1), we first consider the Cauchy problem (1.2) with initial data

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \qquad \theta(0,x) = \theta_0(x), \qquad x \in \mathbb{R}^n. \tag{2.1}$$

For (1.2), (2.1) there exist investigations on the time asymptotic behavior, e.g. on polynomial decay rates, see [2, 17, 18]. In Subsection 2.1, we derive for (1.2), (2.1)

[1]Some techniques presented here can be used for bounded domains too and will lead to numerical estimates for decay rates in bounded domains that are rather sharp (forthcoming paper).

estimates for the associated energy term in Fourier space, leading to L^2 energy estimates and to polynomial decay rates. In Subsection 2.2, we investigate the associated characteristic polynomial and prove the optimality of the decay estimates.

2.1. **Decay estimates.** We start proving pointwise estimates and L^2-decay estimates.

Proposition 2.1. *Let $s \geq 0$ be an integer and suppose that the initial data $(u_1, \Delta u_0, \theta_0)$ belong to H^s. Then the solution (u, θ) satisfies the energy estimate:*

$$\|\partial_x^k(u_t, \Delta u, \theta)(t)\|_{L^2}^2 + \frac{1}{12}\int_0^t \|\partial_x^k \nabla(u_t, \Delta u, \theta)(\sigma)\|_{L^2}^2 d\sigma \leq \frac{13}{3}\|\partial_x^k(u_1, \Delta u_0, \theta_0)\|_{L^2}^2, \tag{2.2}$$

for $t \geq 0$ and $0 \leq k \leq s$.

Moreover, we obtain the following pointwise estimate and, assuming that the initial data belong to L^1, we can get the following polynomial decay estimates.

Theorem 2.2. *The Fourier image $(\hat{u}, \hat{\theta})$ of the solution (u, θ) to the Cauchy problem (1.2), (2.1) satisfies the pointwise estimate:*

$$|\hat{u}_t(t,\xi)| + |\xi|^2|\hat{u}(t,\xi)| + |\hat{\theta}(t,\xi)| \leq \frac{13}{3}e^{-\frac{1}{52}\rho(\xi)t}(|\hat{u}_1(\xi)| + |\xi|^2|\hat{u}_0(\xi)| + |\hat{\theta}_0(\xi)|), \tag{2.3}$$

where $\rho(\xi) := |\xi|^2$. Furthermore, let suppose that the initial data $(u_1, \Delta u_0, \theta_0)$ belong to L^1. Then the solution (u, θ) satisfies the decay estimate:

$$\|\partial_x^k(u_t, \Delta u, \theta)(t)\|_{L^2} \leq Ct^{-n/4-k/2}\|(u_1, \Delta u_0, \theta_0)\|_{L^1} \tag{2.4}$$

for $t > 0$ and $k \geq 0$. Here C is a positive constant not depending on the data or on t.

Remark 1. *From Theorem 2.2, we get in a standard manner for an integer $s \geq 0$ and supposing that the initial data $(u_1, \Delta u_0, \theta_0)$ belong to $H^s \cap L^1$, that the solution (u, θ) satisfies the decay estimate:*

$$\begin{aligned}\|\partial_x^k(u_t, \Delta u, \theta)(t)\|_{L^2} \leq& C(1+t)^{-n/4-k/2}\|(u_1, \Delta u_0, \theta_0)\|_{L^1} \\ &+ Ce^{-ct}\|\partial_x^k(u_1, \Delta u_0, \theta_0)\|_{L^2},\end{aligned} \tag{2.5}$$

for $t \geq 0$ and $0 \leq k \leq s$. The decay estimate (2.5) is same as the standard type decay estimate for the symmetric hyperbolic system with relaxation (cf. [26]). There is "no loss of regularity" since the necessary regularity on the right-hand side is the same of that of the solution at time t on the left-hand side.

Proof of Proposition 2.1 and of Theorem 2.2. We prove Proposition 2.1 and Theorem 2.2 together. By employing the Fourier transform, system (1.2) is described as

$$\begin{aligned}&\hat{u}_{tt} + |\xi|^4\hat{u} - |\xi|^2\hat{\theta} = 0, \\ &\hat{\theta}_t + |\xi|^2\hat{\theta} + |\xi|^2\hat{u}_t = 0.\end{aligned} \tag{2.6}$$

We first derive the basic energy equality for the system (2.6) in the Fourier space. We multiply the equations (2.6) by $\bar{\hat{u}}_t, \bar{\hat{\theta}}$, respectively, and combine the resulting equations, taking real parts. So we obtain for $W(t,\xi) := (|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 + |\hat{\theta}|^2)(t,\xi)$, that

$$\frac{1}{2}\frac{\partial}{\partial t}W(t,\xi) + |\xi|^2|\hat{\theta}|^2 = 0. \tag{2.7}$$

We next create more dissipation terms (negative energy type terms). Multiplying the first equation in (2.6) by $\bar{\hat{u}}$ and taking real parts, we have

$$\frac{\partial}{\partial t}\big(\mathrm{Re}(\hat{u}_t\bar{\hat{u}})\big) - |\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 - |\xi|^2\mathrm{Re}(\hat{\theta}\bar{\hat{u}}) = 0. \tag{2.8}$$

Now we multiply the first and second equations in (2.6) by $\bar{\hat{\theta}}$ and $\bar{\hat{u}}_t$, respectively, and we obtain

$$\frac{\partial}{\partial t}\big(\mathrm{Re}(\hat{u}_t\bar{\hat{\theta}})\big) + |\xi|^2(|\hat{u}_t|^2 - |\hat{\theta}|^2) + |\xi|^2\big(\mathrm{Re}(\hat{\theta}\bar{\hat{u}}_t) + |\xi|^2\mathrm{Re}(\hat{u}\bar{\hat{\theta}})\big) = 0. \tag{2.9}$$

Finally, we multiply (2.8) and (2.9) by $\alpha_1\alpha_2|\xi|^2$ and α_1, respectively, and add these two equations and (2.7), where α_1 and α_2 are positive constants yet to be determined. This yields

$$\frac{\partial}{\partial t}E + D = 0, \tag{2.10}$$

where

$$\begin{aligned} E(t,\xi) &:= \frac{1}{2}(|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 + |\hat{\theta}|^2) + \alpha_1\big(\mathrm{Re}(\hat{u}_t\bar{\hat{\theta}}) + \alpha_2|\xi|^2\mathrm{Re}(\hat{u}_t\bar{\hat{u}})\big), \\ D(t,\xi) &:= |\xi|^2\big[\alpha_1(1-\alpha_2)|\hat{u}_t|^2 + \alpha_1\alpha_2|\xi|^4|\hat{u}|^2 + (1-\alpha_1)|\hat{\theta}|^2\big] \\ &\qquad + \alpha_1|\xi|^2\big[\mathrm{Re}(\hat{\theta}\bar{\hat{u}}_t) + (1-\alpha_2)|\xi|^2\mathrm{Re}(\hat{u}\bar{\hat{\theta}})\big]. \end{aligned}$$

Applying $|\hat{u}_t\bar{\hat{\theta}}| \le \varepsilon_1|\hat{u}_t|^2 + (4\varepsilon_1)^{-1}|\hat{\theta}|^2$ and $|\xi|^2|\hat{u}\bar{\hat{\theta}}| \le \varepsilon_2|\xi|^4|\hat{u}|^2 + (4\varepsilon_2)^{-1}|\hat{\theta}|^2$ to the dissipation term D, we may estimate

$$\begin{aligned} D \ge |\xi|^2\Big[&\alpha_1(1-\alpha_2-\varepsilon_1)|\hat{u}_t|^2 + \alpha_1(\alpha_2 - \varepsilon_2(1-\alpha_2))|\xi|^4|\hat{u}|^2 \\ &+ (1 - \alpha_1(1 + (4\varepsilon_1)^{-1} + (1-\alpha_2)(4\varepsilon_1)^{-1}))|\hat{\theta}|^2\Big]. \end{aligned}$$

Choosing $\alpha_1 = \alpha_2 = \varepsilon_1 = \varepsilon_2 = 1/4$, then the above estimate is rewritten as

$$D \ge |\xi|^2\big(\frac{1}{8}|\hat{u}_t|^2 + \frac{1}{64}|\xi|^4|\hat{u}|^2 + \frac{5}{16}|\hat{\theta}|^2\big) \ge \frac{1}{64}|\xi|^2(|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 + |\hat{\theta}|^2). \tag{2.11}$$

On the other hand, with $\alpha_1 = \alpha_2 = 1/4$, the energy term E is estimated as

$$\begin{aligned} E &\le \frac{13}{16}|\hat{u}_t|^2 + \frac{9}{16}|\xi|^4|\hat{u}|^2 + \frac{3}{4}|\hat{\theta}|^2 \le \frac{13}{16}W(t,\xi), \\ E &\ge \frac{3}{16}|\hat{u}_t|^2 + \frac{7}{16}|\xi|^4|\hat{u}|^2 + \frac{1}{4}|\hat{\theta}|^2 \ge \frac{3}{16}W(t,\xi). \end{aligned} \tag{2.12}$$

By applying the estimates (2.11) and (2.12) to (2.10), we get

$$W(t,\xi) + \frac{1}{12}\int_0^t |\xi|^2 W(\sigma,\xi)d\sigma \le \frac{13}{3}W(0,\xi),$$

and hence

$$\|\partial_x^k(u_t,\partial_x^2u,\theta)(t)\|_{L^2}^2+\frac{1}{12}\int_0^t\|\partial_x^{k+1}(u_t,\partial_x^2u,\theta)(\sigma)\|_{L^2}^2d\sigma\le\frac{13}{3}\|\partial_x^k(u_1,\partial_x^2u_0,\theta_0)\|_{L^2}^2,$$

for $k\ge 0$. This energy estimate gives (2.2).

Furthermore using (2.11), (2.12) and (2.10) again, thus observing that

$$\frac{\partial}{\partial t}E(t,\xi)\le -D(t,\xi)\le -\frac{1}{64}|\xi|^2W(t,\xi)\le -\frac{1}{52}|\xi|^2E(t,\xi),$$

this yields the following pointwise estimate

$$W(t,\xi)\le\frac{13}{3}e^{-\frac{1}{52}|\xi|^2t}W(t,0),$$

and hence we arrive at the desired decay estimate (2.4). □

2.2. **Characteristic values.** In this subsection, we consider the characteristic equation for the system (2.6) and demonstrate the optimality of of the pointwise estimate (2.3) in Theorem 2.2. The characteristic equation is given by

$$P_3(\xi):=\lambda^3+|\xi|^2\lambda^2+2|\xi|^4\lambda+|\xi|^6=0, \tag{2.13}$$

where a zero $\lambda=\lambda(|\xi|)$ of the characteristic polynomial P_3 is called an eigenvalue, and $\xi\in\mathbb{R}^n$ being the Fourier variable. Here, for the Cauchy problem, it is obtained from the differential equations in Fourier space (2.6), regarding ξ as a parameter and looking for the characteristic equation for the remaining system of ordinary differential equations in time.

The equation (2.13) is, for $|\xi|\ne 0$ (w.l.o.g), equivalent to

$$\Big(\frac{\lambda}{|\xi|^2}\Big)^3+\Big(\frac{\lambda}{|\xi|^2}\Big)^2+2\Big(\frac{\lambda}{|\xi|^2}\Big)+1=0. \tag{2.14}$$

Now, by a simple calculation, we find the solutions of $a^3+a^2+2a+1=0$ as

$$a_1=-\frac{1}{3}(1+\alpha),\quad a_2=-\frac{1}{3}\Big(1-\frac{1}{2}\alpha+\frac{\sqrt{3}}{2}\beta i\Big),\quad a_3=-\frac{1}{3}\Big(1-\frac{1}{2}\alpha-\frac{\sqrt{3}}{2}\beta i\Big) \tag{2.15}$$

with

$$\alpha=\sqrt[3]{\frac{1}{2}(3\sqrt{69}+11)}-\sqrt[3]{\frac{1}{2}(3\sqrt{69}-11)},\quad \beta=\sqrt[3]{\frac{1}{2}(3\sqrt{69}+11)}+\sqrt[3]{\frac{1}{2}(3\sqrt{69}-11)}.$$

We note that α satisfies $7/10<\alpha<8/10$, i.e. $\mathrm{Re}(a_j)<0$. Consequently this gives us the solutions of (2.13) as

$$\lambda_j(|\xi|)=a_j|\xi|^2, \tag{2.16}$$

for $j=1,2,3$. In view of the form of ρ in Theorem 2.2 this proves the claimed optimality.

3. The thermoelastic plate equation with Cattaneo's law

In this section we consider the Cauchy problem (1.1) with $\tau > 0$ and without inertial term, i.e. for $\mu = 0$, with initial data

$$u(0,x) = u_0(x),\ \ u_t(0,x) = u_1(x),\ \ \theta(0,x) = \theta_0(x),\ \ q(0,x) = q_0(x), \quad x \in \mathbb{R}^n. \tag{3.1}$$

The well-posedness is easy, cp. [19], and hence the existence of solutions will be assumed now asking for the asymptotic behavior in time. In Subsection 3.1, we prove pointwise estimates and the L^2-decay, in Subsection 3.2, the associated characteristic polynomial will be investigated to prove the optimality of these results, and in Subsection 3.3 the (singular) limit $\tau \to 0$ will be studied.

3.1. **Decay estimates.** For the problem (1.1)-(3.1), we can derive the following energy estimates.

Proposition 3.1. *Let $s \geq 2$ be an integer and suppose that the initial data $(u_1, \Delta u_0, \theta_0, q_0)$ belong to H^s. Then the solution (u, θ, q) satisfies the energy estimate:*

$$\begin{aligned} &\|\partial_x^k(u_t, \Delta u, \theta, \sqrt{\tau} q)(t)\|_{H^2_\tau}^2 + \frac{1}{188}\int_0^t \big(\|\partial_x^{k+1}(u_t, \Delta u, \theta)(\sigma)\|_{L^2}^2 + \|\partial_x^k q(\sigma)\|_{H^2_\tau}^2\big) d\sigma \\ &\qquad \leq \frac{106}{47}\|\partial_x^k(u_1, \Delta u_0, \theta_0, \sqrt{\tau} q_0)\|_{H^2_\tau}^2, \end{aligned} \tag{3.2}$$

for $0 \leq k \leq s-2$.

Above we used the following notation for the modified Sobolev norm $\|\cdot\|_{H^s_a}$: Let $s \geq 0$ be an integer and $a \geq 0$ be a real number, then

$$\|u\|_{H^s_a} := \Big(\sum_{k=0}^s a^k \|\partial_x^k u\|_{L^2}^2\Big)^{1/2}.$$

We observe that $\|\cdot\|_{H^s_0} = \|\cdot\|_{L^2}$.

Moreover, assuming that the initial data belong to L^1, we can get the following pointwise and decay estimates.

Theorem 3.2. *The Fourier image $(\hat{u}, \hat{\theta}, \hat{q})$ of the solution (u, θ, q) to the Cauchy problem (1.1), (3.1) satisfies the pointwise estimate:*

$$\begin{aligned} &|\hat{u}_t(t,\xi)| + |\xi|^2|\hat{u}(t,\xi)| + |\hat{\theta}(t,\xi)| + \tau|\hat{q}(t,\xi)| \\ &\qquad \leq Ce^{-c\eta(\xi)t}(|\hat{u}_1(\xi)| + |\xi|^2|\hat{u}_0(\xi)| + |\hat{\theta}_0(\xi)| + \tau|\hat{q}_0(\xi)|), \end{aligned} \tag{3.3}$$

where

$$\eta(\xi) := |\xi|^2/(1+\tau|\xi|^2)^2.$$

Here C and c are positive constants which are in particular independent of τ. Furthermore, let $s \geq 0$ be an integer and suppose that the initial data $(u_1, \Delta u_0, \theta_0, q_0)$ belong to $H^s \cap L^1$. Then the solution (u, θ, q) satisfies the decay estimate:

$$\begin{aligned} \|\partial_x^k(u_t, \Delta u, \theta, \tau q)(t)\|_{L^2} \leq& C(1+t)^{-n/4-k/2}\|(u_1, \Delta u_0, \theta_0, \tau q_0)\|_{L^1} \\ &+ C(1+t)^{-\ell/2}\|\partial_x^{k+\ell}(u_1, \Delta u_0, \theta_0, \tau q_0)\|_{L^2} \end{aligned} \tag{3.4}$$

for $k \geq 0$, $\ell \geq 0$ and $0 \leq k+\ell \leq s$. Here C is a positive constant which is, in particular, independent of τ.

Remark 2. *The decay estimate* (3.4) *is the same as the regularity-loss type decay estimate for some symmetric hyperbolic systems with relaxation (cf.* [26]*). The "loss of regularity" is visible in the last term of* (3.4) *requiring* $k+\ell$ *derivatives of the data to obtain a decay for* k *derivatives of the solution at time* t*.*

This way, we have a precise description of the dependence on τ visible in the form of η. Comparing this to the corresponding one for $\tau = 0$ given for ρ in Theorem 2.2, we have an essentially different behavior for $\xi \to \infty$ causing the loss of regularity in the case $\tau > 0$.

Proof of Proposition 3.1 and of Theorem 3.2. We prove Proposition 3.1 and Theorem 3.2 together. By employing the Fourier transform, system (1.1) is described as

$$\begin{aligned} \hat{u}_{tt} + |\xi|^4\hat{u} - |\xi|^2\hat{\theta} &= 0, \\ \hat{\theta}_t + i\xi\cdot\hat{q} + |\xi|^2\hat{u}_t &= 0, \\ \tau\hat{q}_t + \hat{q} + i\xi\hat{\theta} &= 0. \end{aligned} \tag{3.5}$$

We first derive the basic energy equality for the system (3.5) in the Fourier space. We multiply the equations (3.5) by $\bar{\hat{u}}_t, \bar{\hat{\theta}}, \bar{\hat{q}}$, respectively, and combine the resulting equations. Moreover, with for $W(t,\xi) := (|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 + |\hat{\theta}|^2 + \tau|\hat{q}|^2)(t,\xi)$, we arrive at the basic energy equality

$$\frac{1}{2}\frac{\partial}{\partial t}W(t,\xi) + |\hat{q}|^2 = 0. \tag{3.6}$$

We next construct further dissipation terms. Multiplying the first equation in (3.5) by $\bar{\hat{u}}$ and taking real parts, we have

$$\frac{\partial}{\partial t}\big(\mathrm{Re}(\hat{u}_t\bar{\hat{u}})\big) - |\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 - |\xi|^2\mathrm{Re}(\hat{\theta}\bar{\hat{u}}) = 0. \tag{3.7}$$

Furthermore we multiply the first and second equations in (3.5) by $\bar{\hat{\theta}}$ and $\bar{\hat{u}}_t$, respectively. Then, combining the resulting equations and taking real parts, we obtain

$$\frac{\partial}{\partial t}\big(\mathrm{Re}(\hat{u}_t\bar{\hat{\theta}})\big) + |\xi|^2(|\hat{u}_t|^2 - |\hat{\theta}|^2) + \xi\cdot\mathrm{Re}(i\hat{q}\bar{\hat{u}}_t) + |\xi|^4\mathrm{Re}(\hat{u}\bar{\hat{\theta}}) = 0. \tag{3.8}$$

Similarly we multiply the second equation in (3.5) by $\tau i\xi\cdot\bar{\hat{q}}$ and take the inner product of the third equation in (3.5) with $-i\xi\bar{\hat{\theta}}$. Then we obtain

$$\frac{\partial}{\partial t}\big(\tau\xi\cdot\mathrm{Re}(i\hat{\theta}\bar{\hat{q}})\big) + |\xi|^2|\hat{\theta}|^2 - \tau|\xi\cdot\hat{q}|^2 - \xi\cdot\mathrm{Re}(i\hat{q}\bar{\hat{\theta}}) + \tau|\xi|^2\xi\cdot\mathrm{Re}(i\hat{u}_t\bar{\hat{q}}) = 0. \tag{3.9}$$

Finally we multiply (4.20), (4.21), (4.22) and (4.23) by $1 + \tau|\xi|^2 + \tau^2|\xi|^4$, $\alpha_1\alpha_2\alpha_3|\xi|^2$, $\alpha_1\alpha_2$, and α_1, respectively, and add these four equations, where α_1, α_2 α_3 are positive constants to be determined. This yields

$$\frac{\partial}{\partial t}E(t,\xi) + D(t,\xi) = 0, \tag{3.10}$$

where we define

$$E(t,\xi) := \frac{1}{2}(1+\tau|\xi|^2+\tau^2|\xi|^4)(|\hat{u}_t|^2+|\xi|^4|\hat{u}|^2+|\hat{\theta}|^2+\tau|\hat{q}|^2) \\ +\alpha_1\tau\xi\cdot\mathrm{Re}(i\hat{\theta}\bar{\hat{q}})+\alpha_1\alpha_2\mathrm{Re}(\hat{u}_t\bar{\hat{\theta}})+\alpha_1\alpha_2\alpha_3|\xi|^2\mathrm{Re}(\hat{u}_t\bar{\hat{u}}),$$

$$D(t,\xi) := \alpha_1\alpha_2\alpha_3|\xi|^6|\hat{u}|^2+\alpha_1\alpha_2(1-\alpha_3)|\xi|^2|\hat{u}_t|^2+\alpha_1(1-\alpha_2)|\xi|^2|\hat{\theta}|^2 \\ +\tau|\xi|^2|\hat{q}|^2-\alpha_1\tau|\xi\cdot\hat{q}|^2+(1+\tau^2|\xi|^4)|\hat{q}|^2 \\ +\alpha_1\alpha_2(1-\alpha_3)|\xi|^4\mathrm{Re}(\hat{u}\bar{\hat{\theta}})+\alpha_1\xi\cdot\mathrm{Re}(i\hat{\theta}\bar{\hat{q}})+\alpha_1(\tau|\xi|^2-\alpha_2)\xi\cdot\mathrm{Re}(i\hat{u}_t\bar{\hat{q}}).$$

Applying $|\xi\cdot\hat{q}|\le|\xi||\hat{q}|$ and

$$|\xi|^2|\hat{u}\bar{\hat{\theta}}|\le\varepsilon_1|\xi|^4|\hat{u}|^2+(4\varepsilon_1)^{-1}|\hat{\theta}|^2, \qquad |\xi||\hat{\theta}\bar{\hat{q}}|\le\varepsilon_2|\xi|^2|\hat{\theta}|^2+(4\varepsilon_2)^{-1}|\hat{q}|^2,$$
$$|\xi||\hat{u}_t\bar{\hat{q}}|\le\varepsilon_3|\xi|^2|\hat{u}_t|^2+(4\varepsilon_3)^{-1}|\hat{q}|^2, \qquad \tau|\xi||\hat{u}_t\bar{\hat{q}}|\le\varepsilon_4|\hat{u}_t|^2+(4\varepsilon_4)^{-1}\tau^2|\xi|^2|\hat{q}|^2$$

to the dissipation term D, we may estimate

$$D\ge\alpha_1\alpha_2\big\{\alpha_3-\varepsilon_1(1-\alpha_3)\big\}|\xi|^6|\hat{u}|^2+\alpha_1\big\{\alpha_2(1-\alpha_3-\varepsilon_3)-\varepsilon_4\big\}|\xi|^2|\hat{u}_t|^2 \\ +\alpha_1\big\{1-\varepsilon_2-\alpha_2\big(1+\frac{1-\alpha_3}{4\varepsilon_1}\big)\big\}|\xi|^2|\hat{\theta}|^2+(1-\alpha_1)\tau|\xi|^2|\hat{q}|^2 \\ +\big\{1-\frac{\alpha_1}{4}\big(\frac{1}{\varepsilon_2}+\frac{\alpha_2}{\varepsilon_3}\big)\big\}|\hat{q}|^2+\big(1-\frac{\alpha_1}{4\varepsilon_4}\big)\tau^2|\xi|^4|\hat{q}|^2.$$

Choosing $\alpha_1=1/20$, $\alpha_2=1/5$, $\alpha_3=1/2$, $\varepsilon_1=1/2$, $\varepsilon_2=1/2$, $\varepsilon_3=1/4$ and $\varepsilon_4=1/40$, then the above estimate can be rewritten as

$$D\ge\frac{1}{800}|\xi|^2|\hat{u}_t|^2+\frac{1}{400}|\xi|^6|\hat{u}|^2+\frac{1}{80}|\xi|^2|\hat{\theta}|^2 \\ +\frac{193}{200}|\hat{q}|^2+\frac{19}{20}\tau|\xi|^2|\hat{q}|^2+\frac{1}{2}\tau^2|\xi|^4|\hat{q}|^2 \\ \ge\frac{1}{800}|\xi|^2\big(|\hat{u}_t|^2+|\xi|^4|\hat{u}|^2+|\hat{\theta}|^2\big)+\frac{19}{40}(1+\tau|\xi|^2)^2|\hat{q}|^2. \tag{3.11}$$

On the other hand, substituting $\alpha_1=1/20$, $\alpha_2=1/5$, $\alpha_3=1/2$ again, the energy term E is estimated as

$$E\le\frac{203}{400}|\hat{u}_t|^2+\frac{201}{400}|\xi|^4|\hat{u}|^2+\frac{53}{100}|\hat{\theta}|^2+\frac{1}{2}\tau|\hat{q}|^2 \\ +\frac{1}{2}\tau|\xi|^2(1+\tau|\xi|^2)(|\hat{u}_t|^2+|\xi|^4|\hat{u}|^2+|\hat{\theta}|^2)+\frac{21}{40}\tau^2|\xi|^2|\hat{q}|^2+\frac{1}{2}\tau^3|\xi|^4|\hat{q}|^2 \\ \le\frac{53}{100}(1+\tau|\xi|^2)^2W(t,\xi),$$
$$E\ge\frac{197}{400}|\hat{u}_t|^2+\frac{199}{400}|\xi|^4|\hat{u}|^2+\frac{47}{100}|\hat{\theta}|^2+\frac{1}{2}\tau|\hat{q}|^2 \\ +\frac{1}{2}\tau|\xi|^2(1+\tau|\xi|^2)(|\hat{u}_t|^2+|\xi|^4|\hat{u}|^2+|\hat{\theta}|^2)+\frac{19}{40}\tau^2|\xi|^4|\hat{q}|^2+\frac{1}{2}\tau^2|\xi|^4|\hat{q}|^2 \\ \ge\frac{47}{200}(1+\tau|\xi|^2)^2W(t,\xi). \tag{3.12}$$

By applying the estimates (3.11) and (3.12) to (3.10), we get

$$(1+\tau|\xi|^2)^2 W(t,\xi) + \frac{1}{188}\int_0^t \big\{|\xi|^2\big(|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 + |\hat{\theta}|^2\big) + (1+\tau|\xi|^2)^2|\hat{q}|^2\big\}\, d\sigma$$
$$\leq \frac{106}{47}(1+\tau|\xi|^2)^2 W(0,\xi),$$

and hence

$$\|\partial_x^k(u_t, \partial_x^2 u, \theta, \tau q)(t)\|_{H^2}^2 + \frac{1}{188}\int_0^t \big(\|\partial_x^{k+1}(u_t, \partial_x^2 u, \theta)(\sigma)\|_{L^2}^2 + \|\partial_x^k q(\sigma)\|_{H^2}^2\big) d\sigma$$
$$\leq \frac{106}{47}\|\partial_x^k(u_1, \partial_x^2 u_0, \theta_0, \tau q_0)\|_{H^2}^2 \tag{3.13}$$

for $0 \leq k \leq s-2$. The estimate (3.13) is equivalent to (3.2).

Furthermore we apply the estimates (3.11) and (3.12) to (3.10) again. This yields

$$\frac{\partial}{\partial t}E(t,\xi) + \frac{1}{424}\frac{|\xi|^2}{(1+\tau|\xi|^2)^2}E(t,\xi) \leq 0.$$

Therefore we can derive the following pointwise estimate

$$W(t,\xi) \leq \frac{106}{47}e^{-\frac{1}{424}\frac{|\xi|^2}{(1+\tau|\xi|^2)^2}t}W(0,\xi),$$

and hence we arrive at the desired decay estimate (3.4). □

3.2. **Characteristic values.** We first consider the characteristic equation for the system (4.19) given by

$$P_4(\xi) := \lambda^4 + \frac{1}{\tau}\lambda^3 + \frac{1}{\tau}(2\tau|\xi|^2+1)|\xi|^2\lambda^2 + \frac{2}{\tau}|\xi|^4\lambda + \frac{1}{\tau}|\xi|^6 = 0. \tag{3.14}$$

This polynomial is also obtained for the bounded domain case, see [19] or [22], where the zeros correspond to eigenvalues of the associated time-independent operator. Here, for the Cauchy problem, it is obtained from the differential equations in Fourier space (4.19), regarding $|\xi|$ as a parameter.

We consider the asymptotic expansion of $\lambda = \lambda(|\xi|)$ for $|\xi| \to 0$ and for $|\xi| \to \infty$. These expansions essentially determine the asymptotic behavior of solutions described in Subsection 3.1.

We first consider the asymptotic expansion for $|\xi| \to 0$:

$$\lambda(|\xi|) = \sum_{k=0}^{\infty} \lambda^{(k)}|\xi|^k. \tag{3.15}$$

Substituting (3.15) into (3.14), we find, after lengthy but straightforward calculations, that

$$\lambda_j(|\xi|) = a_j|\xi|^2 + O(|\xi|^4), \qquad \lambda_4(|\xi|) = -\frac{1}{\tau} + |\xi|^2 + O(|\xi|^4), \tag{3.16}$$

for $j = 1,2,3$, where the a_j are defined by (2.15). This way we also see how the zeros for the polynomial (3.14) approximate, for $j = 1,2,3$, those given in (2.16) for the polynomial (2.13).

On the other hand, we consider the following asymptotic expansion for $|\xi| \to \infty$:

$$\lambda(|\xi|) = \nu^{(2)}|\xi|^2 + \nu^{(1)}|\xi| + \sum_{k=0}^{\infty} \nu^{(-k)}|\xi|^{-k}. \tag{3.17}$$

Then, substituting (3.17) into (3.14), we obtain

$$\begin{aligned} \lambda_j(|\xi|) &= \pm\sqrt{2}i|\xi|^2 \pm \frac{\sqrt{2}i}{8\tau} - \frac{1}{8\tau^2}\Big(1 \mp \frac{3\sqrt{2}i}{16}\Big)|\xi|^{-2} + O(|\xi|^{-4}), \quad j = 1, 2, \\ \lambda_j(|\xi|) &= \pm\sqrt{\frac{1}{2\tau}}i|\xi| - \frac{1}{2\tau} \pm \frac{\sqrt{2\tau}}{4\tau^2}i|\xi|^{-1} + \frac{1}{8\tau^2}|\xi|^{-2} + O(|\xi|^{-3}), \quad j = 3, 4. \end{aligned} \tag{3.18}$$

For $j = 1, 2$, the expansion (3.18) leads to

$$\mathrm{Re}(\lambda_j)(|\xi|) = -\frac{1}{8\tau^2}|\xi|^{-2} + O(|\xi|^{-4}).$$

In these λ_1 and λ_2 the regularity-loss structure for $\tau > 0$ is expressed.

Consequently, the asymptotic expansions (3.16) and (3.18) tell us that the pointwise estimate (3.3) is optimal.

3.3. **The singular limit as $\tau \to 0$: a comparison.** In this subsection, we study, still for $\mu = 0$, the limit as $\tau \to 0$ for the plate equation with Cattaneo's law (1.1). More precisely we derive that the energy of the difference of the solution (u, θ, q) to (1.1), (3.1) and the solution $(\tilde{u}, \tilde{\theta})$ to (1.2), (2.1) vanishes as $\tau \to 0$, provided the initial data are compatible.

For this purpose we introduce the difference (v, ϕ, r) as $v := u - \tilde{u}$, $\phi := \theta - \tilde{\theta}$ and $r := q - \tilde{q}$, where $\tilde{q} := -\nabla\tilde{\theta}$. Then, by using (1.1), (3.1) and (1.2), (2.1), we have the Cauchy problem

$$\begin{aligned} v_{tt} + \Delta^2 v + \Delta\phi &= 0, \\ \phi_t + \operatorname{div} r - \Delta v_t &= 0, \\ \tau r_t + r + \nabla\phi &= \tau\nabla\tilde{\theta}, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} &v(0, x) = v_0(x) := 0, \qquad v_t(0, x) = v_1(x) := 0, \\ &\phi(0, x) = \phi_0(x) := 0, \qquad r(0, x) = r_0(x) := q_0(x) + \nabla\theta_0(x), \qquad x \in \mathbb{R}^n. \end{aligned} \tag{3.20}$$

Now assuming the compatibility condition

$$q_0(x) = -\nabla\theta_0(x), \qquad x \in \mathbb{R}^n, \tag{3.21}$$

the initial data (3.20) satisfy $r_0(x) = 0$ for $x \in \mathbb{R}^n$. For the problem (1.1), (3.1), we can derive the following energy estimate.

Theorem 3.3. *Assume the compatibility condition* (3.21). *Let $s \geq 0$ be an integer and suppose that the initial data $(u_1, \Delta u_0, \theta_0)$ belong to H^s. Then the difference (v, ϕ, r) satisfies the following estimate:*

$$\|\partial_x^k(v_t, \Delta v, \phi, \sqrt{\tau}r)(t)\|_{L^2}^2 + \int_0^t \|\partial_x^k r(\sigma)\|_{L^2}^2 d\sigma \leq \tau^2 C\|\partial_x^k(u_1, \Delta u_0, \theta_0)\|_{L^2}^2 \tag{3.22}$$

for $0 \leq k \leq s$. Here C is a positive constant which is independent of τ.

Proof of Theorem 3.3. To get the desired result, we derive the basic energy equality for the system (3.19). We multiply the first and second equations of (3.19) by v_t and ϕ, respectively, and take the inner product of the third equation with r. Then combining the resulting equations, we arrive at the basic energy equality

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}(|v_t|^2+|\Delta v|^2+|\phi|^2+\tau|r|^2)+|r|^2 \\ +\operatorname{div}\big[v_t\nabla(\Delta v+\phi)-(\Delta v+\phi)\nabla v_t+\phi r\big]=\tau\nabla\tilde{\theta}\cdot r.\end{aligned}$$

Similarly we obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}(|\partial_x^k v_t|^2+|\Delta\partial_x^k v|^2+|\partial_x^k\phi|^2+\tau|\partial_x^k r|^2)+|\partial_x^k r|^2 \\ +\operatorname{div}\big[\partial_x^k v_t\nabla(\Delta\partial_x^k v+\partial_x^k\phi)-(\Delta\partial_x^k v+\partial_x^k\phi)\nabla\partial_x^k v_t+\partial_x^k\phi\partial_x^k r\big]=\tau\nabla\partial_x^k\tilde{\theta}\cdot\partial_x^k r\end{aligned}$$

for $k\ge 0$. Therefore, integrating the above equation over $(0,t)\times\mathbb{R}^n$, this yields

$$\begin{aligned}\|\partial_x^k(v_t,\Delta v,\phi,\sqrt{\tau}r)(t)\|_{L^2}^2+\int_0^t\|\partial_x^k r(\sigma)\|_{L^2}^2d\sigma \\ \le\|\partial_x^k(v_1,\Delta v_0,\phi_0,\sqrt{\tau}r_0)\|_{L^2}^2+\tau^2\int_0^t\|\nabla\partial_x^k\tilde{\theta}(\sigma)\|_{L^2}^2d\sigma.\end{aligned}$$

Finally, using (2.2), (3.20) and (3.21), we arrive at the desired estimate (3.22) and hence complete the proof. □

4. The thermoelastic plate equation with inertial term

We finally consider the general system with inertial term ($\mu>0$), and we will show that there is no regularity-loss, both for Cattaneo's law ($\tau>0$) and for Fourier's law ($\tau=0$), which corresponds again to the fact of exponential stability for bounded domains. We recall the equations from (1.1) as

$$\begin{aligned}u_{tt}+\Delta^2u-\mu\Delta u_{tt}+\Delta\theta=0,\\ \theta_t+\operatorname{div}q-\Delta u_t=0,\\ \tau q_t+q+\nabla\theta=0,\end{aligned}\tag{4.1}$$

with now μ being positive, and with initial data

$$u(0,x)=u_0(x),\ \ u_t(0,x)=u_1(x),\ \ \theta(0,x)=\theta_0(x),\ \ q(0,x)=q_0(x),\quad x\in\mathbb{R}^n,\tag{4.2}$$

where the last initial condition for q is only relevant for the case $\tau>0$.

4.1. **Decay estimates.** For the problem (4.1)-(4.2) with $\tau>0$, we can derive the following decay estimates.

Theorem 4.1. *The Fourier image $(\hat{u},\hat{\theta},\hat{q})$ of the solution (u,θ,q) to the Cauchy problem* (4.1), (4.2) *satisfies the pointwise estimate*

$$\begin{aligned}(1+\mu|\xi|^2)|\hat{u}_t(t,\xi)|+|\xi|^2|\hat{u}(t,\xi)|+|\hat{\theta}(t,\xi)|+\tau|\hat{q}(t,\xi)| \\ \le Ce^{-c\rho(\xi)t}\big\{(1+\mu|\xi|^2)|\hat{u}_1(\xi)|+|\xi|^2|\hat{u}_0(\xi)|+|\hat{\theta}_0(\xi)|+\tau|\hat{q}_0(\xi)|\big\},\end{aligned}\tag{4.3}$$

where

$$\rho(\xi) := \begin{cases} \dfrac{|\xi|^2(1+\tau\mu|\xi|^2)}{(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)}, & \mu \le 1, \\ \dfrac{|\xi|^2}{1+\mu|\xi|^2}, & \mu \ge 1. \end{cases}$$

Here C and c are positive constants which are independent of τ and μ. They are explicitly given by

$$(C, c) := \begin{cases} \left(\dfrac{13}{11}, \dfrac{1}{2730}\right) & \mu \le 1, \\ \left(\dfrac{43}{29}, \dfrac{1}{344}\right), & \mu \ge 1. \end{cases} \tag{4.4}$$

Furthermore, let $s \ge 0$ be an integer and suppose that the initial data $(u_1, \Delta u_0, \theta_0, q_0)$ belong to $H^s \cap L^1$. Then the solution (u, θ, q) satisfies the decay estimate:

$$\begin{aligned} &\|\partial_x^k(u_t, \mu\Delta u_t, \Delta u, \theta, \tau q)(t)\|_{L^2} \\ &\qquad \le C(1+t)^{-n/4-k/2}\|(u_1, \mu\Delta u_1, \Delta u_0, \theta_0, \tau q_0)\|_{L^1} \\ &\qquad\quad + Ce^{-c\tau\mu t}\|\partial_x^k(u_1, \mu\Delta u_1, \Delta u_0, \theta_0, \tau q_0)\|_{L^2}, \qquad \mu \le 1, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\|\partial_x^k(u_t, \mu\Delta u_t, \Delta u, \theta, \tau q)(t)\|_{L^2} \\ &\qquad \le C(1+t/\mu)^{-n/4-k/2}\|(u_1, \mu\Delta u_1, \Delta u_0, \theta_0, \tau q_0)\|_{L^1} \\ &\qquad\quad + Ce^{-ct/\mu}\|\partial_x^k(u_1, \mu\Delta u_1, \Delta u_0, \theta_0, \tau q_0)\|_{L^2}, \qquad \mu \ge 1 \end{aligned} \tag{4.6}$$

for $0 \le k \le s$. Here C and c are also positive constants which are independent of τ and μ.

On the other hand, taking $\tau = 0$ in (4.1) yields

$$\begin{aligned} u_{tt} - \mu\Delta u_{tt} + \Delta^2 u + \Delta\theta = 0, \\ \theta_t - \Delta\theta - \Delta u_t = 0, \end{aligned} \tag{4.7}$$

For the Cauchy problem (4.7) with initial data

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x), \tag{4.8}$$

we can derive the following theorem.

Theorem 4.2. *The Fourier image $(\hat{u}, \hat{\theta})$ of the solution (u, θ) to the Cauchy problem (4.7), (4.8) satisfies the pointwise estimate:*

$$\begin{aligned} &(1+\mu|\xi|^2)|\hat{u}_t(t,\xi)|^2 + |\xi|^4|\hat{u}(t,\xi)|^2 + |\hat{\theta}(t,\xi)|^2 \\ &\qquad \le 5e^{-\frac{1}{10}\rho(\xi)t}\big\{(1+\mu|\xi|^2)|\hat{u}_1(\xi)|^2 + |\xi|^4|\hat{u}_0(\xi)|^2 + |\hat{\theta}_0(\xi)|^2\big\}, \end{aligned} \tag{4.9}$$

where

$$\rho(\xi) := \frac{|\xi|^2}{1+\mu|\xi|^2}.$$

Furthermore, let $s \geq 0$ be an integer and suppose that the initial data $(u_1, \Delta u_0, \theta_0, q_0)$ belong to $H^s \cap L^1$. Then the solution (u, θ, q) satisfies the decay estimate:

$$\begin{aligned} &\|\partial_x^k(u_t, \mu\Delta u_t, \Delta u, \theta)(t)\|_{L^2} \\ &\quad \leq C\big(1 + t/(1+\mu)\big)^{-n/4-k/2}\|(u_1, \mu\Delta u_1, \Delta u_0, \theta_0)\|_{L^1} \\ &\qquad + Ce^{-ct/(1+\mu)}\|\partial_x^k(u_1, \mu\Delta u_1, \Delta u_0, \theta_0)\|_{L^2}, \end{aligned} \tag{4.10}$$

for $0 \leq k \leq s$. Here C and c are positive constants which are independent of μ.

Remark 3. *Both decay estimates (4.5), (4.6) and (4.10) are of standard type for the symmetric hyperbolic system with relaxation. There is no regularity-loss.*

For the pointwise estimates (4.3) and (4.9), letting $\tau \to 0$ or $\mu \to 0$, these estimates tend to the expected estimates discussed in Sections 2 and 3.

Proof of Theorem 4.2. We first prove Theorem 4.2. By employing the Fourier transform, the system (4.7) is described as

$$\begin{aligned} (1 + \mu|\xi|^2)\hat{u}_{tt} + |\xi|^4\hat{u} - |\xi|^2\hat{\theta} = 0, \\ \hat{\theta}_t + |\xi|^2\hat{\theta} + |\xi|^2\hat{u}_t = 0, \end{aligned} \tag{4.11}$$

We first derive the basic energy equality for the system (4.11) in the Fourier space. We multiply (4.11) by $(\bar{\hat{u}}_t, \bar{\hat{\theta}})^T$ and combining the resulting equations, we arrive at the basic energy equality for $W(t,\xi) := \big\{(1 + \mu|\xi|^2)|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 + |\hat{\theta}|^2\big\}(t,\xi)$,

$$\frac{1}{2}\frac{\partial}{\partial t}W(t,\xi) + |\xi|^2|\hat{\theta}|^2 = 0. \tag{4.12}$$

We next construct the dissipation terms. Multiplying the first equation in (4.11) by $\bar{\hat{u}}$, we have

$$(1 + \mu|\xi|^2)\frac{\partial}{\partial t}\big\{\mathrm{Re}(\hat{u}_t\bar{\hat{u}})\big\} - (1 + \mu|\xi|^2)|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 - |\xi|^2\mathrm{Re}(\hat{\theta}\bar{\hat{u}}) = 0. \tag{4.13}$$

Furthermore we multiply the first and second equations in (4.19) by $\bar{\hat{\theta}}$ and $(1+\mu|\xi|^2)\bar{\hat{u}}_t$, respectively. We obtain

$$\begin{aligned} &(1 + \mu|\xi|^2)\frac{\partial}{\partial t}\big(\mathrm{Re}(\hat{u}_t\bar{\hat{\theta}})\big) + |\xi|^2\big\{(1 + \mu|\xi|^2)|\hat{u}_t|^2 - |\hat{\theta}|^2\big\} \\ &\quad + (1 + \mu|\xi|^2)|\xi|^2\mathrm{Re}(\hat{u}_t\bar{\hat{\theta}}) + |\xi|^4\mathrm{Re}(\hat{u}\bar{\hat{\theta}}) = 0. \end{aligned} \tag{4.14}$$

Finally, we multiply (2.7), (4.13) and (4.14) by $1 + \mu|\xi|^2$, $\alpha_1\alpha_2|\xi|^2$, and α_1, respectively, and add these four equations, where α_1, α_2 are positive constants yet to be determined. This yields

$$\frac{\partial}{\partial t}E(t,\xi) + D(t,\xi) = 0, \tag{4.15}$$

where

$$E(t,\xi) := \frac{1}{2}(1+\mu|\xi|^2)\big\{(1+\mu|\xi|^2)|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 + |\hat{\theta}|^2\big\}$$
$$\quad + \alpha_1(1+\mu|\xi|^2)\big\{\mathrm{Re}(\hat{u}_t\bar{\hat{\theta}}) + \alpha_2|\xi|^2\mathrm{Re}(\hat{u}_t\bar{\hat{u}})\big\},$$
$$D(t,\xi) := \alpha_1(1-\alpha_2)(1+\mu|\xi|^2)|\xi|^2|\hat{u}_t|^2 + \alpha_1\alpha_2|\xi|^6|\hat{u}|^2 + (1-\alpha_1)|\xi|^2|\hat{\theta}|^2 + \mu|\xi|^4|\hat{\theta}|^2$$
$$\quad + \alpha_1(1-\alpha_2)|\xi|^4\mathrm{Re}(\hat{u}\bar{\hat{\theta}}) + \alpha_1(1+\mu|\xi|^2)|\xi|^2\mathrm{Re}(\hat{u}_t\bar{\hat{\theta}}).$$

Applying

$$\text{(4.16)}\qquad \begin{aligned}&|\xi|^2|\hat{u}\bar{\hat{\theta}}| \le \varepsilon_1|\xi|^4|\hat{u}|^2 + (4\varepsilon_1)^{-1}|\hat{\theta}|^2,\\ &|\hat{u}_t\bar{\hat{\theta}}| \le \varepsilon_2|\hat{u}_t|^2 + (4\varepsilon_2)^{-1}|\hat{\theta}|^2, \qquad |\xi|^2|\hat{u}_t\bar{\hat{u}}| \le \frac{1}{2}|\hat{u}_t|^2 + \frac{1}{2}|\xi|^4|\hat{u}|^2\end{aligned}$$

to the dissipation term D, we may estimate

$$D \ge \alpha_1(1-\alpha_2-\varepsilon_2)(1+\mu|\xi|^2)|\xi|^2|\hat{u}_t|^2 + \alpha_1\big\{\alpha_2 - \varepsilon_1(1-\alpha_2)\big\}|\xi|^6|\hat{u}|^2$$
$$\quad + \Big\{1-\alpha_1\Big(1+\frac{1-\alpha_2}{4\varepsilon_1}+\frac{1}{4\varepsilon_2}\Big)\Big\}|\xi|^2|\hat{\theta}|^2 + \mu\big(1-\frac{\alpha_1}{4\varepsilon_2}\big)|\xi|^4|\hat{\theta}|^2.$$

Therefore, choosing $\alpha_1 = 1/3$, $\alpha_2 = 1/2$, $\varepsilon_1 = 1/2$, and $\varepsilon_2 = 1/4$, we derive

$$\text{(4.17)}\qquad \begin{aligned}D &\ge \frac{1}{12}(1+\mu|\xi|^2)|\xi|^2|\hat{u}_t|^2 + \frac{1}{12}|\xi|^6|\hat{u}|^2 + \frac{1}{4}|\xi|^2|\hat{\theta}|^2 + \frac{2}{3}\mu|\xi|^4|\hat{\theta}|^2\\ &\ge \frac{1}{12}|\xi|^2\big\{(1+\mu|\xi|^2)|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2\big\} + \frac{1}{4}(1+\mu|\xi|^2)|\xi|^2|\hat{\theta}|^2\end{aligned}$$

and

$$\text{(4.18)}\qquad \begin{aligned}E &\le (1+\mu|\xi|^2)\big\{\frac{2}{3}|\hat{u}_t|^2 + \frac{1}{2}\mu|\xi|^2|\hat{u}_t|^2 + \frac{7}{12}|\xi|^4|\hat{u}|^2 + \frac{5}{6}|\hat{\theta}|^2\big\}\\ &\le \frac{5}{6}(1+\mu|\xi|^2)W(t,\xi),\\ E &\ge (1+\mu|\xi|^2)\big\{\frac{1}{3}|\hat{u}_t|^2 + \frac{1}{2}\mu|\xi|^2|\hat{u}_t|^2 + \frac{5}{12}|\xi|^4|\hat{u}|^2 + \frac{1}{6}|\hat{\theta}|^2\big\}\\ &\ge \frac{1}{6}(1+\mu|\xi|^2)W(t,\xi).\end{aligned}$$

Applying the estimates (4.17) and (4.18) to (4.15) yields

$$\frac{\partial}{\partial t}E(t,\xi) + \frac{1}{10}\frac{|\xi|^2}{1+\mu|\xi|^2}E(t,\xi) \le 0.$$

Thus we can derive the following pointwise estimate

$$W(t,\xi) \le 5e^{-\frac{1}{10}\frac{|\xi|^2}{1+\mu|\xi|^2}t}W(0,\xi),$$

and hence we arrive at the desired pointwise estimate (4.9). □

Remark 4. *As in the previous sections the proof of Theorem 4.2 contains the arguments for the following L^2-estimates for the solutions of* (4.7)–(4.8)*, both for* $0 \le k \le s$.

$$\|\partial_x^k u_t(t)\|_{H^2_\mu}^2 + \|\partial_x^k(\Delta u,\theta)(t)\|_{H^1_\mu}^2 + \frac{1}{2}\int_0^t \big(\|\partial_x^{k+1}(u_t,\theta)(\sigma)\|_{H^1_\mu}^2 + \|\partial_x^{k+1}\Delta u(\sigma)\|_{L^2}^2\big)d\sigma$$
$$\le 5\big(\|\partial_x^k u_1\|_{H^2_\mu}^2 + \|\partial_x^k(\Delta u_0,\theta_0)\|_{H^1_\mu}^2\big).$$

Proof of Theorem 4.1. By employing the Fourier transform, system (1.1) turns into

$$\begin{aligned}(1+\mu|\xi|^2)\hat{u}_{tt} + |\xi|^4\hat{u} - |\xi|^2\hat{\theta} &= 0,\\ \hat{\theta}_t + i\xi\cdot\hat{q} + |\xi|^2\hat{u}_t &= 0,\\ \tau\hat{q}_t + \hat{q} + i\xi\hat{\theta} &= 0.\end{aligned} \tag{4.19}$$

We first derive the basic energy equality for the system (4.19) in the Fourier space. We multiply the equations (4.19) by $\bar{\hat{u}}_t$, $\bar{\hat{\theta}}$, $\bar{\hat{q}}$ and obtain for $W(t,\xi) := \{(1+\mu|\xi|^2)|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 + |\hat{\theta}|^2 + \tau|\hat{q}|^2\}(t,\xi)$, that

$$\frac{1}{2}\frac{\partial}{\partial t}W(t,\xi) + |\hat{q}|^2 = 0. \tag{4.20}$$

Further dissipation terms are obtained as follows. We have (4.13) again, i.e.

$$(1+\mu|\xi|^2)\frac{\partial}{\partial t}\{\mathrm{Re}(\hat{u}_t\bar{\hat{u}})\} - (1+\mu|\xi|^2)|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 - |\xi|^2\mathrm{Re}(\hat{\theta}\bar{\hat{u}}) = 0. \tag{4.21}$$

Furthermore, multiplying the first and second equations in (4.19) by $\bar{\hat{\theta}}$ and $(1+\mu|\xi|^2)\bar{\hat{u}}_t$, respectively. we obtain

$$\begin{aligned}&(1+\mu|\xi|^2)\frac{\partial}{\partial t}\big(\mathrm{Re}(\hat{u}_t\bar{\hat{\theta}})\big) + |\xi|^2\{(1+\mu|\xi|^2)|\hat{u}_t|^2 - |\hat{\theta}|^2\}\\ &\quad + (1+\mu|\xi|^2)\xi\cdot\mathrm{Re}(i\hat{q}\bar{\hat{u}}_t) + |\xi|^4\mathrm{Re}(\hat{u}\bar{\hat{\theta}}) = 0.\end{aligned} \tag{4.22}$$

Similarly we multiply the second equation in (4.19) by $\tau i\xi\cdot\bar{\hat{q}}$ and take the inner product of the third equation in (4.19) with $-i\xi\bar{\hat{\theta}}$. Then we get

$$\frac{\partial}{\partial t}\{\tau\xi\cdot\mathrm{Re}(i\hat{\theta}\bar{\hat{q}})\} + |\xi|^2|\hat{\theta}|^2 - \tau|\xi\cdot\hat{q}|^2 - \xi\cdot\mathrm{Re}(i\hat{q}\bar{\hat{\theta}}) + \tau|\xi|^2\xi\cdot\mathrm{Re}(i\hat{u}_t\bar{\hat{q}}) = 0. \tag{4.23}$$

First case: $\mu \le 1$.

We multiply (4.20), (4.21), (4.22) and (4.23) by $(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)$, $\alpha_1\alpha_2\alpha_3(1+\tau\mu|\xi|^2)|\xi|^2$, $\alpha_1\alpha_2(1+\tau\mu|\xi|^2)$, and $\alpha_1(1+\tau\mu|\xi|^2)$, respectively, where α_1, α_2 α_3 are positive constants to be determined, and add these four equations. This yields

$$\frac{\partial}{\partial t}E_1(t,\xi) + D_1(t,\xi) = 0, \tag{4.24}$$

where

$$E_1(t,\xi) := \frac{1}{2}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)\big\{(1+\mu|\xi|^2)|\hat{u}_t|^2+|\xi|^4|\hat{u}|^2+|\hat{\theta}|^2+\tau|\hat{q}|^2\big\}$$
$$+\alpha_1(1+\tau\mu|\xi|^2)\big\{\tau\xi\cdot\mathrm{Re}(i\hat{\theta}\bar{\hat{q}})+\alpha_2(1+\mu|\xi|^2)\big(\mathrm{Re}(\hat{u}_t\bar{\hat{\theta}})+\alpha_3|\xi|^2\mathrm{Re}(\hat{u}_t\bar{\hat{u}})\big)\big\},$$

$$D_1(t,\xi) := \alpha_1\alpha_2(1-\alpha_3)(1+\tau\mu|\xi|^2)(1+\mu|\xi|^2)|\xi|^2|\hat{u}_t|^2+\alpha_1\alpha_2\alpha_3(1+\tau\mu|\xi|^2)|\xi|^6|\hat{u}|^2$$
$$+\alpha_1(1-\alpha_2)(1+\tau\mu|\xi|^2)|\xi|^2|\hat{\theta}|^2+|\hat{q}|^2+(2\tau+\mu)|\xi|^2|\hat{q}|^2-\alpha_1\tau|\xi\cdot\hat{q}|^2$$
$$+\tau(\tau+\mu)|\xi|^4|\hat{q}|^2-\alpha_1\tau^2\mu|\xi|^2|\xi\cdot\hat{q}|^2+\alpha_1\alpha_2(1-\alpha_3)(1+\tau\mu|\xi|^2)|\xi|^4\mathrm{Re}(\hat{u}\bar{\hat{\theta}})$$
$$+\alpha_1(1+\tau\mu|\xi|^2)\xi\cdot\mathrm{Re}(i\hat{\theta}\bar{\hat{q}})-\alpha_1\big\{\alpha_2-(\tau-\alpha_2\mu)|\xi|^2\big\}(1+\tau\mu|\xi|^2)\xi\cdot\mathrm{Re}(i\hat{u}_t\bar{\hat{q}}).$$

Applying $|\xi\cdot\hat{q}|\le|\xi||\hat{q}|$, (4.16) and

$$\begin{aligned}|\xi||\hat{\theta}\bar{\hat{q}}|&\le\varepsilon_3|\xi|^2|\hat{\theta}|^2+(4\varepsilon_3)^{-1}|\hat{q}|^2, & |\xi||\hat{u}_t\bar{\hat{q}}|&\le\varepsilon_4|\xi|^2|\hat{u}_t|^2+(4\varepsilon_4)^{-1}|\hat{q}|^2,\\ \tau|\xi||\hat{u}_t\bar{\hat{q}}|&\le\varepsilon_5|\hat{u}_t|^2+(4\varepsilon_5)^{-1}\tau^2|\xi|^2|\hat{q}|^2, & \mu|\xi||\hat{u}_t\bar{\hat{q}}|&\le\varepsilon_6\mu^2|\xi|^2|\hat{u}_t|^2+(4\varepsilon_6)^{-1}|\hat{q}|^2\end{aligned}\tag{4.25}$$

to the dissipation term $D_1(t,\xi)$, we may estimate $D_1\ge D_{11}+D_{12}+D_{13}+D_{14}$ with

$$D_{11} := \alpha_1\big\{\alpha_2(1-\alpha_3-\varepsilon_4)-\varepsilon_5\big\}|\xi|^2|\hat{u}_t|^2+\alpha_1\alpha_2(1-\alpha_3-\varepsilon_4)(1+\tau)\mu|\xi|^4|\hat{u}_t|^2$$
$$+\alpha_1\big\{\alpha_2(1-\alpha_3)-\varepsilon_6(\tau+\alpha_2\mu)\big\}\tau\mu^2|\xi|^6|\hat{u}_t|^2,$$

$$D_{12} := \alpha_1\alpha_2\big\{\alpha_3-\varepsilon_1(1-\alpha_3)\big\}(1+\tau\mu|\xi|^2)|\xi|^6|\hat{u}|^2,$$

$$D_{13} := \alpha_1\Big\{1-\varepsilon_3-\alpha_2\Big(1+\frac{1-\alpha_3}{4\varepsilon_1}\Big)\Big\}(1+\tau\mu|\xi|^2)|\xi|^2|\hat{\theta}|^2,$$

$$D_{14} := \Big\{1-\frac{\alpha_1}{4}\Big(\frac{1}{\varepsilon_3}+\frac{\alpha_2}{\varepsilon_4}\Big)\Big\}|\hat{q}|^2+\Big\{(2-\alpha_1)\tau+\Big(1-\frac{\alpha_1\alpha_2(1+\tau)}{4\varepsilon_4}\Big)\mu\Big\}|\xi|^2|\hat{q}|^2$$
$$+\Big\{\Big(1-\frac{\alpha_1}{4}\Big(\frac{1}{\varepsilon_5}+\frac{1}{\varepsilon_6}\Big)\Big)\tau+\Big(1-\alpha_1\Big(\tau+\frac{\alpha_2}{4\varepsilon_6}\Big)\Big)\mu\Big\}\tau|\xi|^4|\hat{q}|^2.$$

Choosing $\alpha_1=1/12$, $\alpha_2=1/5$, $\alpha_3=1/2$, $\varepsilon_1=1/2$, $\varepsilon_3=1/2$, $\varepsilon_4=1/4$, $\varepsilon_5=1/21$, and $\varepsilon_6=1/24$, then D_{11}, D_{12}, D_{13} and D_{14} are estimated by

$$D_{11}\ge\frac{1}{7!}|\xi|^2|\hat{u}_t|^2+\frac{1}{240}(1+\tau)\mu|\xi|^4|\hat{u}_t|^2+\frac{1}{240}\tau\mu^2|\xi|^6|\hat{u}_t|^2$$
$$\ge\frac{1}{7!}(1+\tau\mu|\xi|^2)|\xi|^2(1+\mu|\xi|^2)|\hat{u}_t|^2,$$

$$D_{12}\ge\frac{1}{240}(1+\tau\mu|\xi|^2)|\xi|^6|\hat{u}|^2,\qquad D_{13}\ge\frac{1}{48}(1+\tau\mu|\xi|^2)|\xi|^2|\hat{\theta}|^2,$$

$$D_{14}\ge\frac{113}{120}|\hat{q}|^2+\Big(\frac{23}{12}\tau+\frac{29}{30}\mu\Big)|\xi|^2|\hat{q}|^2+\Big(\frac{1}{16}\tau+\frac{49}{60}\mu\Big)\tau|\xi|^4|\hat{q}|^2$$
$$\ge\frac{1}{16}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{q}|^2.$$

Therefore we conclude

$$
D_1 \geq \frac{1}{7!}(1+\tau\mu|\xi|^2)|\xi|^2\Big\{(1+\mu|\xi|^2)|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 + |\hat{\theta}|^2\Big\} \\
+ \frac{1}{16}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{q}|^2. \tag{4.26}
$$

Similarly, applying (4.16) and (4.25), and substituting $\alpha_1 = 1/12$, $\alpha_2 = 1/5$, $\alpha_3 = 1/2$, $\varepsilon_3 = 1/2$ again, and $\varepsilon_2 = 1/2$, the energy term $E_1(t,\xi)$ is also estimated by $E_1 \leq E_{11} + E_{12} + E_{13} + E_{14}$ and $E_1 \geq \tilde{E}_{11} + \tilde{E}_{12} + \tilde{E}_{13} + \tilde{E}_{14}$ with

$$
\begin{aligned}
E_{11} :=& \frac{1}{2}(1+\tau|\xi|^2)(1+\mu|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{u}_t|^2 \\
&+ \frac{1}{2}\alpha_1\alpha_2(2\varepsilon_2+\alpha_3)(1+\mu|\xi|^2)(1+\tau\mu|\xi|^2)|\hat{u}_t|^2 \\
\leq& \frac{1}{2}(1+\tau|\xi|^2)(1+\mu|\xi|^2)\big(1+\alpha_1\alpha_2(2\varepsilon_2+\alpha_3)+(\tau+\mu)|\xi|^2\big)|\hat{u}_t|^2 \\
\leq& \frac{41}{80}(1+\tau|\xi|^2)(1+\mu|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{u}_t|^2, \\
E_{12} :=& \frac{1}{2}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\xi|^4|\hat{u}|^2 \\
&+ \frac{1}{2}\alpha_1\alpha_2\alpha_3(1+\mu|\xi|^2)(1+\tau\mu|\xi|^2)|\xi|^4|\hat{u}|^2 \\
\leq& \frac{1}{2}(1+\tau|\xi|^2)\big(\tau|\xi|^2+(1+\alpha_1\alpha_2\alpha_3)(1+\mu|\xi|^2)\big)|\xi|^4|\hat{u}|^2 \\
\leq& \frac{121}{240}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\xi|^4|\hat{u}|^2, \\
E_{13} :=& \frac{1}{2}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{\theta}|^2 \\
&+ \frac{\alpha_1\alpha_2}{4\varepsilon_2}(1+\mu|\xi|^2)(1+\tau\mu|\xi|^2)|\hat{\theta}|^2 + \alpha_1\varepsilon_3\tau(1+\tau\mu|\xi|^2)|\xi|^2|\hat{\theta}|^2 \\
\leq& \frac{1}{2}(1+\tau|\xi|^2)\Big((1+2\alpha_1\varepsilon_3)\tau|\xi|^2 + \Big(1+\frac{\alpha_1\alpha_2}{2\varepsilon_2}\Big)(1+\mu|\xi|^2)\Big)|\hat{\theta}|^2 \\
\leq& \frac{13}{24}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{\theta}|^2, \\
E_{14} :=& \frac{1}{2}\tau(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{q}|^2 + \frac{\alpha_1}{4\varepsilon_3}\tau(1+\tau\mu|\xi|^2)|\hat{q}|^2 \\
\leq& \frac{1}{2}\tau(1+\tau|\xi|^2)\Big(1+\frac{\alpha_1}{2\varepsilon_3}+(\tau+\mu)|\xi|^2\Big)|\hat{q}|^2 \\
\leq& \frac{13}{24}\tau(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{q}|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{E}_{11} :=& \frac{1}{2}(1+\tau|\xi|^2)(1+\mu|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{u}_t|^2 \\
&- \frac{1}{2}\alpha_1\alpha_2(2\varepsilon_2+\alpha_3)(1+\mu|\xi|^2)(1+\tau\mu|\xi|^2)|\hat{u}_t|^2 \\
\geq& \frac{39}{80}(1+\tau|\xi|^2)(1+\mu|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{u}_t|^2,
\end{aligned}
$$

$$\begin{aligned}\tilde{E}_{12} :=& \frac{1}{2}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\xi|^4|\hat{u}|^2\\&-\frac{1}{2}\alpha_1\alpha_2\alpha_3(1+\mu|\xi|^2)(1+\tau\mu|\xi|^2)|\xi|^4|\hat{u}|^2\\&\geq \frac{119}{240}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\xi|^4|\hat{u}|^2,\\
\tilde{E}_{13} :=& \frac{1}{2}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{\theta}|^2\\&-\frac{\alpha_1\alpha_2}{4\varepsilon_2}(1+\mu|\xi|^2)(1+\tau\mu|\xi|^2)|\hat{\theta}|^2-\alpha_1\varepsilon_3\tau(1+\tau\mu|\xi|^2)|\xi|^2|\hat{\theta}|^2\\&\geq \frac{11}{24}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{\theta}|^2,\\
\tilde{E}_{14} :=& \frac{1}{2}\tau(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{q}|^2-\frac{\alpha_1}{4\varepsilon_3}\tau(1+\tau\mu|\xi|^2)|\hat{q}|^2\\&\geq \frac{11}{24}\tau(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)|\hat{q}|^2,\end{aligned}$$

Deriving these estimates, we used $\mu \leq 1$ as well as $\tau \leq 1$. Consequently, we obtain

$$\begin{aligned}&E_1(t,\xi) \leq \frac{13}{24}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)W(t,\xi),\\&E_1(t,\xi) \geq \frac{11}{24}(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)W(t,\xi).\end{aligned} \tag{4.27}$$

Applying the estimates (4.26) and (4.27) to (4.24) we get

$$\frac{\partial}{\partial t}E_1(t,\xi)+\frac{1}{2730}\frac{(1+\tau\mu|\xi|^2)|\xi|^2}{(1+\tau|\xi|^2)\big(1+(\tau+\mu)|\xi|^2\big)}E_1(t,\xi) \leq 0.$$

Therefore we conclude the following pointwise estimate

$$W(t,\xi) \leq \frac{13}{11}e^{-\frac{1}{2730}\frac{(1+\tau\mu|\xi|^2)|\xi|^2}{(1+\tau|\xi|^2)(1+(\tau+\mu)|\xi|^2)}t}W(0,\xi),$$

and hence we arrive at the desired decay estimate (4.3) (for $\mu \leq 1$).

Second case: $\mu \geq 1$.

We multiply (4.20), (4.21), (4.22) and (4.23) by $1+\mu|\xi|^2$, $\alpha_1\alpha_2\alpha_3|\xi|^2$, $\alpha_1\alpha_2$, and α_1, respectively, and add these four equations, where α_1, α_2 α_3 are again positive constants yet to be determined. This yields

$$\frac{\partial}{\partial t}E_2(t,\xi)+D_2(t,\xi)=0, \tag{4.28}$$

where

$$\begin{aligned}E_2(t,\xi) :=& \frac{1}{2}(1+\mu|\xi|^2)\big\{(1+\mu|\xi|^2)|\hat{u}_t|^2+|\xi|^4|\hat{u}|^2+|\hat{\theta}|^2+\tau|\hat{q}|^2\big\}\\&+\alpha_1\big\{\tau\xi\cdot\mathrm{Re}(i\hat{\theta}\bar{\hat{q}})+\alpha_2(1+\mu|\xi|^2)\big(\mathrm{Re}(\hat{u}_t\bar{\hat{\theta}})+\alpha_3|\xi|^2\mathrm{Re}(\hat{u}_t\bar{\hat{u}})\big)\big\},\\
D_2(t,\xi) :=& |\xi|^2\big\{\alpha_1\alpha_2(1-\alpha_3)(1+\mu|\xi|^2)|\hat{u}_t|^2+\alpha_1\alpha_2\alpha_3|\xi|^4|\hat{u}|^2+\alpha_1(1-\alpha_2)|\hat{\theta}|^2\big\}\\&+\mu|\xi|^2|\hat{q}|^2-\alpha_1\tau|\xi\cdot\hat{q}|^2+|\hat{q}|^2+\alpha_1\alpha_2(1-\alpha_3)|\xi|^4\mathrm{Re}(\hat{u}\bar{\hat{\theta}})\\&+\alpha_1\xi\cdot\mathrm{Re}(i\hat{\theta}\bar{\hat{q}})-\alpha_1\big\{\alpha_2-(\tau-\alpha_2\mu)|\xi|^2\big\}\xi\cdot\mathrm{Re}(i\hat{u}_t\bar{\hat{q}}).\end{aligned}$$

Applying (4.25) we get

$$
\begin{aligned}
D_2 \geq\, & \alpha_1\alpha_2(1-\alpha_3-\varepsilon_4)|\xi|^2|\hat{u}_t|^2 + \alpha_1\Big\{\alpha_2(1-\alpha_3) - \varepsilon_4\Big(\alpha_2 + \frac{\tau}{\mu}\Big)\Big\}\mu|\xi|^4|\hat{u}_t|^2 \\
& + \alpha_1\alpha_2\big(\alpha_3 - \varepsilon_1(1+\alpha_3)\big)|\xi|^6|\hat{u}|^2 + \alpha_1\Big\{1-\varepsilon_3-\alpha_2\Big(1+\frac{1+\alpha_3}{4\varepsilon_1}\Big)\Big\}|\xi|^2|\hat{\theta}|^2 \\
& + \Big\{1-\frac{\alpha_1}{4}\Big(\frac{1}{\varepsilon_3}+\frac{\alpha_2}{\varepsilon_4}\Big)\Big\}|\hat{q}|^2 + \Big\{1-\alpha_1\Big(1+\frac{1}{4\varepsilon_4}\Big)\frac{\tau}{\mu} - \frac{\alpha_1\alpha_2}{4\varepsilon_4}\Big\}\mu|\xi|^2|\hat{q}|^2.
\end{aligned}
$$

Choosing $\alpha_1 = 1/6$, $\alpha_2 = 1/6$, $\alpha_3 = 1/2$, $\varepsilon_1 = 1/4$, $\varepsilon_3 = 1/2$ and $\varepsilon_4 = 1/16$, then the above estimate is rewritten as

$$
\begin{aligned}
D_2 \geq\, & \frac{7}{24^2}|\xi|^2|\hat{u}_t|^2 + \frac{1}{24^2}\mu|\xi|^4|\hat{u}_t|^2 + \frac{1}{288}|\xi|^6|\hat{u}|^2 \\
& + \frac{1}{72}|\xi|^2|\hat{\theta}|^2 + \frac{29}{36}|\hat{q}|^2 + \frac{1}{18}\mu|\xi|^2|\hat{q}|^2 \\
\geq\, & \frac{1}{24^2}|\xi|^2\big\{(1+\mu|\xi|^2)|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2 + |\hat{\theta}|^2\big\} + \frac{1}{18}(1+\mu|\xi|^2)|\hat{q}|^2.
\end{aligned} \tag{4.29}
$$

Similarly, applying (4.16) and (4.25), and substituting $\alpha_1 = 1/6$, $\alpha_2 = 1/6$, $\alpha_3 = 1/2$, $\varepsilon_2 = 1/2$, and $\varepsilon_3 = 1/2$ again, the energy term E_2 is estimated by

$$
\begin{aligned}
E_2 \leq\, & \frac{1}{2}\big\{1+\alpha_1\alpha_2(2\varepsilon_2+\alpha_3)+\mu|\xi|^2\big\}(1+\mu|\xi|^2)|\hat{u}_t|^2 \\
& + \frac{1}{2}(1+\alpha_1\alpha_2\alpha_3)(1+\mu|\xi|^2)|\xi|^4|\hat{u}|^2 \\
& + \frac{1}{2}\Big(1+\frac{\alpha_1\alpha_2}{2\varepsilon_2}\Big)|\hat{\theta}|^2 + \frac{1}{2}\Big(1+2\alpha_1\varepsilon_3\tau+\frac{\alpha_1\alpha_2}{2\varepsilon_2}\Big)\mu|\xi|^2|\hat{\theta}|^2 \\
& + \frac{1}{2}\Big(1+\frac{\alpha_1}{2\varepsilon_3}\Big)\tau|\hat{q}|^2 + \frac{1}{2}\tau\mu|\xi|^2|\hat{q}|^2 \\
\leq\, & \frac{43}{72}(1+\mu|\xi|^2)W(t,\xi),
\end{aligned} \tag{4.30}
$$

and

$$
\begin{aligned}
E_2 \geq\, & \frac{1}{2}\big\{1-\alpha_1\alpha_2(2\varepsilon_2+\alpha_3)+\mu|\xi|^2\big\}(1+\mu|\xi|^2)|\hat{u}_t|^2 \\
& + \frac{1}{2}(1-\alpha_1\alpha_2\alpha_3)(1+\mu|\xi|^2)|\xi|^4|\hat{u}|^2 \\
& + \frac{1}{2}\Big(1-\frac{\alpha_1\alpha_2}{2\varepsilon_2}\Big)|\hat{\theta}|^2 + \frac{1}{2}\Big(1-2\alpha_1\varepsilon_3\tau-\frac{\alpha_1\alpha_2}{2\varepsilon_2}\Big)\mu|\xi|^2|\hat{\theta}|^2 \\
& + \frac{1}{2}\Big(1-\frac{\alpha_1}{2\varepsilon_3}\Big)\tau|\hat{q}|^2 + \frac{1}{2}\tau\mu|\xi|^2|\hat{q}|^2 \\
\geq\, & \frac{29}{72}(1+\mu|\xi|^2)W(t,\xi).
\end{aligned} \tag{4.31}
$$

Applying the estimates (4.29), (4.30) and (4.31) to (4.28) we obtain

$$
\frac{\partial}{\partial t}E_2(t,\xi) + \frac{1}{344}\frac{|\xi|^2}{(1+\mu|\xi|^2)^2}E_2(t,\xi) \leq 0.
$$

This leads to the pointwise estimate

$$W(t,\xi) \le \frac{43}{29} e^{-\frac{1}{344}\frac{|\xi|^2}{1+\mu|\xi|^2}t} W(0,\xi),$$

and hence we arrive at the desired decay estimate (4.3)(for $\mu \ge 1$).

The decay estimate (4.5) and (4.6) follow from (4.3) as usual. This completes the proof. □

Remark 5. *As in the previous sections the proof of Theorem 4.1 contains the arguments for the following L^2-estimates for the solutions of* (4.1)–(4.2)*, both for $0 \le k \le s$.*

For $\mu \le 1$:

$$\begin{aligned} &\|\partial_x^k u_t(t)\|^2_{H^2_{\tau,\mu}} + \|\partial_x^k(\Delta u,\theta,\tau q)(t)\|^2_{H^1_\tau} \\ &\quad + \frac{1}{2310}\int_0^t \big(\|\partial_x^{k+1}u_t(\sigma)\|^2_{\mathcal{H}^1_\mu} + \|\partial_x^{k+1}(\Delta u,\theta)(\sigma)\|^2_{\mathcal{L}^2} + \|\partial_x^k q(\sigma)\|^2_{H^1_\tau}\big)d\sigma \\ &\le \frac{13}{11}\big(\|\partial_x^k u_1\|^2_{H^2_{\tau,\mu}} + \|\partial_x^k(\Delta u_0,\theta,\tau q_0)\|^2_{H^1_\tau}\big). \end{aligned} \tag{4.32}$$

For $\mu \ge 1$:

$$\begin{aligned} &\|\partial_x^k u_t(t)\|^2_{H^2_\mu} + \|\partial_x^k(\Delta u,\theta,\tau q)(t)\|^2_{H^1_\mu} \\ &\quad + \frac{1}{232}\int_0^t \big(\|\partial_x^{k+1}u_t(\sigma)\|^2_{H^1_\mu} + \|\partial_x^{k+1}(\Delta u,\theta)(\sigma)\|^2_{L^2} + \|\partial_x^k q(\sigma)\|^2_{H^1_\mu}\big)d\sigma \\ &\le \frac{43}{29}\big(\|\partial_x^k u_1\|^2_{H^2_\mu} + \|\partial_x^k(\Delta u_0,\theta_0,\tau q_0)\|^2_{H^1_\mu}\big). \end{aligned} \tag{4.33}$$

for $k \ge 0$.

Here we introduce the following notations for the weighted Sobolev norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{H}^s_a}$: Let $s \ge 0$ be an integer and $a \ge 0$ be a real number, then

$$\|u\|_{\mathcal{L}^2} := \|\phi\hat{u}\|_{L^2}, \qquad \phi(\xi) := \Big\{\frac{1+\tau\mu|\xi|^2}{1+(\tau+\mu)|\xi|^2}\Big\}^{1/2},$$

and

$$\|u\|_{\mathcal{H}^s_a} := \Big(\sum_{k=0}^s a^k \|\partial_x^k u\|^2_{\mathcal{L}^2}\Big)^{1/2}.$$

We observe that $\|\cdot\|_{\mathcal{H}^s_0} = \|\cdot\|_{\mathcal{L}^2}$, and

$$\frac{\tau\mu}{\tau+\mu}\|u\|_{H^s_a} \le \|u\|_{\mathcal{H}^s_a} \le \|u\|_{H^s_a}, \tag{4.34}$$

which comes from the fact that $\tau\mu/(\tau+\mu) \le \phi^2 \le 1$ for $0 \le \tau \le 1$ and $\mu \ge 0$. Furthermore we also used the following modified Sobolev norm $\|\cdot\|_{H^s_{a,b}}$: Let $s \ge 0$ be an even integer and $a \ge 0$ and $b \ge 0$ be real numbers, then

$$\|u\|_{H^s_{a,b}} := \Big(\sum_{j=0}^{s/2}\sum_{k=0}^{s/2} a^j b^k \|\partial_x^{j+k}u\|^2_{L^2}\Big)^{1/2}.$$

In particular, by using (4.34), we can conclude for $\mu \le 1$ from (4.32) that

$$\begin{aligned}(4.35)\qquad &\|\partial_x^k u_t(t)\|_{H^2_{\tau,\mu}}^2 + \|\partial_x^k(\Delta u,\theta,\tau q)(t)\|_{H^1_\tau}^2 + \frac{1}{2310}\int_0^t \|\partial_x^k q(\sigma)\|_{H^1_\tau}^2 d\sigma \\ &+ \frac{1}{2310}\frac{\tau\mu}{\tau+\mu}\int_0^t \big(\|\partial_x^{k+1}u_t(\sigma)\|_{H^1_\mu}^2 + \|\partial_x^{k+1}(\Delta u,\theta)(\sigma)\|_{L^2}^2\big)d\sigma \\ &\le \frac{13}{11}\big(\|\partial_x^k u_1\|_{H^2_{\tau,\mu}}^2 + \|\partial_x^k(\Delta u_0,\theta_0,\tau q_0)\|_{H^1_\tau}^2\big).\end{aligned}$$

4.2. **Characteristic values.** In this section, we first consider the characteristic equation for the system (2.6). This equation is given by

$$(4.36)\qquad Q_3(\xi) := (1+\mu|\xi|^2)\lambda^3 + |\xi|^2(1+\mu|\xi|^2)\lambda^2 + 2|\xi|^4\lambda + |\xi|^6 = 0,$$

where a zero $\lambda = \lambda(|\xi|)$ of the characteristic polynomial Q_3 is called an eigenvalue, and $\xi \in \mathbb{R}^n$ is again the Fourier variable.

Similarly as in Subsection 3.2, we consider the asymptotic expansion of $\lambda(|\xi|)$ for $|\xi| \to 0$ and for $|\xi| \to \infty$. We first consider the asymptotic expansion for $|\xi| \to 0$:

$$(4.37)\qquad \lambda(|\xi|) = \sum_{k=0}^{\infty} \lambda^{(k)}|\xi|^k.$$

Substituting (4.37) into (4.36), we find

$$(4.38)\qquad \lambda_j(|\xi|) = a_j|\xi|^2 + O(\xi^4),$$

for $j = 1,2,3$, where the a_j are defined by (2.15). We remark that $\mathrm{Re}(a_j) < 0$.

On the other hand, we consider the following asymptotic expansion for $|\xi| \to \infty$:

$$(4.39)\qquad \lambda(|\xi|) = \nu^{(2)}|\xi|^2 + \nu^{(1)}|\xi| + \sum_{k=0}^{\infty} \nu^{(-k)}|\xi|^{-k}.$$

Then, substituting (4.39) into (4.36), we conclude

$$\begin{aligned}(4.40)\qquad &\lambda_j(|\xi|) = \pm\sqrt{\frac{1}{\mu}}i|\xi| - \frac{1}{3\mu} + O(|\xi|^{-1}), \quad j = 1,2, \\ &\lambda_3(|\xi|) = -|\xi|^2 + \frac{1}{\mu} + O(|\xi|^{-1}).\end{aligned}$$

Consequently, the asymptotic expansions (4.38) and (4.40) tell us that the pointwise estimate (4.9) is optimal.

On the other hand, the characteristic equation for the system (4.19) is given by

$$\begin{aligned}(4.41)\qquad Q_4(\xi) := {}&\tau(1+\mu|\xi|^2)\lambda^4 + (1+\mu|\xi|^2)\lambda^3 \\ &+ \big\{1 + (2\tau+\mu)|\xi|^2\big\}|\xi|^2\lambda^2 + 2|\xi|^4\lambda + |\xi|^6 = 0.\end{aligned}$$

ξ as a parameter. We consider the asymptotic expansion of $\lambda = \lambda(|\xi|)$ for $|\xi| \to 0$ and for $|\xi| \to \infty$ again. We first consider the asymptotic expansion for $|\xi| \to 0$. Substituting the expansion (4.37) into (4.41), we find

$$(4.42)\qquad \lambda_j(|\xi|) = a_j|\xi|^2 + O(|\xi|^4), \qquad \lambda_4(|\xi|) = -\frac{1}{\tau} + |\xi|^2 + O(|\xi|^4),$$

for $j=1,2,3$, with the sme a_j defined by (2.15). For $|\zeta|\to\infty$ we obtain

$$
\begin{aligned}
\lambda_j(\zeta) &= \pm\sqrt{b_{1+}}\,i|\xi| - \frac{1}{4\tau}(1+b_2) + O(|\xi|^{-1}), \quad j=1,2,\\
\lambda_j(\zeta) &= \pm\sqrt{b_{1-}}\,i|\xi| - \frac{1}{4\tau}(1-b_2) + O(|\xi|^{-1}), \quad j=3,4.
\end{aligned} \tag{4.43}
$$

where

$$
b_{1+} := \frac{2\tau+\mu+\sqrt{4\tau^2+\mu^2}}{2\tau\mu}, \quad b_{1-} := \frac{2\tau+\mu-\sqrt{4\tau^2+\mu^2}}{2\tau\mu}, \quad b_2 := \frac{\mu-2\tau}{\sqrt{4\tau^2+\mu^2}}.
$$

We note that $b_{1+}>0$, $b_{1-}>0$ and $|b_2|<1$ for $\tau>0,\ \mu>0$. In these representations for the values λ_j, for $j=1,2,3,4$, the standard type decay structure (no regularity-loss) for $\tau>0$, $\mu>0$ is also clearly expressed.

Consequently, the asymptotic expansions (4.42) and (4.43) tell us that the pointwise estimate (4.3) is optimal.

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