COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH REVISED MAXWELL'S LAW

YUXI HU AND REINHARD RACKE

Abstract. We investigate the compressible Navier-Stokes equations where the constitutive law for the stress tensor given by Maxwell's law is revised to a system of relaxation equations for two parts of the tensor. The global well-posedness is proved as well as the compatibility with the classical compressible Navier-Stokes system in the sense that, for vanishing relaxation parameters, the solutions to the Maxwell system are shown to converge to solutions of the classical system.

Keywords: compressible Navier-Stokes, Maxwell fluid, global existence, singular limit
AMS classification code: 35B25, 76N10

1. Introduction

The classical compressible Navier-Stokes equations in $\mathbb{R}^n$, $n = 2, 3$, are given by

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div}(S), \\ \partial_t(\rho(e + \frac{1}{2}u^2)) + \operatorname{div}(\rho u(e + \frac{1}{2}u^2) + up) - \kappa \triangle \theta = \operatorname{div}(uS), \end{cases} \tag{1.1}$$

with the constitutive law for a Newtonian fluid,

$$S = \mu(\nabla u + \nabla u^T - \frac{2}{n}\operatorname{div} u\, I_n) + \lambda \operatorname{div} u\, I_n. \tag{1.2}$$

Here, ρ, $u = (u_1, \cdots, u_n)$, p, S, e and θ represent fluid density, velocity, pressure, stress tensor, specific internal energy per unit mass and temperature, respectively. I_n denotes the identity matrix in $\mathbb{R}^n$. The equations are the consequence of conservation of mass, momentum and energy, respectively. κ, μ, λ are positive constants.

Maxwell's relaxation replaces (1.2) by the differential equation

$$\tau \partial_t S + S = \mu(\nabla u + \nabla u^T - \frac{2}{n}\operatorname{div} u\, I_n) + \lambda \operatorname{div} u\, I_n, \tag{1.3}$$

with the relaxation parameter $\tau > 0$. For $\tau \to 0$ we formally recover (1.2). For *incompressible* Navier-Stokes equations this relaxation has been discussed by Racke & Saal [20, 21] and Schöwe [23, 24] proving global well-posedness for small data and rigorously investigating the singular limit as $\tau \to 0$.

A splitting of the tensor S was discussed by Yong [28] in the *isentropic* case leading to the following system with a revised Maxwell law, now for the *non-isentropic* case, that we are going to further investigate here:

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = \mathrm{div}(S_1) + \nabla S_2, \\ \partial_t(\rho(e + \frac{1}{2}u^2)) + \mathrm{div}(\rho u(e + \frac{1}{2}u^2) + up) - \kappa\triangle\theta = \mathrm{div}(u(S_1 + S_2 I_n)), \\ \tau_1\partial_t S_1 + S_1 = \mu(\nabla u + \nabla u^T - \frac{2}{n}\mathrm{div} u I_n), \\ \tau_2\partial_t S_2 + S_2 = \lambda \mathrm{div} u, \end{cases} \tag{1.4}$$

where S_1 is a $n\times n$ square matrix and symmetric and traceless if it was initially, and S_2 is a scalar variable.

A similar revised Maxwell model was considered by Chakraborty & Sader [1] for a compressible viscoelastic fluid (isentropic case), where τ_1 counts for the shear relaxation time, and τ_2 counts for the compressional relaxation time. The importance of this model for describing high frequency limits is underlined together with the presentation of numerical experiments. The authors conclude that it provides a general formalism with which to characterize the fluid-structure interaction of nanoscale mechanical devices vibrating in simple liquids.

We consider the more complex non-isentropic case with general equations of state assuming that the pressure $p = p(\rho,\theta)$ and $e = e(\rho,\theta)$ are smooth functions of (ρ,θ) satisfying

$$\rho^2 e_\rho(\rho,\theta) = p(\rho,\theta) - \theta p_\theta(\rho,\theta), \tag{1.5}$$

where θ denotes the absolute temperature. In particular, the case of a polytropic gas $p = R\rho\theta, e = c_v\theta$ is included here.

We investigate the Cauchy problem for the functions

$$(\rho, u, \theta) : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$$

with initial condition

$$(\rho(x,0), u(x,0), \theta(x,0)) = (\rho_0, u_0, \theta_0). \tag{1.6}$$

In [28] a local existence result is presented exploiting a entropy dissipation structure found. Here we first present a local existence theorem in suggesting an explicit transformation to a symmetric-hyperbolic system. Moreover, we prove a global existence theorem for small data. The strategy follows our paper [7].

As second topic we consider the singular limit $\tau := \tau_1 = \tau_2 \to 0$, being more complex than the local in time singular limit studied in [28] for the isentropic case. For $\tau = 0$, the relaxed system (1.4) turns into the classical Newtonian compressible Navier-Stokes system (1.1), (1.2). For the latter, because of its physical importance and mathematical challenges, the well-posedness has been widely studied, see [2, 3, 4, 5, 6, 8, 10, 11, 12, 14, 16, 17, 18, 22, 26]. In particular, the local existence and uniqueness of smooth solutions was established by Serrin [22] and Nash [18] for initial data far away from vacuum. Later, Matsumura and Nishida [16] got global smooth solutions for small initial data without vacuum. For large data, Xin [26], Cho and Jin [2] showed that smooth solutions must blow up in finite time if the initial data has a vacuum state.

We will show the convergence of solutions to the relaxed system (1.4) to the the classical system (1.1), (1.2) rigorously and also obtain the convergence order with respect to τ. The energy method is used extending [7, 28].

To summarize the main new contributions, we mention

- a first discussion of the non-isentropic compressible Navier-Stokes equations with revised Maxwell's law,
- the proof of global well-posedness via finding appropriate symmetric structures,
- the description of the singular limit to the classical Newtonian case in terms of order of convergence in the relaxation parameter τ.

The paper is organized as follows. In Section 2 we prove the local well-posedness as well as a global existence result for small data for the Cauchy problem (1.4), (1.6). The singular limit as $\tau \to 0$ is subject of Section 3, where a convergence result is proved. In the Appendix in Section 4, we provide Moser-type inequalities.

Finally, we introduce some notation. $W^{m,p} = W^{m,p}(\mathbb{R}^n)$, $0 \le m \le \infty$, $1 \le p \le \infty$, denotes the usual Sobolev space with norm $\|\cdot\|_{m,p}$. For convenience, H^m and L^p stand for $W^{m,2}(\Omega)$ and $W^{0,p}(\Omega)$ with norms $\|\cdot\|_m$ and $\|\cdot\|_{L^p}$, respectively. For $p=2$, we denote the norm $\|\cdot\|_{L^2}$ by $\|\cdot\|$.

2. Local and Global Well-Posedness

In this part, we prove the local and the global well-posedness for the Cauchy problem (1.4), (1.6). For this we need the following assumptions A.1 and A.2. As in [7] we try to transform the system with symmetrizers to finally be able to apply the results from Kawashima, see [13] or [25].

- A.1. The initial data satisfy
$$\{(\rho_0, u_0, \theta_0, S_{10}, S_{20})(x) : x \in \mathbb{R}^n\} \subset [\rho_*, \rho^*] \times [-C_1, C_1]^n \times [\theta_*, \theta^*] \times [-C_1, C_1]^{n\times n} \times [-C_1, C_1] =: G_0,$$
where $C_1 > 0$ as well as $0 < \rho_* < 1 < \rho^* < \infty$ and $0 < \theta_* < 1 < \theta^* < \infty$ are constants.
- A.2. For each given G_1 satisfying $G_0 \subset\subset G_1 \subset\subset G$, $\forall(\rho, u, \theta, S_1, S_2) \in G_1$, the pressure p and the internal energy e satisfy
$$p(\rho,\theta), p_\theta(\rho,\theta), p_\rho(\rho,\theta), e_\theta(\rho,\theta) > C(G_1) > 0,$$
where $C(G_1)$ is a positive constants depending on G_1.

For the standard assumption A.2 see for example [9, 17].

Theorem 2.1. *(Local existence) Let $s \ge s_0 + 1$ with $s_0 \ge [\frac{n}{2}] + 1$ be integers. Suppose that the Assumptions A.1 and A.2 hold and that the initial data $(\rho_0 - 1, u_0, \theta_0 - 1, S_{10}, S_{20})$ are in H^s. Then, for each convex open subset G_1 satisfying $G_0 \subset\subset G_1 \subset\subset G$, there exists $T_{ex} > 0$ such that the system (1.4) has an unique classical solution $(\rho, u, \theta, S_1, S_2)$ satisfying*
$$\begin{cases} (\rho - 1, u, S_1, S_2) \in C([0, T_{ex}], H^s) \cap C^1([0, T_{ex}], H^{s-1}), \\ \theta - 1 \in C([0, T_{ex}], H^s) \cap C^1([0, T_{ex}], H^{s-2}) \end{cases} \tag{2.1}$$
and
$$(\rho, u, \theta, S_1, S_2)(x, t) \in G_1, \quad \forall (x, t) \in \mathbb{R}^n \times [0, T_{ex}].$$

Proof. First we consider the three-dimensional case $n = 3$. Using (1.5), we rewrite the system (1.4) as
$$\begin{cases} \partial_t \rho + u\nabla\rho + \rho \mathrm{div} u = 0, \\ \rho\partial_t u + \rho u \nabla u + p_\theta \nabla\theta + p_\rho \nabla \rho = \mathrm{div} S_1 + \nabla S_2, \\ \rho e_\theta \partial_t \theta + \rho e_\theta u \nabla\theta + \theta p_\theta \mathrm{div} u = \kappa \triangle \theta + (S_1 + S_2 I_3)\nabla u, \\ \tau_1 \partial_t S_1 + S_1 = \mu(\nabla u + (\nabla u)^T - \frac{2}{3}\mathrm{div} u I_3), \\ \tau_2 \partial_t S_2 + S_2 = \lambda \mathrm{div} u. \end{cases} \tag{2.2}$$

Without loss of generality, we assume S_1 to take the following form:
$$S_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & -a_{11} - a_{22} \end{pmatrix}. \tag{2.3}$$

Let $\omega=(\rho,u,a_{11},a_{12},a_{13},a_{22},a_{23},S_2)$. Then, we have

$$\begin{cases} A_0(\omega)\omega_t+\sum\limits_{j=1}^{3}A_j(\omega)\partial_{x_j}\omega+L(\omega)\omega=f_1(\omega,\theta,\nabla\theta),\\ \rho e_\theta\partial_t\theta-\kappa\triangle\theta=f_2(\omega,\theta,\nabla\omega,\nabla\theta). \end{cases} \tag{2.4}$$

Here, $f_1(\omega,\theta,\nabla\theta)=(0,p_\theta\nabla\theta,0,0,0,0,0,0)$, $f_2(\omega,\theta,\nabla\omega,\nabla\theta)=S\nabla u-\rho e_\theta u\nabla\theta-\theta p_\theta \mathrm{div}u$ and

$$A_0(\omega)=\mathrm{diag}\left\{\frac{p_\rho}{\rho},\rho,\rho,\rho,\frac{3\tau_1}{4\mu},\frac{\tau_1}{\mu},\frac{\tau_1}{\mu},\frac{3\tau_1}{4\mu},\frac{\tau_1}{\mu},\frac{\tau_2}{\lambda}\right\},$$

$$L(\omega)=\mathrm{diag}\left\{0,0,0,0,\frac{3}{4\mu},\frac{1}{\mu},\frac{1}{\mu},\frac{3}{4\mu},\frac{1}{\mu},\frac{1}{\lambda}\right\},$$

$$\sum_{j=1}^{3}A_j(\omega)\xi_j=\begin{pmatrix} \frac{p_\rho}{\rho}u\xi & p_\rho\xi & 0_{1\times5} & 0\\ p_\rho\xi^T & \rho u\xi I_3 & C_{3\times5}(\xi) & -\xi^T\\ 0_{5\times1} & D_{5\times3}(\xi) & 0_{5\times5} & 0\\ 0 & -\xi & 0_{1\times5} & 0 \end{pmatrix},$$

where

$$C_{3\times5}(\xi)=\begin{pmatrix} -\xi_1 & -\xi_2 & -\xi_3 & 0 & 0\\ 0 & -\xi_1 & 0 & -\xi_2 & -\xi_3\\ \xi_3 & 0 & \xi_3-\xi_1 & 0 & -\xi_2 \end{pmatrix}, D_{5\times3}(\xi)=\begin{pmatrix} -\xi_1 & \frac{\xi_2}{2} & \frac{\xi_3}{2}\\ -\xi_2 & -\xi_1 & 0\\ -\xi_3 & 0 & -\xi_1\\ \frac{\xi_1}{2} & -\xi_2 & \frac{\xi_3}{2}\\ 0 & -\xi_3 & -\xi_2 \end{pmatrix}$$

for each $\xi\in\mathbb{S}^3$.

Note that the matrix $\sum\limits_{j=1}^{3}A_j\xi_j$ is not symmetric. Therefore, the theory of symmetric hyperbolic parabolic system does not apply directly. Fortunately, we can perform a transformation to overcome this problem. Let $b_{11}:=\frac{a_{11}+a_{22}}{2}, b_{22}:=\frac{a_{11}-a_{22}}{2}$. This particularly implies $a_{11}=b_{11}+b_{22}, a_{22}=b_{11}-b_{22}$. Let $\tilde{\omega}:=(\rho,u,b_{11},a_{12},a_{13},b_{22},a_{23},S_2)$. Then system (2.4) can be rewritten as

$$\begin{cases} \tilde{A}_0(\tilde{\omega})\tilde{\omega}_t+\sum\limits_{j=1}^{3}\tilde{A}_j(\tilde{\omega})\partial_{x_j}\tilde{\omega}+\tilde{L}(\tilde{\omega})\tilde{\omega}=f_1(\tilde{\omega},\theta,\nabla\theta),\\ \rho e_\theta\partial_t\theta-\kappa\triangle\theta=f_2(\tilde{\omega},\nabla\tilde{\omega},\theta,\nabla\theta). \end{cases} \tag{2.5}$$

Here, $f_1(\tilde{\omega},\theta,\nabla\theta)=(0,p_\theta\nabla\theta,0,0,0,0,0,0)$, $f_2(,\nabla\tilde{\omega},\theta,\nabla\theta)=S\nabla u-\rho e_\theta u\nabla\theta-\theta p_\theta\mathrm{div}u$ and

$$\tilde{A}_0(\tilde{\omega})=\mathrm{diag}\left\{\frac{p_\rho}{\rho},\rho,\rho,\rho,\frac{3\tau_1}{\mu},\frac{\tau_1}{\mu},\frac{\tau_1}{\mu},\frac{\tau_1}{\mu},\frac{\tau_1}{\mu},\frac{\tau_2}{\lambda}\right\},\tilde{L}(\tilde{\omega})=\mathrm{diag}\left\{0,0,0,0,\frac{3}{\mu},\frac{1}{\mu},\frac{1}{\mu},\frac{1}{\mu},\frac{1}{\mu},\frac{1}{\lambda}\right\}$$

and

$$\sum_{j=1}^{3}\tilde{A}_j(\tilde{\omega})\xi_j=\begin{pmatrix} \frac{p_\rho}{\rho}u\xi & p_\rho\xi & 0 & 0\\ p_\rho\xi^T & \rho u\xi I_3 & \tilde{C}_{3\times5}(\xi) & -\xi^T\\ 0 & \tilde{D}_{5\times3}(\xi) & 0 & 0\\ 0 & -\xi & 0 & 0 \end{pmatrix},$$

where

$$\tilde{C}_{3\times5}(\xi)=\begin{pmatrix} -\xi_1 & -\xi_2 & -\xi_3 & -\xi_1 & 0\\ -\xi_2 & -\xi_1 & 0 & \xi_2 & -\xi_3\\ 2\xi_3 & 0 & -\xi_1 & 0 & -\xi_2 \end{pmatrix}, \tilde{D}_{5\times3}(\xi)=\begin{pmatrix} -\xi_1 & -\xi_2 & 2\xi_3\\ -\xi_2 & -\xi_1 & 0\\ -\xi_3 & 0 & -\xi_1\\ -\xi_1 & \xi_2 & 0\\ 0 & -\xi_3 & -\xi_2 \end{pmatrix}.$$

Note that $\tilde{C}_{3\times5}(\xi) = \tilde{D}^T_{5\times3}(\xi)$ for each $\xi \in \mathcal{S}^3$. Therefore, the system (2.5) is a symmetric hyperbolic parabolic system and the local existence theorem follows, see [15, 13, 19].

In the two-dimensional case $n = 2$, we only remark that one can easily check that the system can be written in a symmetric form immediately. This is different from the 3-d case, for which we needed further transformations to get a system in a symmetric form. □

Remark 2.1. *In the* isentropic *case, Yong* [28] *proved a local existence theorem by checking that the system satisfies an entropy dissipation condition. A global existence theorem is not proved. In contrast to* [28]*, our method is to write out the corresponding system explicitly for each component, see* (2.4) *and to try to find a symmetrizer explicitly, see* (2.5)*. This methods allow us to deal with the non-isentropic case and more importantly, to get the global solutions by checking the so called Kawashima condition, see Theorem 2.2 below.*

Theorem 2.2. *(Global existence) Let $s \geq s_0 + 1$ with $s_0 \geq [\frac{n}{2}] + 1$ be integers. Suppose that the initial data satisfy $(\rho_0 - 1, u_0, \theta_0 - 1, S_{10}, S_{20}) \in H^s$. Then there exists a positive constant δ such that if $\|(\rho_0 - 1, u_0, \theta_0 - 1, S_{10}, S_{20})\|_s \leq \delta$, there exists a global unique solution $(\rho, u, \theta, S_1, S_2)$ satisfying*

$$\begin{cases} (\rho - 1, u, S_1, S_2) \in C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-1}), \\ (\theta - 1) \in C([0,\infty), H^s) \cap C^1([0,\infty, H^{s-2}). \end{cases} \tag{2.6}$$

Proof. Again the interesting case is the tree-dimensional case $n = 3$.

Let $U = (\rho, u, \theta, b_{11}, a_{12}, a_{13}, b_{22}, a_{23}, S_2)$. Linearizing the system (2.5) around the steady state $\bar{U} = (\bar{\rho}, \bar{u}, \bar{\theta}, \bar{b}_{11}, \bar{a}_{12}, \bar{a}_{13}, \bar{b}_{22}, \bar{a}_{23}, \bar{S}_2) = (1, 0, 1, 0, 0, 0, 0, 0, 0)$, one gets

$$B_0(\bar{U})\partial_t U + \sum_{j=1}^{3} B_j(\bar{U})\partial_{x_j} U + \sum_{j=1}^{3}\sum_{k=1}^{3} D_{jk}(\bar{U})\partial_{x_j x_k} U + L(\bar{U})U = 0. \tag{2.7}$$

Here, $B_0(\bar{U}) = \mathrm{diag}\left\{\bar{p}_\rho, 1, 1, 1, \bar{e}_\theta, \frac{3\tau_1}{\mu}, \frac{\tau_1}{\mu}, \frac{\tau_1}{\mu}, \frac{\tau_1}{\mu}, \frac{\tau_1}{\mu}, \frac{\tau_2}{\lambda}\right\}$, $L(\bar{U}) = \mathrm{diag}\left\{0, 0, 0, 0, 0, \frac{3}{\mu}, \frac{1}{\mu}, \frac{1}{\mu}, \frac{1}{\mu}, \frac{1}{\mu}, \frac{1}{\lambda}\right\}$, $\sum_{j=1}^{3}\sum_{k=1}^{3} D_{jk}(\bar{U})\xi_j\xi_k = \mathrm{diag}\{0, 0, 0, 0, \kappa, 0, 0, 0, 0, 0, 0\}$ and

$$\sum_{j=1}^{3} B_j(\bar{U})\xi_j = \begin{pmatrix} 0 & \bar{p}_\rho\xi & 0 & 0_{1\times5} & 0 \\ \bar{p}_\rho\xi^T & 0_{3\times3} & \bar{p}_\theta\xi^T & A_{3\times5}(\xi) & \xi^T \\ 0 & \bar{p}_\theta\xi & 0 & 0_{1\times5} & 0 \\ 0_{5\times1} & A^T_{3\times5}(\xi) & 0 & 0_{5\times5} & 0 \\ 0 & \xi & 0 & 0_{1\times5} & 0 \end{pmatrix},$$

where $\bar{p}_\rho := p_\rho(1,1), \bar{e}_\theta := e_\theta(1,1)$ and

$$A_{3\times5}(\xi) = \begin{pmatrix} -\xi_1 & -\xi_2 & -\xi_3 & -\xi_1 & 0 \\ -\xi_2 & -\xi_1 & 0 & \xi_2 & -\xi_3 \\ 2\xi_3 & 0 & -\xi_1 & 0 & -\xi_2 \end{pmatrix}.$$

Define

$$\sum_{j=1}^{3} K_j\xi_j = \alpha \begin{pmatrix} 0 & \bar{c}^2\xi & 0 & 0 & 0 \\ -\xi^T & 0 & 0 & P_{3\times5} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -(PM)^T_{5\times3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.8}$$

where $\bar{c} := \sqrt{p_\rho}$ and $M = \operatorname{diag}\left\{\frac{3\tau}{\mu}, \frac{\tau}{\mu}, \frac{\tau}{\mu}, \frac{\tau}{\mu}, \frac{\tau}{\mu}\right\}$. The positive parameter α and the matrix $P_{3\times 5}$ will be determined later. A simple calculation gives

$$\sum_{j=1}^{3} K_j\xi_j A_0 = \alpha \begin{pmatrix} 0 & \bar{c}^2\xi & 0 & 0 & 0 \\ -\bar{c}^2\xi^T & 0 & 0 & PM & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -(PM)^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.9}$$

which is an anti-symmetric square matrix. On the other hand, we have

$$\begin{aligned} Q &:= \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\left(K_j\xi_jA_k\xi_k + (K_j\xi_jA_k\xi_k)^T\right) + \sum_{j=1}^{3}\sum_{k=1}^{3} D_{jk}\xi_j\xi_k + L \\ &= \alpha \begin{pmatrix} \bar{c}^4 & 0 & \frac{\bar{c}^2}{2}p_\theta & \frac{\bar{c}^2}{2}\xi(A-PM) & \frac{\bar{c}^2}{2} \\ 0 & \frac{1}{2}(PA^T+AP^T)-\bar{c}^2\xi^T\xi & 0 & 0 & 0 \\ \frac{\bar{c}^2}{2}p_\theta & 0 & \frac{\kappa}{\alpha} & -\frac{p_\theta}{2}\xi PM & 0 \\ \frac{\bar{c}^2}{2}(A^T-(PM)^T)\xi^T & 0 & -\frac{p_\theta}{2}(PM)^T\xi^T & \frac{1}{\alpha}J-\frac{1}{2}((PM)^TA+A^TPM) & -\frac{1}{2}(PM)^T\xi^T \\ \frac{\bar{c}^2}{2} & 0 & 0 & -\frac{1}{2}\xi(PM) & \frac{1}{\alpha\lambda} \end{pmatrix}, \end{aligned}$$

where $J = \operatorname{diag}\left\{\frac{3}{\mu}, \frac{1}{\mu}, \frac{1}{\mu}, \frac{1}{\mu}, \frac{1}{\mu}\right\}$. We need to show that the matrix Q is a symmetric positive definite matrix in order to explore the theory of symmetric hyperbolic parabolic system, see [13, 25]. Let $\eta = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)$ where $\eta_1, \eta_3, \eta_5 \in \mathbb{R}^1$ and $\eta_3 \in \mathbb{R}^3, \eta_4 \in \mathbb{R}^5$. Then we have

$$\begin{aligned} \eta Q \eta^T &= \left[\bar{c}^4\eta_1 + \frac{1}{2}\bar{c}^2p_\theta\eta_3 + \frac{1}{2}\bar{c}^2\eta_4(A^T\xi^T - (PM)^T\xi^T) + \frac{1}{2}\bar{c}^2\eta_5\right]\eta_1 \\ &+ \left[\eta_2(\frac{1}{2}(PA^T + AP^T) - \bar{c}^2\xi^T\xi)\right]\eta_2^T \\ &+ \left[\frac{1}{2}\bar{c}^2p_\theta\eta_1 + \frac{1}{\alpha}\kappa\eta_3 - \frac{p_\theta}{2}\eta_4(PM)^T\xi^T\right]\eta_3 \\ &+ \left[\frac{1}{2}\bar{c}^2\eta_1(\xi A - \xi PM) - \frac{p_\theta}{2}\eta_3\xi PM + \eta_4\left(\frac{1}{\alpha}J - \frac{1}{2}((PM)^TA + A^TPM)\right) - \frac{1}{2}\eta_5\xi PM\right]\eta_4^T \\ &+ \left[\frac{1}{2}\bar{c}^2\eta_1 - \frac{1}{2}\eta_4(PM)^T\xi^T + \frac{1}{\alpha\lambda}\eta_5\right]\eta_5 \\ &= \bar{c}^4\eta_1^2 + \frac{\kappa}{\alpha}\eta_3^2 + \eta_4\left(\frac{1}{\alpha}J - \frac{1}{2}((PM)^TA + A^TPM)\right)\eta_4^T + \frac{1}{\alpha\lambda}\eta_5^2 \\ &+ \bar{c}^2p_\theta\eta_1\eta_3 + \bar{c}^2\eta_4(A^T\xi^T - (PM)^T\xi^T)\eta_1 + \bar{c}^2\eta_1\eta_5 - p_\theta\eta_4(PM)^T\xi^T\eta_3 - \eta_5\xi PM\eta_4^T \\ &+ \eta_2\left(\frac{1}{2}(PA^T + AP^T) - \bar{c}^2\xi^T\xi\right)\eta_2^T. \end{aligned}$$

From the above formula, by choosing α sufficiently small, we see that the positive definiteness of Q is equivalent to the positive definiteness of $\frac{1}{2}(PA^T + AP^T) - \bar{c}^2\xi^T\xi$. Therefore, our aim is to choose P such that $\frac{1}{2}(PA^T + AP^T) - \bar{c}^2\xi^T\xi$ is a positive definite matrix for each $\xi \in S^3$. Let

$$P = \bar{c}^2\begin{pmatrix} -\xi_1 & -\xi_2 & -\xi_3 & -\xi_1 & 0 \\ -\xi_2 & -\xi_1 & 0 & \xi_2 & -\xi_3 \\ \xi_3 & 0 & -\xi_1 & 0 & -\xi_2 \end{pmatrix}.$$

One easily calculates

$$\frac{1}{2}(PA^T + AP^T) - \bar{c}^2\xi^T\xi = \bar{c}^2 \begin{pmatrix} 1 & 0 & -\frac{3}{2}\xi_1\xi_3 \\ 0 & 1 & -\frac{3}{2}\xi_2\xi_3 \\ -\frac{3}{2}\xi_1\xi_3 & -\frac{3}{2}\xi_2\xi_3 & 1 \end{pmatrix}.$$

Note that the first and second leading principal minors of the above matrix is 1. The third leading principal minors is

$$1 - \frac{9}{4}\xi_2^2\xi_3^2 - \frac{9}{4}\xi_1^2\xi_3^2 = 1 - \frac{9}{4}\xi_3^2(1-\xi_3^2).$$

Define a function $f(x) = 1 - \frac{9}{4}x(1-x), 0 \le x \le 1$. It is not difficult to see that $\min_{0\le x\le 1} f(x) = f\left(\frac{1}{2}\right) = \frac{7}{16} > 0$. Therefore, the matrix $\frac{1}{2}(PA^T + AP^T) - \bar{c}^2\xi^T\xi$ is is a positive definite matrix for each $\xi \in S^3$. So, Kawashima's condition follows and the proof is completed. □

Remark 2.2. *We note that a smallness condition on the L^p-norm of the initial data is not necessary since there are no quadratic terms of the type $|(\rho-1, u, \theta-1, S_1, S_2)|^2$ in our system, see* [13]*. In fact, one can see that in our system* (2.2) *the nonlinear terms appear in the form $U \cdot \nabla U$. We also have that the conditions $n \ge 3$ and $s \ge s_0 + 2$ there are changed into $n \ge 2$ and $s \ge s_0 + 1$ here since there are no quadratic terms of this type.*

Remark 2.3. *Kawashima's results also imply decay properties of the solutions, that is,*

$$\|(\rho-1, u, \theta-1, S_1, S_2)\|_{s-(s_0+1)} \to 0, \qquad \text{as} \quad t \to \infty.$$

Moreover, for $n = 3$, if we further assume $s \ge s_0 + 2$ and $\|(\rho-1, u, \theta-1, S_1, S_2)\|_{L^p} \le \delta$ where $p \in [1, \frac{3}{2}]$, then the solutions have the following decay

$$\|(\rho-1, u, \theta-1, S_1, S_2)\|_{s-1} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}\|(\rho_0-1, u_0, \theta_0-1, S_{10}, S_{20})\|_{s-1,p},$$

where the constant C is neither depending on t nor on the data.

3. Convergence Results

In this part, we show the compatibility of the revised Maxwell law with the Newtonian law. This has been done for a similar singular limit in the isentropic case in [28], and for a singular limit for compressible Navier-Stokes equations with hyperbolic heat conduction in [7]. There and here, the energy method combined with sophisticated estimates of the nonlinear terms is used.

For simplicity, we assume $\tau_1 = \tau_2 \equiv \tau$. We shall show the uniform convergence of the system (1.4) to the classical compressible Navier-Stokes system as τ go to zero. To this end, we need the following natural compatibility condition on the initial data, that is we assume

$$S_{10} = \mu(\nabla u_0 + (\nabla u_0)^T - \frac{2}{n}\mathrm{div}u_0\, I_n),\ S_{20} = \lambda \mathrm{div}u_0. \tag{3.1}$$

Denote by $(\rho^\tau, u^\tau, \theta^\tau, S_1^\tau, S_2^\tau)$ the solutions given by Theorem 2.1 with G_1 satisfying $G_0 \subset\subset G_1 \subset\subset G$. Denote

$$T_\tau = \sup\left\{T > 0, (\rho^\tau - 1, u^\tau, \theta^\tau - 1, S_1^\tau, S_2^\tau) \in C([0,T], H^s), (\rho^\tau, u^\tau, \theta^\tau, S_1^\tau, S_2^\tau) \in G_1\right\}.$$

Then we have the following theorem.

Theorem 3.1. *Let (ρ, u, θ) be a smooth solution to the classical compressible Navier-Stokes equations with $(\rho(x,0), u(x,0), \theta(x,0)) = (\rho_0, u_0, \theta_0)$ satisfying*

$$\rho \in C([0,T_*], H^{s+3}) \cap C^1([0,T_*], H^{s+2}),\ (u,\theta) \in C([0,T_*], H^{s+3}) \cap C^1([0,T_*], H^{s+1})$$

with $T_* > 0$ (finite). Then there are positive constants τ_0 and C such that for $\tau \le \tau_0$,

$$\|(\rho^\tau, u^\tau, \theta^\tau)(t,\cdot) - (\rho, u, \theta)(t,\cdot)\|_s \le C\tau \tag{3.2}$$

and

$$\|S_1^\tau(t,\cdot)-\mu\left(\nabla u+(\nabla u)^T-\frac{2}{n}\mathrm{div}uI_n\right)(t,\cdot)\|_s\le C\tau^{\frac12},\quad \|S_2^\tau(t,\cdot)-\lambda\mathrm{div}u(t,\cdot)\|_s\le C\tau^{\frac12}\tag{3.3}$$

for $t\in[0,\min\{T_*,T_\tau\})$*, where* C *does not depend on* τ*.*

Theorem 3.1 in particular implies that T_τ is independent of τ, see [7, 27, 28].

Theorem 3.2. *Under the condition of Theorem 3.1, for any* G_1 *satisfying*

$$G_0\cup\tilde G_0\subset\subset G_1\subset\subset G,$$

where $\tilde G_0=\left\{\cup(\rho,u,\theta,\mu(\nabla u+(\nabla u)^T-\frac{2}{n}\mathrm{div}uI_n),\lambda\mathrm{div}u)(x,t),\ (x,t)\in\mathbb{R}^n\times[0,T_*]\right\}$*, we have that* $T_\tau>T_*$ *holds for* $\tau>0$ *sufficiently small.*

Remark 3.1. *We note that if the initial data are sufficiently small, there exists a global solution for classical compressible Navier-Stokes equations, see* [16]*. Therefore, we can establish a convergence results for any fixed interval* $[0,T_{ex}]$ *for small data.*

Proof. (of Theorem 3.1) We introduce the variables $S_1^0:=\mu(\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div}uI_n), S_2^0:=\lambda\mathrm{div}u$ and define

$$\rho^d:=\frac{\rho^\tau-\rho}{\tau},u^d:=\frac{u^\tau-u}{\tau},\theta^d:=\frac{\theta^\tau-\theta}{\tau},S_1^d:=\frac{S_1^\tau-S_1^0}{\tau},S_2^d:=\frac{S_2^\tau-S_2^0}{\tau}.\tag{3.4}$$

Our aim is to show that, for small τ and for $t<\min\{T_*,T_\tau\}$,

$$\|(\rho^d,u^d,\theta^d)(t,\cdot)\|_s\le C,\quad \|\sqrt{\tau}(S_1^d,S_2^d)(t,\cdot)\|_s\le C,\tag{3.5}$$

where $C>0$ denotes constants not depending on τ or t. The equations for the difference variables $(\rho^d,u^d,\theta^d,S_1^d,S_2^d)$ can be written as

$$\begin{cases}\partial_t\rho^d+u^\tau\nabla\rho^d+\rho^\tau\mathrm{div}u^d=-u^d\nabla\rho-\rho^d\mathrm{div}u=:f_1,\\ \partial_tu^d+u^\tau\nabla u^d+\frac{p_\theta^\tau}{\rho^\tau}\nabla\theta^d+\frac{p_\rho^\tau}{\rho^\tau}\nabla\rho^d-\frac{1}{\rho^\tau}(\mathrm{div}(S_1^d)+\nabla S_2^d)\\ =-\frac{1}{\rho^\tau}\rho^du_t-\frac{1}{\tau\rho^\tau}\left\{(\rho^\tau u^\tau-\rho u)\nabla u+(p_\rho^\tau-p_\rho)\nabla\rho+(p_\theta^\tau-p_\theta)\nabla\theta\right\}=:f_2,\\ \partial_t\theta^d+u^\tau\nabla\theta^d+\frac{\theta^\tau p_\theta^\tau}{\rho^\tau e_\theta^\tau}\mathrm{div}u^d-\frac{1}{\rho^\tau e_\theta^\tau}\kappa\triangle\theta^d-\frac{1}{\rho^\tau e_\theta^\tau}(S_1^\tau\nabla u^d+S_1^d\nabla u+S_2^\tau\mathrm{div}u^d+S_2^d\mathrm{div}u)\\ \qquad=\frac{1}{\tau\rho^\tau e_\theta^\tau}\left\{(\rho^\tau e_\theta^\tau-\rho e_\theta)\partial_t\theta+(\rho^\tau e_\theta^\tau u^\tau-\rho e_\theta u)\nabla\theta+(\theta^\tau p_\theta^\tau-\theta p_\theta)\mathrm{div}u\right\}=:f_3,\\ \tau\partial_tS_1^d+S_1^d-\mu(\nabla u^d+(\nabla u^d)^T-\frac{2}{n}\mathrm{div}u^dI_n)=-\partial_tS_1^0=:f_4,\\ \tau\partial_tS_2^d+S_2^d-\lambda\mathrm{div}u^d=-\partial_tS_2^0=:f_5.\end{cases}\tag{3.6}$$

Here, we note that the expression for f_3 is different from that in our previous paper [7], f_3 there does not include the term "$\frac{1}{\rho^\tau e_\theta^\tau}(S_1^\tau\nabla u^d+\cdots)$". This is due to the fact that the velocity u^d in our case does not have enough dissipation compared to that in [7] and the term $\|\nabla^{s+1}u^d\|$ can not be controlled. Instead, we shall use the dissipation of θ to control such terms in the following estimates. Now we define

$$E:=\sup_{0\le t\le T}\|(\rho,u,\theta)\|_{s+3}+\sup_{0\le t\le T}\|\rho_t\|_{s+2}+\sup_{0\le t\le T}\|(u_t,\theta_t)\|_{s+1}$$

and

$$E^d:=\sup_{0\le t\le T}\|(\rho^d,u^d,\theta^d,\sqrt{\tau}S_1^d,\sqrt{\tau}S_2^d)\|_s.$$

Note that

$$E\le C\tag{3.7}$$

and

$$\|(\rho^\tau,u^\tau,\theta^\tau)\|_s\le C+\tau E^d,\ \|(S_1^\tau,S_2^\tau)\|_s\le C+\sqrt{\tau}E^d.\tag{3.8}$$

Here and in the sequel we often use the Moser-type inequalities from Lemma 4.1 in the Appendix. We also need the following two lemmas in order to continue the proof of Theorem 3.1. We will use the letter C to denote various positive constants.

Lemma 3.3. *For $0 \le |\alpha| \le s$, we have the following estimates*

$$\|\nabla^\alpha f_1\| \le CE^d, \|\nabla^\alpha f_2\| \le C(E^d + \tau(E^d)^2), \|\nabla^\alpha f_3\| \le C(E^d + \tau(E^d)^2). \tag{3.9}$$

Proof. The proof of Lemma 3.3 can be found in our previous paper [7], we recall it here for completeness. First, by Sobolev's imbedding theorem and the Moser-type inequalities, using (3.7), we have

$$\begin{aligned}&\|\nabla^\alpha f_1\| = \|\nabla^\alpha(-u^d\nabla\rho - \rho^d \mathrm{div} u\| \\ &\le \|\nabla\rho\|_{L^\infty}\|\nabla^\alpha u^d\| + \|u^d\|_{L^\infty}\|\nabla^{\alpha+1}\rho\| + \|\mathrm{div} u\|_{L^\infty}\|\nabla^\alpha \rho^d\| + \|\rho^d\|_{L^\infty}\|\nabla^{\alpha+1}u\| \le CE^d.\end{aligned}$$

Remember that both (ρ, u, θ) and $(\rho^\tau, u^\tau, \theta^\tau, S_1^\tau, S_2^\tau)$ take values in a convex compact subset of the state space, we have

$$\begin{aligned}&\left\|\nabla^\alpha(-\frac{1}{\rho^\tau}\rho^d u_t)\right\| \\ &\le \|u_t\|_{L^\infty}\left\|\nabla^\alpha(\frac{\rho^d}{\rho^\tau})\right\| + \left\|\frac{\rho^d}{\rho^\tau}\right\|_{L^\infty}\|\nabla^\alpha u_t\| \\ &\le CE^d + C\|\rho^d\|_{L^\infty}\|\nabla^\alpha\rho^\tau\| \le C(E^d + \tau(E^d)^2).\end{aligned}$$

Similarly, we have

$$\left\|\nabla^\alpha\left(\frac{1}{\tau\rho^\tau}(\rho^\tau u^\tau - \rho u)\nabla u\right)\right\| \le C(E^d + \tau(E^d)^2).$$

Recalling that $p(\rho, \theta)$ is a smooth function of (ρ, θ) and using the mean value theorem, we obtain

$$\begin{aligned}&\left\|\nabla^\alpha\left(\frac{1}{\tau\rho^\tau}\left((p_\rho^\tau - p_\rho)\nabla\rho + (p_\theta^\tau - p_\theta)\nabla\theta\right)\right)\right\| \\ &\le C\left\|\nabla^\alpha\left(\frac{1}{\rho^\tau}(\rho^d + \theta^d)(\nabla\rho + \nabla\theta)\right)\right\| \le C(E^d + \tau(E^d)^2).\end{aligned}$$

By assumption A.2 and using the mean value theorem, we have

$$\begin{aligned}&\left\|\nabla^\alpha\left(\frac{1}{\tau\rho^\tau e_\theta^\tau}(\rho^\tau e_\theta^\tau - \rho e_\theta)\theta_t\right)\right\| \\ &\le \left\|\nabla^\alpha\left(\frac{1}{\tau e_\theta^\tau}(e_\theta^\tau - e_\theta)\right)\right\| + \left\|\nabla^\alpha\left(\frac{\rho^d}{\rho^\tau e_\theta^\tau}e_\theta\theta_t\right)\right\| \\ &\le C(E^d + \tau(E^d)^2),\end{aligned}$$

where we used the fact that

$$\|\nabla^\alpha(\rho^\tau e_\theta^\tau)\| \le \|\rho^\tau\|_{L^\infty}\|\nabla^\alpha e_\theta^\tau\| + \|e_\theta^\tau\|_{L^\infty}\|\nabla\rho^\tau\| \le C + \tau E^d.$$

Similarly, we get

$$\left\|\nabla^\alpha\left(\frac{1}{\tau\rho^\tau e_\theta^\tau}(\theta^\tau p_\theta^\tau - \theta p_\theta)\mathrm{div} u\right)\right\| \le C(E^d + \tau(E^d)^2)$$

and

$$\left\|\nabla^\alpha\left(\frac{1}{\tau\rho^\tau e_\theta^\tau}(\rho^\tau e_\theta^\tau u^\tau - \rho e_\theta u)\nabla\theta\right)\right\| \le C(E^d + \tau(E^d)^2).$$

This completes the proof of Lemma 3.3. □

Lemma 3.4. *We have for $\tau \le 1$ that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(E^d)^2 \le C(1+(E^d)^2+\tau(E^d)^4). \tag{3.10}$$

Proof. Applying ∇^α to the equations (3.6) and multiplying the result by $\frac{p_\rho^\tau}{\rho^\tau}\nabla^\alpha\rho^d$, $\rho^\tau\nabla^\alpha u^d$, $\frac{\rho^\tau e_\theta^\tau}{\theta^\tau}\nabla^\alpha\theta^d$, $\frac{1}{2\mu}\nabla^\alpha S_1^d$, $\frac{1}{\lambda}\nabla^\alpha S_2^d$, respectively, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left\{\frac{p_\rho^\tau}{\rho^\tau}(\nabla^\alpha\rho^d)^2+\rho^\tau(\nabla^\alpha u^d)^2+\frac{\rho^\tau e_\theta^\tau}{\theta^\tau}(\nabla^\alpha\theta^d)^2+\frac{\tau}{2\mu}(\nabla^\alpha S_1^d)^2+\frac{\tau}{\lambda}(\nabla^\alpha S_2^d)^2\right\}\mathrm{d}x\\
&+\int\left\{\frac{\kappa}{\theta^\tau}(\nabla^{\alpha+1}\theta^d)^2+\frac{1}{2\lambda}(\nabla^\alpha S_1^d)^2+\frac{1}{\mu}(\nabla^\alpha S_2^d)^2\right\}\mathrm{d}x\\
&\le\sum_{i=1}^5 F_i+\sum_{i=1}^3 T_i+\sum_{i=1}^{10}G_i+D+N,
\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}
&F_1=\int\nabla^\alpha f_1\frac{p_\rho^\tau}{\rho^\tau}\nabla^\alpha\rho^d\mathrm{d}x,\ F_2=\int\nabla^\alpha f_2\rho^\tau\nabla^\alpha u^d\mathrm{d}x,\\
&F_3=\int\nabla^\alpha f_3\frac{\rho^\tau e_\theta^\tau}{\theta^\tau}\nabla^\alpha\theta^d\mathrm{d}x, F_4=\int\nabla^\alpha f_4\nabla^\alpha S_1^d\mathrm{d}x, F_5=\int\nabla^\alpha f_5\nabla^\alpha S_2^d\mathrm{d}x\\
&T_1=\int\left(\frac{p_\rho^\tau}{\rho^\tau}\right)_t(\nabla^\alpha\rho^d)^2\mathrm{d}x,\ T_2=\int\rho_t^\tau(\nabla^\alpha u^d)^2\mathrm{d}x, T_3=\int\left(\frac{\rho^\tau e_\theta^\tau}{\theta^\tau}\right)_t(\nabla^\alpha\theta^d)^2\mathrm{d}x,\\
&D=\int\left\{\left(\nabla^\alpha(\frac{\kappa}{\rho^\tau e_\theta^\tau}\triangle\theta^d)-\frac{\kappa}{\rho^\tau e_\theta^\tau}\nabla^\alpha(\triangle\theta^d)\right)\frac{\rho^\tau e_\theta^\tau}{\theta^\tau}\nabla^\alpha\theta^d+\nabla(\frac{\kappa}{\theta^\tau})\nabla^{\alpha+1}\theta^d\nabla^\alpha\theta^d\right\}\mathrm{d}x,\\
&N=\int\nabla^\alpha\left(\frac{1}{\rho^\tau e_\theta^\tau}(S_1^\tau\nabla u^d+S_1^d\nabla u+S_2^\tau\mathrm{div}u^d+S_1^d\mathrm{div}u)\right)\nabla^\alpha\theta\mathrm{d}x,\\
&G_1=\int\nabla^\alpha(u^\tau\nabla\rho^d)\frac{p_\rho^\tau}{\rho^\tau}\nabla^\alpha\rho^d\mathrm{d}x,\ G_2=\int\nabla^\alpha(\rho^\tau\mathrm{div}u^d)\frac{p_\rho^\tau}{\rho^\tau}\nabla^\alpha\rho^d\mathrm{d}x,\\
&G_3=\int\nabla^\alpha(u^\tau\nabla u^d)\rho^\tau\nabla^\alpha u^d\mathrm{d}x,\ G_4=\int\nabla^\alpha(\frac{p_\rho^\tau}{\rho^\tau}\nabla\rho^d)\rho^\tau\nabla^\alpha u^d\mathrm{d}x,\\
&G_5=\int\nabla^\alpha(\frac{p_\theta^\tau}{\rho^\tau}\nabla\theta^d)\rho^\tau\nabla^\alpha u^d\mathrm{d}x,\ G_6=\int\nabla^\alpha\left(\frac{1}{\rho^\tau}(\mathrm{div}S_1^d+\nabla S_2^d)\right)\rho^\tau\nabla^\alpha u^d\mathrm{d}x,\\
&G_7=\int\nabla^\alpha(u^\tau\nabla\theta^d)\frac{\rho^\tau e_\theta^\tau}{\theta^\tau}\nabla^\alpha\theta^d\mathrm{d}x, G_8=\int\nabla^\alpha\left(\frac{\theta^\tau p_\theta^\tau}{\rho^\tau e_\theta^\tau}\mathrm{div}u^d\right)\frac{\rho^\tau e_\theta^\tau}{\theta^\tau}\nabla^\alpha\theta^d\mathrm{d}x,\\
&G_9=\int\frac{1}{2}\nabla^\alpha\left(\nabla u^d+(\nabla u^d)^T-\frac{2}{n}\mathrm{div}u^dI_n\right)\nabla^\alpha S_1^d\mathrm{d}x,\ G_{10}=\int\nabla^\alpha(\mathrm{div}u^d)\nabla^\alpha S_2^d\mathrm{d}x.
\end{aligned}$$

In the sequel, we keep in mind that some inequalities such as the Cauchy-Schwarz inequality, the Hölder inequality or the Moser-type inequalities will be frequently used without being mentioned explicitly (for exemplarily detailed estimates see (3.12) - (3.16) below). From Lemma 3.3 we know that

$$F_i\le C((E^d)^2+\tau(E^d)^3),$$

for each $i=1,2,3$ and $F_4+F_5\le C(\varepsilon)+\varepsilon(\|\nabla^\alpha S_1^d\|^2+\|\nabla^\alpha S_2^d\|^2)$ (for $\varepsilon>0$, with $C(\varepsilon)$ at most depending on ε; here we use the fact that $\partial_t(S_1^0)=\mu(\nabla u_t+(\nabla u_t)^T-\frac{2}{3}\mathrm{div}u_tI)$, $\partial_tS_2^0=\lambda\mathrm{div}u_t$

and $\|u_t\|_{s+1} \leq C$). Moreover, we have

$$
\begin{aligned}
|D| \leq & \left\| \nabla^\alpha \left(\frac{\kappa}{\rho^\tau e_\theta^\tau} \triangle \theta^d \right) - \frac{\kappa}{\rho^\tau e_\theta^\tau} \nabla^\alpha (\triangle \theta^d) \right\| \left\| \frac{\rho^\tau e_\theta^\tau}{\theta^\tau} \nabla^\alpha \theta^d \right\| \\
& + \left\| \nabla \left(\frac{\kappa}{\theta^\tau} \right) \right\|_{L^\infty} \|\nabla^{\alpha+1} \theta^d\| \|\nabla^\alpha \theta^d\| \\
\leq & C \left(\left\| \nabla \left(\frac{\kappa}{\rho^\tau e_\theta^\tau} \right) \right\|_{L^\infty} \|\nabla^{\alpha-1} \Delta \theta^d\| + \|\Delta \theta^d\|_{L^\infty} \left\| \nabla^\alpha \left(\frac{\kappa}{\rho^\tau e_\theta^\tau} \right) \right\| \right) \left\| \frac{\rho^\tau e_\theta^\tau}{\theta^\tau} \nabla^\alpha \theta^d \right\| \\
& \qquad \text{(Moser inequalities Lemma 4.1 (ii))} \\
& + \left[\varepsilon \|\nabla^{\alpha+1} \theta^d\|^2 + C(\varepsilon) \left(\left\| \nabla \left(\frac{\kappa}{\theta^\tau} \right) \right\|_{L^\infty}^2 \|\nabla^\alpha \theta^d\|^2 \right) \right] \\
\equiv & A + [B] \qquad (3.12)
\end{aligned}
$$

with

$$
B \leq \varepsilon \|\nabla^{\alpha+1} \theta^d\|^2 + C(\varepsilon) \left((E^d)^2 + \tau (E^d)^3 + \tau^2 (E^d)^4 \right) \qquad (3.13)
$$

and

$$
\begin{aligned}
A \quad \leq \quad & \left\| \nabla \left(\frac{\kappa}{\rho^\tau e_\theta^\tau} \right) \right\|_{L^\infty} \|\nabla^{\alpha+1} \theta^d\| \left\| \frac{\rho^\tau e_\theta^\tau}{\theta^\tau} \nabla^\alpha \theta^d \right\| \\
& + \|\Delta \theta^d\|_{L^\infty} \left\| \nabla^\alpha \left(\frac{\kappa}{\rho^\tau e_\theta^\tau} \right) \right\| \left\| \frac{\rho^\tau e_\theta^\tau}{\theta^\tau} \nabla^\alpha \theta^d \right\| \\
\equiv \quad & A_1 + A_2. \qquad (3.14)
\end{aligned}
$$

We have

$$
A_1 \leq \varepsilon \|\nabla^{\alpha+1} \theta^d\|^2 + C(\varepsilon) \left((E^d)^2 + \tau (E^d)^3 + \tau^2 (E^d)^4 \right) \qquad (3.15)
$$

and

$$
A_2 \leq C (E^d)^2 \left\| \nabla^\alpha \left(\frac{\kappa}{\rho^\tau e_\theta^\tau} \right) \right\| \leq C (E^d)^2 (1 + \tau E^d), \qquad (3.16)
$$

where we used

$$
\left\| \nabla^\alpha \left(\frac{1}{\rho^\tau} \right) \right\| \leq C \frac{1}{\|\rho^\tau\|_{L^\infty}^2} \|\nabla^\alpha \rho^\tau\| \leq C \|\rho^\tau\|_s
$$

which follows from the Moser-type inequalities Lemma 4.1 (i). Summarizing (3.12) - (3.16) we have

$$
|D| \leq \varepsilon \|\nabla^{\alpha+1} \theta^d\|^2 + C(\varepsilon) \left((E^d)^2 + \tau (E^d)^3 + \tau^2 (E^d)^4 \right). \qquad (3.17)
$$

The term N can be divided into two terms:

$$
\begin{aligned}
N = & \int \nabla^\alpha \left(\frac{1}{\rho^\tau e_\theta^\tau} (S_1^\tau \nabla u^d + S_1^d \nabla u + S_2^\tau \mathrm{div} u^d + S_1^d \mathrm{div} u) \right) \nabla^\alpha \theta \mathrm{d}x \\
= & \int \nabla^\alpha \left(\frac{1}{\rho^\tau e_\theta^\tau} (S_1^\tau \nabla u^d + S_2^\tau \mathrm{div} u^d) \right) \nabla^\alpha \theta \mathrm{d}x + \int \nabla^\alpha \left(\frac{1}{\rho^\tau e_\theta^\tau} (S_1^d \nabla u + S_2^d \mathrm{div} u) \right) \nabla^\alpha \theta \mathrm{d}x \\
=: & N_1 + N_2.
\end{aligned}
$$

We estimate the term N_1 as follows: for $\alpha = 0$, we have

$$
\begin{aligned}
|N_1| = & \left| \int \frac{1}{\rho^\tau e_\theta^\tau} \left(S_1^\tau \nabla u^d + S_2^\tau \mathrm{div} u^d \right) \theta^d \mathrm{d}x \right| \\
\leq & \|(\rho^\tau, \theta^\tau, S_1^\tau, S_2^\tau)\|_{L^\infty} \|\nabla u^d\| \|\theta^d\| \leq C((E^d)^2 + \sqrt{\tau} (E^d)^3);
\end{aligned}
$$

for $1 \le \alpha \le s$, we have for $\tau \le 1$ that

$$\begin{aligned}|N_1| &= \left|\int \nabla^{\alpha-1}\left(\frac{1}{\rho^\tau e_\theta^\tau}(S_1^\tau \nabla u^d + S_2^\tau \mathrm{div} u^d)\right)\nabla^{\alpha+1}\theta^d \mathrm{d}x\right| \\ &\le \left(\|(\rho^\tau,\theta^\tau,S_1^\tau,S_2^\tau)\|_{L^\infty}\|\nabla^\alpha u^d\| + \|\nabla u^d\|_{L^\infty}\|(\nabla^{\alpha-1}(\rho^\tau,\theta^\tau,S_1^\tau,S_2^\tau)\|\right)\|\nabla^{\alpha+1}\theta^d\| \\ &\le \varepsilon\|\nabla^{\alpha+1}\theta^d\|^2 + C(\varepsilon)\left((E^d)^2 + \sqrt{\tau}(E^d)^3 + \tau(E^d)^4\right).\end{aligned}$$

For the term N_2, we can get

$$\begin{aligned}|N_2| &\le \left(\|(S_1^d,S_2^d)\|_{L^\infty}\|\nabla^\alpha(\frac{1}{\rho^\tau e_\theta^\tau}\nabla u)\| + \|\frac{1}{\rho^\tau e_\theta^\tau}\nabla u\|_{L^\infty}\|\nabla^\alpha(S_1^d,S_2^d)\|\right)\|\nabla^\alpha\theta^d\| \\ &\le \varepsilon\|(S_1^d,S_2^d)\|_s^2 + C(\varepsilon)\left((E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4\right).\end{aligned}$$

Therefore, we obtain

$$|N| \le \varepsilon(\|\nabla^{\alpha+1}\theta^d\|^2 + \|(S_1^d,S_2^d)\|_s^2) + C(\varepsilon)((E^d)^2 + \tau(E^d)^4).$$

Now, we estimate G_i for each i.

$$\begin{aligned}|G_1| &= \left|\int\left\{(\nabla^\alpha(u^\tau\nabla\rho^d) - u^\tau\nabla^{\alpha+1}\rho^d)\frac{p_\rho^\tau}{\rho^\tau}\nabla^\alpha\rho^d + \frac{u^\tau p_\rho^\tau}{\rho^\tau}\nabla^{\alpha+1}\rho^d\nabla^\alpha\rho^d\right\}\mathrm{d}x\right| \\ &\le C(\|\nabla\rho^d\|_{L^\infty}\|\nabla^\alpha u^\tau\| + \|\nabla u^\tau\|_{L^\infty}\|\nabla^\alpha\rho^d\|)\|\nabla^\alpha\rho^d\| + \left\|\nabla\left(\frac{u^\tau p_\rho^\tau}{\rho^\tau}\right)\right\|_{L^\infty}\|\nabla^\alpha\rho^d\|^2 \\ &\le C((E^d)^2 + \tau(E^d)^3).\end{aligned}$$

G_3 and G_7 can be estimated in the same way.

$$\begin{aligned}&|G_2+G_4| \\ &= \left|\int(\nabla^\alpha(\rho^\tau\mathrm{div}u^d) - \rho^\tau\mathrm{div}\nabla^\alpha u^d)\frac{p_\rho^\tau}{\rho^\tau}\nabla^\alpha\rho^d\mathrm{d}x\right. \\ &\qquad \left. + \int\left\{\nabla^\alpha(\frac{p_\rho^\tau}{\rho^\tau}\nabla\rho^d)\rho^\tau\nabla^\alpha u^d - p_\rho^\tau\nabla^{\alpha+1}\rho^d\nabla^\alpha u^d\right\}\mathrm{d}x - \int\nabla p_\rho^\tau\nabla^\alpha u^d\nabla^\alpha\rho^d\mathrm{d}x\right| \\ &\le C\|(\mathrm{div}u^d,\nabla\rho^d)\|_{L^\infty}(\|\nabla^\alpha\rho^\tau\| + \left\|\nabla(\frac{p_\rho^\tau}{\rho^\tau})\right\|)(\|\nabla^\alpha\rho^d\| + \|\nabla^\alpha u^d\|) \\ &+ \left\|(\nabla\rho^d,\nabla p_\rho^\tau,\frac{p_\rho^\tau}{\rho^\tau})\right\|_{L^\infty}\|\nabla^\alpha u^d\|\|\nabla^\alpha\rho^d\| \le C((E^d)^2 + \tau(E^d)^3).\end{aligned}$$

$G_5 + G_8$ can also be estimated similarly, while

$$\begin{aligned}&|G_6+G_9+G_{10}| \\ &= \left|\int\left\{\nabla^\alpha\left(\frac{1}{\rho^\tau}(\mathrm{div}S_1^d + \nabla S_2^d)\right) - \frac{1}{\rho^\tau}\left(\mathrm{div}\nabla^\alpha S_1^d + \nabla^{\alpha+1}S_2^d\right)\right\}\rho^\tau\nabla^\alpha u^d\mathrm{d}x\right| \\ &\le \|\nabla(S_1^d,S_2^d)\|_{L^\infty}\|\nabla^\alpha\rho^\tau\|\|\nabla^\alpha u^d\| + \|\nabla\rho^\tau\|_{L^\infty}\|\nabla^\alpha(S_1^d,S_2^d)\|\|\nabla^\alpha u^d\| \\ &\le \varepsilon\|(S_1^d,S_2^d)\|_s^2 + C(\varepsilon)((E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4),\end{aligned}$$

where we used the cancelation relations

$$\int(\nabla^\alpha\mathrm{div}S_1^d)\nabla^\alpha u^d\mathrm{d}x = -\int\frac{1}{2}\nabla^\alpha(\nabla u^d + (\nabla u^d)^T - \frac{2}{n}\mathrm{div}u^d I_n)\nabla^\alpha S_1^d\mathrm{d}x$$

and

$$\int\nabla^\alpha(\nabla S_2^d)\nabla^\alpha u^d\mathrm{d}x = -\int\nabla^\alpha(\mathrm{div}u^d)\nabla^\alpha S_2^d\mathrm{d}x,$$

which can be easily shown to hold by doing partial integration and using the fact that S_1^d is a symmetric and traceless matrix. Such cancelation relations are essential in our estimates which implies the necessity of dividing the stress tensor S into S_1 and $S_2 I_n$ and do relaxation for S_1 and S_2, respectively. In fact, it is not difficult to see that if we only do a relaxation for S, then the cancelation relation fails and the validity of the estimates will be destroyed.

For the terms $T_i, i = 1, 2, 3$, we have

$$\begin{aligned}|T_i| &\le \|(\rho_t^\tau, \theta_t^\tau)\|_{L^\infty}(E^d)^2 \le C(1 + \tau\|(\rho_t^d, \theta_t^d)\|_{L^\infty})(E^d)^2 \\ &\le C(1 + \tau(E^d + \tau(E^d)^2 + \|(S_1^d, S_2^d)\|_s))(E^d)^2 \\ &\le C(\varepsilon)((E^d)^2 + \tau(E^d)^3 + \tau^2(E^d)^4) + \varepsilon\|(S_1^d, S_2^d)\|_s^2.\end{aligned}$$

By choosing ε sufficiently small, using assumption A.2 and summing α for 0 to s, we conclude for $\tau \le 1$ that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(E^d)^2 + \|\nabla\theta^d\|_s^2 + \|(S_1^d, S_2^d)\|_s^2 \\ &\le C(1 + (E^d)^2 + \sqrt{\tau}(E^d)^3 + \tau(E^d)^4) \le C(1 + (E^d)^2 + \tau(E^d)^4).\end{aligned}$$

Therefore, (3.10) holds and this completes the proof of Lemma 3.4. □

Using Lemma 3.4, we can finally show that E^d is uniformly bounded. Let $g := (E^d)^2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} g \le C(1 + g + \tau g^2), \tag{3.18}$$

We assume a priori that

$$g \le e^{2CT} - 1. \tag{3.19}$$

We will show that $g \le \frac{1}{2}(e^{2CT} - 1)$ holds if we choose τ sufficiently small. This justifies the a priori estimate (3.19) and thus proves our result. In fact, if $\tau \le \frac{1}{e^{2CT}-1}$, we get $g \le \frac{1}{\tau}$ and thus $\tau g^2 \le g$. Hence, the inequality (3.18) turns into

$$\frac{\mathrm{d}}{\mathrm{d}t} g \le C(1 + 2g).$$

Solving the above inequality, we immediately get $g \le \frac{1}{2}(e^{2CT} - 1)$. This finishes the proof of the main Theorem 3.1. □

4. Appendix

The following Moser-type inequalities have been used in Section 3 and can be found as a standard tool for example in [15, 19].

Lemma 4.1. *(i) Let $r, m, n \in \mathbb{N}$, $1 < p \le \infty$, $h \in C^r(\mathbb{R}^m)$, $B := \|h\|_{C^r(\overline{B(0,1)})}$. Then there is a constant $c = c(r, m, n, p) > 0$ such that for all $w = (w_1, \ldots, w_m) \in W^{r,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $\|w\|_{L^\infty} \le 1$ the inequality*

$$\|\nabla^r h(w)\|_{L^p} \le c\, B\, \|\nabla^r w\|_{L^p} \tag{4.1}$$

holds.

(ii) Let $m \in \mathbb{N}$. Then there is a constant $c = c(m, n) > 0$ such that for all $f, g \in W^{m,2} \cap L^\infty$ and $\alpha \in \mathbb{N}_0^n, |\alpha| \le m$, the following inequalities hold:

$$\|\nabla^\alpha(fg)\|_2 \leq c(\|f\|_{L^\infty}\|\nabla^m g\|_2 + \|\nabla^m f\|_2\|g\|_{L^\infty}), \tag{4.2}$$

$$\|\nabla^\alpha(fg) - f\nabla^\alpha g\|_2 \leq c(\|\nabla f\|_{L^\infty}\|\nabla^{m-1} g\|_2 + \|\nabla^m f\|_2\|g\|_{L^\infty}). \tag{4.3}$$

References

[1] D. Chakraborty and J.E. Sader, Constitutive models for linear compressible viscoelastic flows of simple liquids at nanometer length scales, *Physics of Fluids* **27** (2015), 052002-1–052002-13.

[2] Y. Cho and B.J. Jin, Blow-up of viscous heat-conducting compressible flows, *J. Math. Anal. Appl.* **320** (2) (2006), 819-826.

[3] H.J. Choe and H. Kim, Strong solutions of the Navier-Stokes equations for isentropic compressible fluids, *J. Differ. Eqs.* **190** (2003), 504-523.

[4] D. Hoff, Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data, *Trans. Amer. Math. Soc.* **303** (1) (1987), 169-181.

[5] D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differ. Eqs.* **120** (1) (1995), 215-254.

[6] E. Feireisl, A. Novotny and H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, *J. Math. Fluid Mech.* **3** (2001), 358-392.

[7] Y. Hu and R. Racke, Compressible Navier-Stokes equations with hyperbolic heat conduction, *J. Hyper. Diff. Equations* (2015), to appear.

[8] X.D. Huang, J. Li and Z.P. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations, *Comm. Pure. Appl. Math.* **65** (2012), 549-585.

[9] S. Jiang, On the asymptotic behavior of the motion of a viscous, heat-conducting, one-dimensional real gas, *Math. Z.* **216** (1994), 317-336.

[10] S. Jiang and P. Zhang, Global spherically symmetry solutions of the compressible isentropic Navier-Stokes equations, *Comm. Math. Phys.* **215** (2001), 559-581.

[11] S. Jiang and P. Zhang, Axisymmetric solutions of the 3-D Navier-Stokes equations for compressible isentropic fluids, *J. Math. Pures. Appl.* **82** (2003), 949-973.

[12] P.L. Lions, *Mathematical Topics in Fluid Mechanics*, Vol.I, Incompressible Models. Clarendon Press, Oxford (1996).

[13] S. Kawashima, Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics, Thesis, Kyoto University (1983).

[14] P.L. Lions, *Mathematical Topics in Fluid Mechanics*, Vol.II, Compressible Models. Clarendon Press, Oxford (1998).

[15] A. Majda *Compressible fluid flow and systems of conservation laws in several space variables.* Appl. Math. Sci. **53**. Springer, New York et al. (1984).

[16] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.* **20** (1) (1980), 67-104.

[17] A. Matsumura and T. Nishida, Initial Boundary Value Problems for the Equations of Motion of Compressible Viscous and Heat-Conductive Fluids, *Comm. Math. Phys.* **89** (1983), 445-464.

[18] J. Nash, Le problème de Cauchy pour les équations différentielles d'un fluide général, *Bull. Soc. Math. France* **90** (1962), 487-497.

[19] R. Racke, *Lectures on Nonlinear Evolution Equations. Initial Value Problems,* 2nd edition, Birkhäuser, Basel (2015).

[20] R. Racke and J. Saal, Hyperbolic Navier-Stokes equations I: Local well-posedness, Evolution Equations and Control Theory **1** (2012), 195-215.

[21] R. Racke and J. Saal. Hyperbolic Navier-Stokes equations II: Global existence of small solutions, *Evolution Equations and Control Theory* **1** (2012), 217-234.

[22] J. Serrin, On the uniqueness of compressible fluid motion, *Arch. Rational Mech. Anal.* **3** (1959), 271-288.

[23] A. Schöwe, A quasilinear delayed hyperbolic Navier-Stokes system: global solution, asymptotics and relaxation limit, *Meth. Appl. Anal.* **19** (2012), 99-118.

[24] A. Schöwe, Global strong solution for large data to the hyperbolic Navier-Stokes equation, *arxiv.org/abs/1409.7797* (2014).

[25] Y. Shizuta and S. Kawashima, Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.* **14** (1985), 249-275.
[26] Z.P. Xin, Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density, *Comm. Pure. Appl. Math.* **51** (1998), 229-240.
[27] W.-A. Yong, A note on the zero mach number limit of compressible euler equations, *Proc. Amer. Math. Soc.*, **133** (2005), 3079-3085.
[28] W.-A. Yong, Newtonian limit of Maxwell fluid flows, *Arch. Rational Mech. Anal.* **214** (2014) 913-922.

Yuxi Hu, Department of Mathematics, China University of Mining and Technology, Beijing, 100083, P.R. China, yxhu86@163.com

Reinhard Racke, Department of Mathematics and Statistics, University of Konstanz, 78457 Konstanz, Germany, reinhard.racke@uni-konstanz.de