Optimal Decay Rates and Global Existence for a Semilinear Timoshenko System with Two Damping Effects

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Abstract

In this paper, we study a semilinear Timoshenko system having two damping effects. The observation that two damping effects might lead to smaller decay rates for solutions in comparison to one damping effect is rigorously proved here in providing optimality results. Moreover the global well-posedness for small data in a low regularity class is presented for a larger class of nonlinearities than previously known and proved by a simpler approach.

Key words: semilinear Timoshenko system, optimal decay estimate

1 Introduction.

We consider the following one-dimensional semilinear Timoshenko system in all of $\mathbb{R}$ with two damping terms,

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x + \alpha\varphi_t = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \mu\psi_t = |\psi|^r, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (\varphi, \varphi_t, \psi, \psi_t)(0,x) = (\varphi_0, \varphi_1, \psi_0, \psi_1), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $r > 8$, with associated linearized case

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x + \alpha\varphi_t = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \mu\psi_t = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (\varphi, \varphi_t, \psi, \psi_t)(0,x) = (\varphi_0, \varphi_1, \psi_0, \psi_1), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

which is known to be the damped classical Timoshenko system. Here t denotes the time variable and x denotes the space variable. The functions φ and ψ represent the transversal displacement and the rotation angle of a beam, respectively, a, α and μ are positive constants.

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The system describes a vibrating beam with damping terms both in the rotation angle ($\mu\psi_t$) and in the transversal displacement ($\alpha\varphi_t$). This system goes back to Timoshenko [29]. He formulated it this way in a bounded region in space:

$$\begin{cases} \rho\varphi_{tt} = (K(\varphi_x - \psi))_x, & \text{in } (0,\infty)\times(0,L), \\ I_\rho\psi_{tt} = (EI\psi_x)_x + K(\varphi_x - \psi), & \text{in } (0,\infty)\times(0,L), \end{cases} \tag{1.3}$$

where the coefficients ρ, I_ρ, E, I and K are the density, the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus. The system is completed with initial conditions and with the boundary conditions

$$EI\psi_x|_{x=0}^{x=L} = 0, \qquad K(\varphi_x - \psi)|_{x=0}^{x=L} = 0.$$

It is conservative, the associated total energy of the beam remains constant in time. The question which kind of damping effects stabilize the system in an exponential or a polynomial manner has drawn a lot of attention in recent years. Stability has been discussed for Timoshenko systems with different damping terms mainly in bounded domains. Timoshenko system with frictional damping is discussed in [12, 15, 25]. The relation to the wave speeds is investigated in giving stability results related to these speeds in [6, 7, 24, 28]. For the stability of memory type Timoshenko systems we refer to [1, 4, 11, 16]. For the stability of Timoshenko systems with thermal dissipation we mention [3, 12, 13, 17]. Thermal dissipation with heat conduction models using the Cattaneo law instead of the Fourier law is considered in [3, 12].

For the Cauchy problem we have few results even for the linearized system see, for example, [21, 22, 23, 27, 31]. Introducing $U := (\varphi_x - \psi, \varphi_t, a\psi_x, \psi_t)$, the linearized system (1.2) turns into the following first-oder system,

$$\begin{cases} \partial_t U + A\partial_x U + B_\alpha U = 0, \\ U(0,x) = U_0(x), \end{cases} \tag{1.4}$$

where A is a real symmetric matrix, the matrix B_α satisfies $\mathrm{Re}\,\langle B_\alpha U, U\rangle \geq 0$, and $U_0 = (\varphi_x(0) - \psi_0, \varphi_1, a\psi_x(0), \psi_1)$. Semigroup theory gives the solution as $U(t,x) = (e^{t\Phi}U_0)(x)$, where

$$\Phi = -A\partial_x - B_\alpha.$$

If $\alpha = 0$, this is just the case studied by Ide Haramoto and Kawashima[5], they obtained the decay estimates, when $a = 1$,

$$\left\|\partial_x^k U(t)\right\|_2 \leq C(1+t)^{-1/4-k/2}\left\|U_0\right\|_1 + Ce^{-ct}\left\|\partial_x^k U_0\right\|_2,$$

and when $a \neq 1$,

$$\left\|\partial_x^k U(t)\right\|_2 \leq C(1+t)^{-1/4-k/2}\left\|U_0\right\|_1 + C(1+t)^{-l/2}\left\|\partial_x^{k+l} U_0\right\|_2,$$

where k and l are nonnegative integers, C and c are positive constants. In Racke and Said-Houari [22], decay estimates could be improved under an extra condition on the initial data. If w is an odd function, $\gamma \in [0,1]$, then, when $a = 1$,

$$\left\|\partial_x^k e^{t\Phi} w\right\|_2 \leq C(1+t)^{-1/4-k/2-\gamma/2}\left\|w\right\|_{1,\gamma} + Ce^{-ct}\left\|\partial_x^k w\right\|_2, \tag{1.5}$$

and when $a \neq 1$,

$$\left\|\partial_x^k e^{t\Phi} w\right\|_2 \leq C(1+t)^{-1/4-k/2-\gamma/2}\left\|w\right\|_{1,\gamma} + C(1+t)^{-l/2}\left\|\partial_x^{k+l} w\right\|_2, \tag{1.6}$$

where $\|v\|_i := \|v\|_{L^i(\mathbb{R})}, i = 1, 2$, $\|u\|_{1,\gamma} := \int_{\mathbb{R}}(1+|x|)^\gamma |u(x)|dx$, for $u \in L^1(\mathbb{R})$. These results imply that the system is of so-called regularity-loss type when $a \neq 1$. While Ueda, Duan and Kawashima [31] considered the Cauchy problem for a more general first-order linear symmetric-hyperbolic systems, they formulated a new structural condition extending the Kawashima-Shizuta condition, and got a similar decay result for the Timoshenko system with one damping term. However, it seems that their method cannot be carried over to our case immediately.

For the case of two damping terms (both α and μ being positive), Soufyane and Said-Houari [27] investigated the decay rates of solutions to the linearized system and found rates that are smaller than the ones known for the system with only one damping. In [27, Remark 7, p. 737] the question is raised on the optimality of these rates describing the interesting phenomenon that two damping terms might have a weaker effect than only one damping term. We shall answer this question here and prove the optimality. For this purpose we will use a detailed analysis of the low frequency behavior of the solution in Fourier space, since the decay of the solution is mainly determined by the low frequency part. This was also used, for example, in [2, 8, 9, 32].

For the semilinear system (1.1) with $\alpha = 0$, Racke and Said-Houari [22] proved a global existence theorem for $r > 12$ using the more complicated method of weighted multipliers going back to Todorova and Yordanov [30]. Here we can both improve the admissible values of r to the condition $r > 8$ as well as present a simpler proof. Additionally, decay rates for solutions also to the semilinear problem are provided. To summarize our main new contributions, we have

- a proof of the optimality of the striking result that two damping effects have a weaker effect than only one,
- a larger class of admissible nonlinearities for the semilinear problem and
- a simpler proof for the following global well-posedness result avoiding weight functions.

Theorem 1.1. *Assume $r > 8$. Then there is a constant $\delta_0 > 0$ such that if*

$$E_0^2 \equiv \|(\varphi_t, \varphi_x, \psi_t, \psi_x, \psi)(0,\cdot)\|_2^2 + \|\psi(0,\cdot)\|_1^2 < \delta_0, \tag{1.7}$$

then there exists a unique global weak solution $U := (\varphi_t, \varphi_x - \psi, \psi_t, a\psi_x)$ of (1.1). U satisfies for all $t \geq 0$:

$$\|U(t,\cdot)\|_2 \leq C\, E_0(1+t)^{-\frac{1}{8}}, \tag{1.8}$$

where the positive constant C does not depend on t or on the initial data.

The paper is organized as follows. Section 2 provides decay estimates for the linearized system. In Section 3 the optimality of the decay rates is proved. Section 4 presents the local existence result for the semilinear problem, and in Section 5 the global well-posedness result (Theorem 1.1) is proved.

We use the following notation. Let $\hat{f}$ denote the Fourier transform of f:

$$\hat{f}(\xi) \equiv \mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx,$$

and let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. For $1 \leq p \leq \infty$ let $\|\cdot\|_p$ denote the norm in the Lebesgue space $L^p(\mathbb{R})$. For $\gamma \in [0, \infty)$, the weighted function space

$L^{1,\gamma}(\mathbb{R})$ with norm $\|u\|_{1,\gamma}$ is defined as

$$L^{1,\gamma}(\mathbb{R}) := \left\{ u \in L^1(\mathbb{R}) : \left\|u\right\|_{1,\gamma} \equiv \int_{\mathbb{R}} (1+|x|)^{\gamma} |u(x)| dx < \infty \right\}.$$

The convolution of f and g is given as usual by

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y) g(y) dy.$$

The following two lemmas will be useful too. The first one can be found in [10, 20] or in [26], the second one in [18].

Lemma 1.2. *Let $\alpha > 0$ and $\beta > 0$ be given. If $\max(\alpha, \beta) > 1$, then there is a constant $C > 0$ such that for all $t \geq 0$*

$$\int_0^t (1+t-\tau)^{-\alpha} (1+\tau)^{-\beta} d\tau \leq C(1+t)^{-min(\alpha,\beta)}.$$

Lemma 1.3. *Let u and f be nonnegative continuous functions defined for $t \geq 0$. If*

$$u^2(t) \leq c^2 + 2 \int_0^t f(s) u(s) ds,$$

for $t \geq 0$, where $c \geq 0$ is a constant, then, for $t \geq 0$,

$$u(t) \leq c + \int_0^t f(s) ds.$$

2 Decay estimates for the linear system

Taking the Fourier transform of system (1.4), we have

$$\begin{cases} \hat{U}_t + i\xi A \hat{U} + B_\alpha \hat{U} = 0, \\ \hat{U}(0,x) = \hat{U}_0, \end{cases} \tag{2.1}$$

where

$$A = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix}, \quad B_\alpha = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \mu \end{pmatrix}.$$

We use [27, Theorem 7.3].

Lemma 2.1. *Let $\hat{\Phi}_\alpha(i\xi) = -(i\xi A + B_\alpha)$. Then the corresponding matrix $e^{t\hat{\Phi}_\alpha(i\xi)}$ satisfies the following estimate for any $t \geq 0$ and $\xi \in \mathbb{R}$:*

$$\left| e^{t\hat{\Phi}_\alpha(i\xi)} \right| \leq C e^{-c\rho(\xi)t}, \tag{2.2}$$

where $\rho(\xi) = \frac{\xi^4}{(1+\xi^2)^2}$, and C, c are positive constants.

Based on Lemma 2.1, we get the first decay estimates as in [27], now for any $\gamma \in [0,1]$.

Theorem 2.2. *Let $\gamma \in [0,1]$, and let $e^{t\Phi_\alpha}$ be the semigroup associated with the system* (1.2)*. Then, if w is an odd funtion, we have the following decay estimate:*

$$\left\|\partial_x^k e^{t\Phi_\alpha} w\right\|_2 \leq C(1+t)^{-1/8-k/4-\gamma/4}\left\|w\right\|_{1,\gamma} + Ce^{-ct}\left\|\partial_x^k w\right\|_2, \tag{2.3}$$

where k is nonnegative integer, and C, c are two positive constants.

Proof: Applying Plancherel's theorem and Lemma 2.1, we get,

$$\begin{aligned}\left\|\partial_x^k e^{t\Phi} w\right\|_2^2 =& \frac{1}{2\pi}\int_{\mathbb{R}} |\xi|^{2k}|e^{t\hat{\Phi}(i\xi)}\hat{w}(\xi)|^2 d\xi \\ \leq& C\int_{\mathbb{R}} |\xi|^{2k} e^{-c\rho(\xi)t}|\hat{w}(\xi)|^2 d\xi.\end{aligned}$$

With a partition in Fourier space, we have,

$$\begin{aligned}\left\|\partial_x^k e^{t\Phi_\alpha} w\right\|_2^2 \leq& C\int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\rho(\xi)t}|\hat{w}(\xi)|^2 d\xi + C\int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\rho(\xi)t}|\hat{w}(\xi)|^2 d\xi \\ =&: I_1 + I_2.\end{aligned}$$

For the high frequency part I_2 we easily have

$$I_2 \leq Ce^{-ct}\left\|\partial_x^k w\right\|_2^2. \tag{2.4}$$

For the low frequency part I_1 we can follow [22] to obtain

$$I_1 \leq C(1+t)^{-1/2-(k+\gamma)}\left\|w\right\|_{1,\gamma}^2. \tag{2.5}$$

Together this proves the Theorem. □

Remark. Without the damping term ($\alpha\phi_t$) we know from (1.5) and (1.6) that the solution will have a regularity loss for $a \neq 1$, but in our case this damping term allows for considering any a without regularity loss. On the other hand, we encounter a worse decay compared to the case $\alpha = 0$.

In the following discussion $R > 0$ will be arbitrary but chosen appropriately large in different places. In the sequel, a decompostion of functions u based on a decomposition in Fourier space is useful:

$$u = u_H + u_L, \tag{2.6}$$

where

$$\hat{u_H} := \chi_1\hat{u}, \quad \hat{u_L} := \chi_2\hat{u}, \tag{2.7}$$

with χ_1 being smooth, $0 \leq \chi_1 \leq 1$, and

$$\chi_1 = \chi_1(\xi) = \begin{cases} 1, & |\xi| \geq 2R, \\ 0, & |\xi| \leq R. \end{cases} \tag{2.8}$$

Let $\chi_2 := 1 - \chi_1$.

Lemma 2.3. *With χ_1 and χ_2 defined as above, we have the following estimate for the semigroup, for any $t \geq 0$ and a constant $C > 0$,*

$$\|\chi_1 e^{t\hat{\Phi}_\alpha(i\xi)}\|_\infty \leq Ce^{-ct}, \tag{2.9}$$

$$\|\chi_2 e^{t\hat{\Phi}_\alpha(i\xi)}\|_2 \leq C(1+t)^{-\frac{1}{8}}. \tag{2.10}$$

Proof: Lemma 2.1 gives

$$\left| e^{t\hat{\Phi}_\alpha(i\xi)} \right| \leq Ce^{-c\rho(\xi)t}, \tag{2.11}$$

where $\rho = \frac{\xi^4}{(1+\xi^2)^2}$.
For sufficiently large R we have for $|\xi| \geq R$

$$e^{-ct} \leq e^{-c\rho(\xi)t} \leq e^{-\frac{c}{4}t}. \tag{2.12}$$

Hence, we have

$$\|\chi_1 e^{t\hat{\Phi}_\alpha(i\xi)}\|_\infty \leq Ce^{-ct}. \tag{2.13}$$

Moreover, Lemma 2.1 implies

$$\|\chi_2 e^{t\hat{\Phi}_\alpha(i\xi)}\|_2^2 = \int_{\mathbb{R}} |\chi_2|^2 |e^{t\hat{\Phi}_\alpha(i\xi)}|^2 d\xi \leq \int_{\mathbb{R}} |\chi_2|^2 |e^{-c\rho(\xi)t}|^2 d\xi. \tag{2.14}$$

Since for $|\xi| \leq 2R$, there exist constant $c_1, c_2 \in (0, \infty)$ such that $c_1\xi^4 \leq \rho(\xi) \leq c_2\xi^4$, we get

$$\int_{\mathbb{R}} |\chi_2|^2 |e^{-c\rho(\xi)t}|^2 d\xi \leq \int_{\mathbb{R}} |\chi_2|^2 |e^{-\tilde{c}\xi^4 t}|^2 d\xi, \tag{2.15}$$

for some $\tilde{c} > 0$. For $t \geq 1$ we conclude

$$\int_{\mathbb{R}} |\chi_2|^2 |e^{-c\xi^4 t}|^2 d\xi \leq Ct^{-\frac{1}{4}} \leq C(1+t)^{-\frac{1}{4}}. \tag{2.16}$$

For $t < 1$ we have

$$\int_{\mathbb{R}} |\chi_2|^2 |e^{-c\xi^4 t}|^2 d\xi \leq 4R \leq C(1+t)^{-\frac{1}{4}}. \tag{2.17}$$

Summarizing, we have proved the Lemma. □

3 Low frequency analysis

We recall the result from [5], which corresponds to our system (1.1) with $\alpha = 0$. Observe

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \mu \end{pmatrix}.$$

Lemma 3.1. *Let* $\hat{\Phi}_0(i\xi) := -(i\xi A + B_0)$*. Then the corresponding matrix* $e^{t\hat{\Phi}_0(i\xi)}$ *satisfies the following estimate for any* $t \geq 0$ *and* $\xi \in \mathbb{R}$*:*
When $a = 1$*, we have*

$$\left|e^{t\hat{\Phi}_0(i\xi)}\right| \leq Ce^{-c\rho_1(\xi)t}. \tag{3.1}$$

When $a \neq 1$*, we have*

$$\left|e^{t\hat{\Phi}_0(i\xi)}\right| \leq Ce^{-c\rho_2(\xi)t}, \tag{3.2}$$

where $\rho_1(\xi) = \frac{\xi^2}{1+\xi^2}, \rho_2(\xi) = \frac{\xi^2}{(1+\xi^2)^2}$*, and* C, c *are positive constants.*

We notice that for the low frequency part, $e^{t\hat{\Phi}_0(i\xi)}$ in Lemma 3.1 provides the same decay as for a heat kernel, while in Lemma 2.1 the low frequency part of $e^{t\hat{\Phi}_\alpha(i\xi)}$ provides a decay like $e^{-c\xi^4 t}$. This interesting phenomenon shows that that an additional damping term $\alpha\varphi_t$ in the system (1.1) does not make the system decaying faster, instead, making it decaying more slowly. This phenomenon was also observed in [27]. There, the decay estimate is based on inewquality (2.2), leaving the question of optimality open. In this section, we prove the optimality by a careful examination of the low frequency behavior, sinces the low frequency part near $|\xi| = 0$ determines the decay rates. Representing the solution in Fourier space, we have an eigenfunction expansion of the solution, and we know that the decay rate of the low frequency part of the solution is determined by the eigenvalue which provides the slowest decay. This way, the question of optimality turns into an investigation of eigenvalues. Define

$$\chi_3 = \chi_3(\xi) := \begin{cases} 1, & |\xi| \leq \varepsilon \ll 1, \\ 0, & |\xi| \geq 2\varepsilon, \end{cases}$$

where ε is chosen small enough later on. We shall prove:
When $\alpha \neq 0$,

$$|\chi_3 e^{t\hat{\Phi}_\alpha(i\xi)}| \sim e^{-c\xi^4 t}, \tag{3.3}$$

and when $\alpha = 0$,

$$|\chi_3 e^{t\hat{\Phi}_0(i\xi)}| \sim e^{-c\xi^2 t}. \tag{3.4}$$

For this purpose, we notice that the Fourier representation of a solution $\Phi_\alpha(i\xi)\,\hat{U}_0$ is given by

$$e^{t\hat{\Phi}_\alpha(i\xi)}\,\hat{U}_0 = \sum_{j=1}^{4} Q_j(\xi)e^{\lambda_j t}, \tag{3.5}$$

where $\lambda_j = \lambda_j(\xi)$ is an eigenvalue of $\hat{\Phi}_\alpha(i\xi)$, and Q_j depends on the data $\hat{U}_0$, $j = 1, 2, 3, 4$. Up to a set of measure zero, the eigenvalues are different, cp. the expansions below. Since we can use the characteristic polynomial to show there are no multiple eigenvalues. Computing the characteristic polynomial,

$$f(\lambda) = \begin{vmatrix} \lambda+\alpha & -i\xi & 0 & 0 \\ -i\xi & \lambda & 1 & 0 \\ 0 & -1 & \lambda+\mu & -i\xi a \\ 0 & 0 & -i\xi a & \lambda \end{vmatrix} = \lambda^4 + (\alpha+\mu)\lambda^3 + (a^2\xi^2 + \xi^2 + \alpha\mu + 1)\lambda^2$$
$$+(\alpha a^2\xi^2 + \mu\xi^2 + \alpha)\lambda + a^2\xi^4,$$

we notice that λ_j solves

$$\lambda^4 + (\alpha+\mu)\lambda^3 + (a^2\xi^2 + \xi^2 + \alpha\mu + 1)\lambda^2 + (\alpha a^2\xi^2 + \mu\xi^2 + \alpha)\lambda + a^2\xi^4 = 0. \quad (3.6)$$

Comparing the polynomials f and $\frac{d}{d\lambda}f$, we notice that there are non-trivial common divisors at most for values of ξ in a set of measure zero. Hence, outside this set we have only simple roots of f .

Observing that (3.6) only has terms with even order of ξ, we assume for an eigenvalue

$$\lambda(\xi) = a_0 + a_1\xi^2 + a_2\xi^4 + O(\xi^6), \quad (3.7)$$

which corresponds to the Taylor expansion of $\lambda(\xi)$ with respect to ξ^2 at $\xi^2 = 0$. Substituting this expansion into (3.6), we have,

$$\begin{aligned}(a_0 + a_1\xi^2 + a^2\xi^4 + O(\xi^6))^4 &+ (\alpha+\mu)(a_0 + a_1\xi^2 + a^2\xi^4 + O(\xi^6))^3 \\ &+ (a^2\xi^2 + \xi^2 + \alpha\mu + 1)(a_0 + a_1\xi^2 + a^2\xi^4 + O(\xi^6))^2 \\ &+ (\alpha a^2\xi^2 + \mu\xi^2 + \alpha)(a_0 + a_1\xi^2 + a^2\xi^4 + O(\xi^6)) + a^2\xi^4 = 0.\end{aligned}$$

By comparing coefficients, we get

$$(a_0^4 + (\alpha+\mu)a_0^3 + (\alpha\mu+1)a_0^2 + \alpha a_0) = 0, \quad (3.8)$$

$$\begin{aligned}(a_0(\alpha a^2 + \mu) + a_1\alpha + a_0^2(a^2+1) &+ 4a_0^3a_1 + 2a_0a_1(\alpha\mu+1) \\ &+ 3a_0^2a_1(\alpha+\mu)) = 0, \quad (3.9)\end{aligned}$$

$$\begin{aligned}(a_1(\alpha a^2 + \mu) + a_2\alpha + (a_1^2 + 2a_0a_2)(\alpha\mu+1) &+ 4a_0^3a_2 + a^2 + 6a_0^2a_1^2 \\ +2a_0a_1(a^2+1) + 3a_0a_1^2\alpha + 3a_0^2a_2\alpha &+ 3a_0a_1^2\mu + 3a_0^2a_2\mu) = 0. \quad (3.10)\end{aligned}$$

By a simple calculation, we obtain from (3.8)

$$a_0(a_0+\alpha)(a_0^2 + \mu a_0 + 1) = 0. \quad (3.11)$$

Hence we know a_0 has four solutions $a_{01} = 0$, $a_{02} = -\alpha$, $a_{03} = -\frac{\mu}{2} + \frac{(\mu^2-4)^{1/2}}{2}$, $a_{04} = -\frac{\mu}{2} - \frac{(\mu^2-4)^{1/2}}{2}$. Due to assumption (3.7), we have

$$\lambda_j(\xi) = a_{0j} + a_{1j}\xi^2 + a_{2,j}\xi^4 + O(\xi^6).$$

For $|\xi| \le \varepsilon$, λ_2, λ_3 and λ_4 lead to exponential decay. λ_1 provides the worst decay, which is our main concern. Equality (3.9) gives

$$a_1 = \frac{-a_0[\alpha a^2 + \mu + a_0(a^2+1)]}{\alpha + 4a_0^3 + 2a_0\alpha\mu + 2a_0(\alpha\mu+1) + 3a_0^2(\alpha+\mu)}. \quad (3.12)$$

Substituting $a_0 = 0$ into (3.12), we obtain $a_1 = 0$. Then we substitute $a_0 = 0$ and $a_1 = 0$ into equality (3.10), and we have

$$a_2\alpha + a^2 = 0. \quad (3.13)$$

Therefore, $a_2 = -\frac{a^2}{\alpha}$. Now we can write down the eigenvalues λ_j as

$$\begin{aligned}
\lambda_1 &= 0 + 0 - \frac{a^2}{\alpha}\xi^4 + O(\xi^6),\\
\lambda_2 &= -\alpha + \frac{\alpha - \mu}{\alpha^2 - \mu\alpha + 1}\xi^2 + a_{22}\xi^4 + O(\xi^6),\\
\lambda_3 &= -\frac{\mu}{2} + \frac{(\mu^2-4)^{1/2}}{2} + a_{13}\xi^2 + a_{23}\xi^4 + O(\xi^6),\\
\lambda_4 &= -\frac{\mu}{2} - \frac{(\mu^2-4)^{1/2}}{2} + a_{14}\xi^2 + a_{24}\xi^4 + O(\xi^6),
\end{aligned}$$

This way we have proved the sharp asymptotic behavior of the solution given in the following theorem:

Theorem 3.2.

$$|\chi_3 e^{t\hat{\Phi}_\alpha(i\xi)}| \sim e^{-c\xi^4 t}$$

for some constant $c > 0$

Similarly, we get for the case $\alpha = 0$ the following Theorem.

Theorem 3.3.

$$|\chi_3 e^{t\hat{\Phi}(i\xi)}| \sim e^{-c\xi^2 t},$$

for some constant $c > 0$.

Proof: We keep track of α and use the calculations above, and we write the eigenvalues of the low frequency part of the system (1.2) with $\alpha = 0$ again in the form

$$\tilde{\lambda} = \tilde{a}_0 + \tilde{a}_1\xi^2 + \tilde{a}_2\xi^4 + O(\xi^6). \tag{3.14}$$

Then (3.8), (3.9) and (3.10) yield

$$\begin{aligned}
(\tilde{a}_0^4 + (\alpha+\mu)\tilde{a}_0^3 + (\alpha\mu+1)\tilde{a}_0^2 + \alpha\tilde{a}_0) &= 0\\
(\tilde{a}_0(\alpha a^2 + \mu) + a_1\alpha + \tilde{a}_0^2(a^2+1) + 4\tilde{a}_0^3\tilde{a}_1 + 2\tilde{a}_0\tilde{a}_1(\alpha\mu+1) + 3\tilde{a}_0^2\tilde{a}_1(\alpha+\mu)) &= 0\\
(\tilde{a}_1(\alpha a^2 + \mu) + \tilde{a}_2\alpha + (\tilde{a}_1^2 + 2\tilde{a}_0\tilde{a}_2)(\alpha\mu+1) + 4\tilde{a}_0^3\tilde{a}_2 + a^2 + 6\tilde{a}_0^2\tilde{a}_1^2&\\
+2\tilde{a}_0\tilde{a}_1(a^2+1) + 3\tilde{a}_0\tilde{a}_1^2\alpha + 3\tilde{a}_0^2\tilde{a}_2\alpha + 3\tilde{a}_0\tilde{a}_1^2\mu + 3\tilde{a}_0^2\tilde{a}_2\mu) &= 0
\end{aligned}$$

For $\alpha = 0$, the coefficients of the eigenvalues are now calculated from the following equations,

$$\tilde{a}_0^4 + \mu\tilde{a}_0^3 + \tilde{a}_0^2 = 0, \tag{3.15}$$

$$\tilde{a}_0\mu + \tilde{a}_0^2(a^2+1) + 4\tilde{a}_0^3\tilde{a}_1 + 2\tilde{a}_0\tilde{a}_1 + 3\tilde{a}_0^2\tilde{a}_1\mu = 0, \tag{3.16}$$

$$\tilde{a}_1\mu + (\tilde{a}_1^2 + 2\tilde{a}_0 a_2) + 4\tilde{a}_0^3\tilde{a}_2 + a^2 + 6\tilde{a}_0^2\tilde{a}_1^2 + 2\tilde{a}_0 a_1(\tilde{a}^2+1) + 3\tilde{a}_0\tilde{a}_1^2\mu + 3\tilde{a}_0^2\tilde{a}_2\mu = 0. \tag{3.17}$$

We observe from (3.15) that $\tilde{a}_0 = 0, \tilde{a}_0 = -\frac{\mu}{2} + \frac{(\mu^2-4)^{1/2}}{2}, \tilde{a}_0 = -\frac{\mu}{2} - \frac{(\mu^2-4)^{1/2}}{2}$ are solutions. From (3.16) we can obtain that when $\tilde{a}_0 = 0$, and any $\tilde{a}_1$ satisfies (3.16), but from (3.17), we see

$$\tilde{a}_1\mu + \tilde{a}_1^2 + a^2 = 0. \tag{3.18}$$

Hence, we have if $\tilde{a}_0 = 0$, then $\tilde{a}_1 = -\frac{\mu}{2} + \frac{(\mu^2-4a^2)^{1/2}}{2}$, or $\tilde{a}_1 = -\frac{\mu}{2} - \frac{(\mu^2-4a^2)^{1/2}}{2}$. Therefore we obtain

$$\tilde{\lambda}_1 = 0 - (\frac{\mu}{2} + \frac{(\mu^2-4a^2)^{1/2}}{2})\xi^2 + \tilde{a}_{21}\xi^4 + O(\xi^6), \tag{3.19}$$

$$\tilde{\lambda}_2 = 0 - (\frac{\mu}{2} - \frac{(\mu^2-4a^2)^{1/2}}{2})\xi^2 + \tilde{a}_{22}\xi^4 + O(\xi^6), \tag{3.20}$$

$$\tilde{\lambda}_3 = -\frac{\mu}{2} + \frac{(\mu^2-4)^{1/2}}{2} + \tilde{a}_{13}\xi^2 + \tilde{a}_{23}\xi^4 + O(\xi^6), \tag{3.21}$$

$$\tilde{\lambda}_4 = -\frac{\mu}{2} - \frac{(\mu^2-4)^{1/2}}{2} + \tilde{a}_{14}\xi^2 + \tilde{a}_{24}\xi^4 + O(\xi^6). \tag{3.22}$$

This proves the theorem. □

Remark. Theorem 3.2 and Theorem 3.3 prove the optimality of the decay rates obtained for the solution to the linear Timoshenko system under investigation, and hence also prove the striking effect that two damping terms can lead to a weaker decay than just one damping term.

4 Local existence for the semilinear system

In this section, we use a fixed point theorem to prove the local existence of the solution of the semilinear Timoshenko system (1.1). We present an approach improving [22] by not using weight functions.

Let us first recall the notion of a weak solution to this system according to [22]. Rewriting system (1.1) as

$$\begin{cases} U_t + (A\partial_x + L)U = F, & (4.1a) \\ U(0,\cdot) = U_0, & (4.1b) \\ F = (0,0,0,f), & (4.1c) \end{cases}$$

for $U = (\varphi_x - \psi, \phi_t, a\psi_x, \psi_t)$, $U_0 = (\varphi_{0,x} - \psi_0, \phi_1, a\psi_{0,x}, \psi_1)$, $f = |\psi|^r$, the operator $\bar{A} := A\partial_x + L$ with domain $D(\bar{A}) := (H^1(\mathbb{R}))^4 \subset (L^2(\mathbb{R}))^4 \to (L^2(\mathbb{R}))^4$ is, as mentioned in the linear part, the generator of a contraction semigroup $(e^{-t\bar{A}})_{t\geq 0}$, and for $U_0 \in D(\bar{A})$ and $f = f(t,x) \in C^1([0,\infty), L^2(\mathbb{R}))$ we have a classical solution

$$U \in C^1([0,\infty), L^2(\mathbb{R})) \cap C^0([0,\infty), H^1(\mathbb{R}))$$

satisfying

$$U(t) = e^{-t\bar{A}}U_0 + \int_0^t e^{-(t-s)\bar{A}}F(s)ds. \tag{4.2}$$

A weak solution is given by an approximation process. Letting (j_n^1) and (j_n^2) be fixed two Dirac sequences of mollifiers with respect to x and t, respectively, we define for $U_0 \in L^2(\mathbb{R})$ and $F \in C^0([0,\infty), L^2(\mathbb{R}))$, approximations $U_{0,n} := j_n^1 * U_0$ and $F_n := j_n^2 * F$ satisfying

$$U_{0,n} \to U_0 \ \text{ in } L^2(\mathbb{R}), \tag{4.3}$$

$$F_n \to F \ \text{ in } C^0([0,\infty), L^2(\mathbb{R})). \tag{4.4}$$

This way we get a sequence of classical solutions U_n from (4.2) with $U_{0,n}$ and F_n. Due to (4.3) and (4.4), U_n converges to some U in $C^0([0,\infty), L^2(\mathbb{R}))$, and U satisfy (4.2). This U is called a (the) weak solution.

Theorem 4.1 (Local existence)**.** *Let $(\varphi_0, \varphi_1, \psi_0, \psi_1)$ satisfy $U_0 \in H^1$, $\psi_0 \in L^2(\mathbb{R})$, and*

$$J := \|U_0\|_2 + \|\psi_0\|_2 < \infty. \tag{4.5}$$

Then there exists a maximal existence time $T_{max} = T_{max}(J) > 0$, such that problem (1.1) has a unique solution $U \in C([0, T_{max}), H^1(\mathbb{R}))$ satisfying

$$\sup_{[0,T]} \left\{ \|U(t,\cdot)\|_2 + \|\psi(t,\cdot)\|_2 \right\} < \infty, \tag{4.6}$$

where $0 \le T < T_{max}$.

Proof: We apply a similar method as the one used in [22], now without a weight function. Define

$$B_T^K := \{\bar{V} = (\bar{\varphi}, \bar{\psi}) : (\bar{\varphi_x} - \bar{\psi}, \bar{\varphi_t}, \bar{\psi_x}, \bar{\psi_t}) \in \ (C([0,T], L^2(\mathbb{R})))^4, \ \text{ and } \ \|\bar{V}\|_T \le K\}, \tag{4.7}$$

where

$$\begin{aligned} \|\bar{V}\|_T := \|(\bar{\varphi}, \bar{\psi})\|_T := \sup_{[0,T]} \{ &\|\bar{\varphi}_t(t,\cdot)\|_2 + \|(\bar{\varphi_x} - \bar{\psi})(t,\cdot)\|_2 \\ &+ \|\bar{\psi_t}(t,\cdot)\|_2 + \|\bar{\psi}(t,\cdot)\|_2 + \|\bar{\psi_x}(t,\cdot)\|_2 \}. \end{aligned} \tag{4.8}$$

Let $X := \{(\bar{\varphi}, \bar{\psi}) : (\bar{\varphi_x} - \bar{\psi}, \bar{\varphi_t}, \bar{\psi_x}, \bar{\psi_t}) \in \ (C([0,T], L^2(\mathbb{R})))^4\}$, then X with norm $\|\cdot\|_T$ is a Banach space.

We fix the initial data $U_0 \in H^1(\mathbb{R})$, $\psi_0 \in L^2(\mathbb{R})$. For a fixed $\bar{V} = (0, \bar{\psi})^\tau \in B_T^K$, define $\Gamma : B_T^K \to X$, $\Gamma(\bar{V}) := (\varphi, \psi)^\tau$, where $(\varphi, \psi)^\tau$ is the weak solution to

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x + \alpha\varphi_t = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \mu\psi_t = |\bar{\psi}|^r, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ (\varphi, \varphi_t, \psi, \psi_t)(0,x) = (\varphi_0, \varphi_1, \psi_0, \psi_1), & x \in \mathbb{R}. \end{cases} \tag{4.9}$$

Our aim is to show that, T chosen suitably, Γ is a contraction map, and $\Gamma(B_T^K) \subset B_T^K$. We first prove the estimate in the class of classical solutions, then one can obtain the same for the weak solution by approximation as in [22].

First we show $\Gamma(B_T^K) \subset B_T^K$. We multiply the first equation of (4.9) by ϕ_t, multiply the second equation of (4.9) by ψ_t, and sum up the results to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\varphi_t^2 - \frac{d}{dx}((\varphi_x - \psi)\varphi_t) + (\varphi_x - \psi)\varphi_{tx} + \alpha\phi_t^2 \\ &\quad + \frac{d}{dt}(\frac{1}{2}\psi_t^2 + \frac{a^2}{2}\psi_x^2) - a^2\frac{d}{dx}(\psi_x\psi_t) - (\varphi_x - \psi)\psi_t + \mu\psi_t^2 = |\bar{\psi}|^r\psi_t. \end{aligned} \tag{4.10}$$

Hence we have

$$\begin{aligned} &\frac{d}{dt}(\frac{1}{2}(\varphi_t^2 + \psi_t^2 + a^2\psi_x^2 + (\varphi_x - \psi)^2)) \\ &\quad - \frac{d}{dx}((\varphi_x - \psi)\varphi_t) - a^2\frac{d}{dx}(\psi_x\psi_t) + \mu\psi_t^2 + \alpha\varphi_t^2 = |\bar{\psi}|^r\psi_t. \end{aligned} \tag{4.11}$$

Since the term $+\mu\psi_t^2$ and $+\alpha\varphi_t^2$ is non-negative this gives

$$\begin{aligned}&\frac{d}{dt}(\frac{1}{2}(\varphi_t^2+\psi_t^2+a^2\psi_x^2+(\varphi_x-\psi)^2))\\&\quad-\frac{d}{dx}((\varphi_x-\psi)\varphi_t)-a^2\frac{d}{dx}(\psi_x\psi_t)\le|\bar{\psi}|^r\psi_t.\end{aligned}\tag{4.12}$$

Integrating over $[0,t]\times\mathbb{R}$, we obtain

$$E_{\varphi,\psi}(t)\le E_{\varphi,\psi}(0)+\int_0^t\int_{\mathbb{R}}|\bar{\psi}(s,x)|^r\psi_s(s,x)dxds,\tag{4.13}$$

where

$$E_{\varphi,\psi}(t):=\frac{1}{2}(\|\varphi_t\|_2^2+\|\psi_t\|_2^2+a^2\|\psi_x\|_2^2+\|(\varphi_x-\psi)\|_2^2).$$

So we get

$$\begin{aligned}E_{\varphi,\psi}(t)&\le E_{\varphi,\psi}(0)+\int_0^t\|(\bar{\psi})^r\|_2\|\psi_s\|_2ds\\&\le E_{\varphi,\psi}(0)+\sqrt{2}\int_0^t\|(\bar{\psi})^r\|_2(E_{\varphi,\psi}(s))^{1/2}ds.\end{aligned}\tag{4.14}$$

Applying Lemma 1.3 we conclude

$$(E^{\phi}_{\varphi,\psi}(t))^{1/2}\le(E^{\phi}_{\varphi,\psi}(0))^{1/2}+\frac{1}{\sqrt{2}}\int_0^t\|(\bar{\psi})^r\|_2ds.\tag{4.15}$$

Since

$$\|(\bar{\psi})^r\|_2^2\le\|\bar{\psi}\|_\infty^{2r-2}\int_{\mathbb{R}}|\bar{\psi}|^2dx\le CK^{2r}\tag{4.16}$$

we obtain

$$(E_{\varphi,\psi}(t))^{1/2}\le(E_{\varphi,\psi}(0))^{1/2}+CTK^r,\tag{4.17}$$

hence

$$\|\psi(t)\|_2\le\|\psi(0)\|_2+\sqrt{2}(E_{\varphi,\psi}(0))^{1/2}T+CK^rT^2.\tag{4.18}$$

From (4.17) and (4.18), we conclude

$$\|(\varphi,\psi)\|_T\le(E_{\varphi,\psi}(0))^{1/2}+CTK^r+\|\psi(0)\|_2+\sqrt{2}(E_{\varphi,\psi}(0))^{1/2}T+CK^rT^2.\tag{4.19}$$

Choosing K large enough such that $(E_{\varphi,\psi}(0))^{1/2}+\|e^{\phi(0)}\psi(0)\|_2\le K/2$, and then choosing T small enough such that

$$CTK^r+\sqrt{2}(E_{\varphi,\psi}(0))^{1/2}T+CK^rT^2\le K/2\tag{4.20}$$

we obtain

$$\|(\varphi,\psi)\|_T\le K\tag{4.21}$$

proving $(\varphi,\psi)^\tau \in B_T^K$. Next, we show the contraction property. Let $(\varphi,\psi) = \Gamma(0,\bar\psi) = \Gamma(\bar V)$, and $(\hat\varphi,\hat\psi) = \Gamma(0,\hat{\bar\psi}) = \Gamma(\hat V)$. Set $\tilde\varphi = \varphi - \hat\varphi$, $\tilde\psi = \psi - \hat\psi$, then $(\tilde\varphi,\tilde\psi)$ satisfy

$$\begin{cases} \tilde\varphi_{tt} - (\tilde\varphi_x - \tilde\psi)_x + \alpha\tilde\varphi_t = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R} \\ \tilde\psi_{tt} - a^2\tilde\psi_{xx} - (\tilde\varphi_x - \tilde\psi) + \mu\tilde\psi_t = |\bar\psi|^r - |\hat{\bar\psi}|^r, & (t,x) \in \mathbb{R}^+ \times \mathbb{R} \\ (\tilde\varphi,\tilde\varphi_t,\tilde\psi,\tilde\psi_t)(0,x) = (0,0,0,0). & x \in \mathbb{R} \end{cases} \tag{4.22}$$

We analogously get

$$E_{\tilde\varphi,\tilde\psi}(t) \le \int_0^t \int_{\mathbb{R}} \left| (|\bar\psi(s,x)|^r - |\hat{\bar\psi}(s,x)|^r)\tilde\psi_s(s,x) \right| dxds. \tag{4.23}$$

Since

$$\left| |\bar\psi(s,x)|^r - |\hat{\bar\psi}(s,x)|^r \right| \le r|\bar\psi(s,x) - \hat{\bar\psi}(s,x)|(|\bar\psi(s,x)| + |\hat{\bar\psi}(s,x)|)^{r-1}, \tag{4.24}$$

we conclude from (4.23)

$$\begin{aligned} E_{\tilde\varphi,\tilde\psi}(t) &\le r\int_0^t \int_{\mathbb{R}} |\bar\psi(s,x) - \hat{\bar\psi}(s,x)|(|\bar\psi(s,x)| + |\hat{\bar\psi}(s,x)|)^{r-1}|\tilde\psi_s(s,x)|dxds \\ &\le C\int_0^t (E_{\tilde\varphi,\tilde\psi}(s))^{1/2}\|(\bar\psi(s,x) - \hat{\bar\psi}(s,x))\|_{2r}\|(|\bar\psi(s,x)| + |\hat{\bar\psi}(s,x)|)\|_{2r}^{r-1}ds, \end{aligned} \tag{4.25}$$

implying by Lemma 1.3 that

$$(E_{\tilde\varphi,\tilde\psi}(t))^{1/2} \le C\int_0^t \|(\bar\psi(s,x) - \hat{\bar\psi}(s,x))\|_{2r}(\|\bar\psi(s,x)\|_{2r} + \|\hat{\bar\psi}(s,x)\|_{2r})^{r-1}ds. \tag{4.26}$$

We have

$$\|\bar\psi(s,x)\|_{2r}^{2r} \le \|\bar\psi\|_\infty^{2r-2}\int_{\mathbb{R}} |\bar\psi|^2 dx \le CK^{2r}. \tag{4.27}$$

Similarly, we have $\|\hat{\bar\psi}(s,x)\|_{2r} \le CK$, so we get, using the Gagliardo-Nirenberg inequality (see e.g. [22]),

$$\begin{aligned} \|\bar\psi(s,x) - \hat{\bar\psi}(s,x)\|_{2r} &\le C\|\bar\psi(s,x) - \hat{\bar\psi}(s,x)\|_2^{1-(\frac12 - \frac{1}{2r})}\|\partial_x(\bar\psi(s,x) - \hat{\bar\psi}(s,x))\|_2^{\frac12 - \frac{1}{2r}} \\ &\le C\|\bar V - \hat V\|_T. \end{aligned} \tag{4.28}$$

Applying (4.27) (4.28) to (4.26), we conclude

$$(E_{\tilde\varphi,\tilde\psi}(t))^{1/2} \le CTK^{r-1}\|\bar V - \hat V\|_T. \tag{4.29}$$

Observing now

$$\|\tilde\psi\|_2 \le \int_0^t \|\tilde\psi_s(s,x)\|_2 ds \le CT^2K^{r-1}\|\bar V - \hat V\|_T^\phi, \tag{4.30}$$

(4.29) and (4.30) imply

$$\|(\varphi - \hat{\varphi}, \psi - \hat{\psi})\|_T \leq C(1+T)TK^{r-1}\|\bar{V} - \hat{V}\|_T. \tag{4.31}$$

We can choose T small enough such that

$$C(1+T)TK^{r-1} < \frac{1}{2}. \tag{4.32}$$

proving that Γ is a contraction map having a unique fixed point (φ, ψ). Together with the representation (4.2) we see for $U_0 \in H^1(\mathbb{R})$, that we have a classical solution. $\square$

5 Global well-posedness of the semilinar system

5.1 Weighted a priori estimate.

As in Section 2 we use the functions χ_1 and χ_2 to decompose a funtion u into

$$u = u_H + u_L, \tag{5.1}$$

where

$$\hat{u_H} = \chi_1 \hat{u}, \quad \hat{u_L} = \chi_2 \hat{u}. \tag{5.2}$$

Let U be the solution of the semilinear problem (4.1), then we have again the representation

$$\hat{U}(t,x) = e^{t\hat{\Phi}_\alpha(i\xi)}\hat{U_0} + \int_0^t e^{(t-s)\hat{\Phi}_\alpha(i\xi)}\hat{F}(U)(s)ds, \tag{5.3}$$

where $e^{t\hat{\Phi}_\alpha(i\xi)}$ is defined as in Lemma 2.1.

Lemma 5.1. *Let (φ, ψ) be the local solution according to Theorem 4.1, let*

$$\Lambda(t) := \sup_{0 \leq s \leq t}\{(1+s)^{\frac{1}{4}}\|(\varphi_t, \varphi_x, \psi_t, \psi_x, \psi)(s,\cdot)\|_2^2\},$$

and

$$E_0^2 := \|(\varphi_t, \varphi_x, \psi_t, \psi_x, \psi)(0,\cdot)\|_2^2 + \|\psi(0,\cdot)\|_1^2.$$

Then, if E_0 is sufficiently small, we have for all $t \in [0, T_{max})$

$$\Lambda(t) \leq CE_0^2, \tag{5.4}$$

where $C > 0$ is a positive constant not depending on t or on the data.

Proof: Observing

$$\|\psi\|_{2r} \leq C\|\psi\|_{H^1}$$

we have

$$\|\psi(t,\cdot)\|_{2r} \leq C(1+t)^{-\frac{1}{8}}\Lambda(t)^{1/2}. \tag{5.5}$$

For the high frequency part we obtain

$$\|\chi_1\hat{U}(t,\cdot)\|_2 \leq \|\chi_1 e^{t\hat{\Phi}_\alpha(i\xi)}\hat{U_0}\|_2 + \int_0^t \|\chi_1 e^{(t-s)\hat{\Phi}_\alpha(i\xi)}\hat{F}(U)(s)\|_2 ds. \tag{5.6}$$

Therefore, using Lemma 2.3, (5.5) and Lemma 1.2, we obtain

$$\begin{aligned}
\|\chi_1\hat{U}(t,\cdot)\|_2 &\le \|\chi_1 e^{t\hat{\Phi}_\alpha(i\xi)}\|_\infty \|\hat{U_0}\|_2 + \int_0^t \|\chi_1 e^{(t-s)\hat{\Phi}_\alpha(i\xi)}\hat{F}(U)(s)\|_2 ds \\
&\le Ce^{-ct}E_0 + C\int_0^t e^{-c(t-s)}\|\psi\|_{2r}^r ds \\
&\le Ce^{-ct}E_0 + C\int_0^t e^{-c(t-s)}(1+s)^{-\frac{r}{8}}\Lambda(s)^{r/2} ds \\
&\le Ce^{-ct}E_0 + C(1+t)^{-\frac{r}{8}}\Lambda(t)^{r/2}, \qquad (5.7)
\end{aligned}$$

where we used $r > 8$. Thus

$$\|\chi_1\hat{U}(t,\cdot)\|_2^2 \le C(1+t)^{-\frac{r}{4}}(E_0 + \Lambda(t)^r). \qquad (5.8)$$

Simlilarly, we obtain for the low frequency part

$$\begin{aligned}
\|\chi_2\hat{U}(t,\cdot)\|_2 &\le \|\chi_2 e^{t\hat{\Phi}_\alpha(i\xi)}\|_2 \|\hat{U_0}\|_\infty + \int_0^t \|\chi_2 e^{(t-s)\hat{\Phi}_\alpha(i\xi)}\|_2 \|\hat{F}(U)(s)\|_\infty ds \\
&\le C(1+t)^{-\frac{1}{8}}\|U_0\|_1 + C\int_0^t (1+t-s)^{-\frac{1}{8}}\|\psi^r\|_1 ds \\
&\le C(1+t)^{-\frac{1}{8}}E_0 + C\int_0^t (1+t-s)^{-\frac{1}{8}}(1+s)^{-\frac{r}{8}}\Lambda^{\frac{r}{2}}(t) ds \\
&\le C(1+t)^{-\frac{1}{8}}(E_0 + \Lambda^{\frac{r}{2}}(t)). \qquad (5.9)
\end{aligned}$$

Combining (5.8) and (5.9) we conclude

$$(1+t)^{\frac{1}{4}}\|(\varphi_t,\varphi_x,\psi_t,\psi_x,\psi)\|_2^2 \le CE_0^2 + C\Lambda^r(t), \qquad (5.10)$$

implying

$$\Lambda(t) \le CE_0^2 + C\Lambda^r(t). \qquad (5.11)$$

If E_0 is sufficently small this implies

$$\Lambda(t) \le CE_0^2. \qquad (5.12)$$

(Actually, $\Lambda(t)$ is bounded by the first zero of the function f where $f(x) := CE_0^2 + Cx^r - x$ with the constant C from (5.11).) □

5.2 Global existence – proof of Theorem 1.1

Using Lemma 5.1, we can continue the local solution obtained in Theorem 4.1 globally in time, because

$$\|U(t,\cdot)\|_2^2 + \|\psi(t,\cdot)\|_2^2 \le C\Lambda(t),$$

on the interval of local existence. Observing the dependence of the length of the interval, T, on the initial data given in the local existence theorem, the latter can be used again at time T, and so on. In particular, we have the claimed decay estimate (1.8). This completes the proof of Theorem 1.1. □

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References

[1] F. Ammar-Khodja, A. Benabdallah, J.E. Muñoz Rivera, and R. Racke. Energy decay for Timoshenko systems of memory type. *J. Differential Equations*, 194(1):82–115, 2003.

[2] J. Chen and W. Wang. The point-wise estimates for the solution of damped wave equation with nonlinear convection in multi-dimensional space. *Commun. Pure Appl. Anal.*, 13(1):307–330, 2014.

[3] H.D. Fernández Sare and R. Racke. On the stability of damped Timoshenko systems: Cattaneo versus Fourier law. *Arch. Ration. Mech. Anal.*, 194(1):221–251, 2009.

[4] A. Guesmia and S.A. Messaoudi. A general stability result in a Timoshenko system with infinite memory: a new approach. *Math. Methods Appl. Sci.*, 37(3):384–392, 2014.

[5] K. Ide, K. Haramoto, and S. Kawashima. Decay property of regularity-loss type for dissipative Timoshenko system. *Math. Models Methods Appl. Sci.*, 18(5):647–667, 2008.

[6] M. Kafini, S.A. Messaoudi, and M.I. Mustafa. Energy decay rates for a Timoshenko-type system of thermoelasticity of type III with constant delay. *Appl. Anal.*, 93(6):1201–1216, 2014.

[7] M. Kirane, B. Said-Houari, and M.N. Anwar. Stability result for the Timoshenko system with a time-varying delay term in the internal feedbacks. *Commun. Pure Appl. Anal.*, 10(2):667–686, 2011.

[8] F. Li and W. Wang. The pointwise estimate of solutions to the parabolic conservation law in multi-dimensions. *Nonlinear Differ. Equ. Appl.*, 21(1):87–103, 2014.

[9] Y. Liu and W. Wang. The pointwise estimates of solutions for dissipative wave equation in multi-dimensions. *Discrete Cont. Dyn.-A*, 20(4):1013–1028, 2008.

[10] A. Matsumura. On the asymptotic behavior of solutions of semi-linear wave equations. *Publ. Res. Inst. Math. Sci.*, 12(1):169–189, 1976.

[11] S.A. Messaoudi and M.I. Mustafa. A stability result in a memory-type Timoshenko system. *Dynam. Systems Appl.*, 18(3-4):457–468, 2009.

[12] S.A. Messaoudi, M. Pokojovy, and B. Said-Houari. Nonlinear damped Timoshenko systems with second sound—global existence and exponential stability. *Math. Methods Appl. Sci.*, 32(5):505–534, 2009.

[13] S.A. Messaoudi and B. Said-Houari. Energy decay in a Timoshenko-type system of thermoelasticity of type III. *J. Math. Anal. Appl.*, 348(1):298–307, 2008.

[14] N. Mori and S. Kawashima. Decay property for the Timoshenko system with Fourier's type heat conduction. *J. Hyperbolic Differ. Equ.*, 11(1):135–157, 2014.

[15] J.E. Muñoz Rivera and R. Racke. Global stability for damped Timoshenko systems. *Discrete Contin. Dyn. Syst.*, 9(6):1625–1639, 2003.

[16] J.E. Muñoz Rivera and H.D. Fernández Sare. Stability of Timoshenko systems with past history. *J. Math. Anal. Appl.*, 339(1):482–502, 2008.

[17] J.E. Muñoz Rivera and R. Racke. Mildly dissipative nonlinear Timoshenko systems—global existence and exponential stability. *J. Math. Anal. Appl.*, 276(1):248–278, 2002.

[18] B.G. Pachpatte. Inequalities for differential and integral equations. Academic Press: London, 1998.

[19] P. Pei, M.A. Rammaha, and D. Toundykov. Global well-posedness and stability of semilinear Mindlin-Timoshenko system. *J. Math. Anal. Appl.*, 418(2):535–568, 2014.

[20] R. Racke. Lectures on nonlinear evolution equations. Initial value problems. Second edition. Birkhäuser: Basel, 2015.

[21] R. Racke and B. Said-Houari. Global existence and decay property of the Timoshenko system in thermoelasticity with second sound. *Nonlinear Anal.*, 75(13):4957–4973, 2012.

[22] R. Racke and B. Said-Houari. Decay rates and global existence for semilinear dissipative Timoshenko systems. *Quart. Appl. Math.*, 71(2):229–266, 2013.

[23] B. Said-Houari and A. Kasimov. Damping by heat conduction in the Timoshenko system: Fourier and Cattaneo are the same. *J. Differential Equations*, 255(4):611–632, 2013.

[24] B. Said-Houari and Y. Laskri. A stability result of a Timoshenko system with a delay term in the internal feedback. *Appl. Math. Comput.*, 217(6):2857–2869, 2010.

[25] M.L. Santos, D.S. Almeida Júnior, J.H. Rodrigues, and F.A. Falcão Nascimento. Decay rates for Timoshenko system with nonlinear arbitrary localized damping. *Differential Integral Equations*, 27(1-2):1–26, 2014.

[26] I.E. Segal. Dispersion for non-linear relativistic equations, ii. *Ann. Sci. Ecole Norm. Sup.*, 1(4):459–497, 1968.

[27] A. Soufyane and B. Said-Houari. The effect of the wave speeds and the frictional damping terms on the decay rate of the bresse system. *Evolution Equations and Control Theory*, 3(4):713–738, 2014.

[28] A. Soufyane and A. Wehbe. Uniform stabilization for the Timoshenko beam by a locally distributed damping. *Electron. J. Differential Equations*, pages No. 29, 14 pp. (electronic), 2003.

[29] S. Timoshenko. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Philisophical Magazine*, 41:744–746, 1921.

[30] G. Todorova and B. Yordanov. Critical exponent for a nonlinear wave equation with damping. *J. Differential Equations*, 174:464–489. 2001.

[31] Y. Ueda, R. Duan, and S. Kawashima. Decay structure for symmetric hyperbolic systems with non-symmetric relaxation and its application. *Arch. Ration. Mech. Anal.*, 205(1):239–266, 2012.

[32] W. Wang and T. Yang. The pointwise estimates of solutions for euler equations with damping in multi-dimensions. *J. Differential Equations*, 173(2):410 – 450, 2001.