Stability for thermoelastic plates with two temperatures

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Abstract: We investigate the well-posedness, the exponential stability, or the lack thereof, of thermoelastic systems in materials where, in contrast to classical thermoelastic models for Kirchhoff type plates, two temperatures are involved, related by an elliptic equation. The arising initial boundary value problems for different boundary conditions deal with systems of partial differential equations involving Schrödinger like equations, hyperbolic and elliptic equations, which have a different character compared to the classical one with the usual single temperature. Depending on the model – with Fourier or with Cattaneo type heat conduction – we obtain exponential resp. non-exponential stability, thus providing another examples where the change from Fourier's to Cattaneo's law leads to a loss of exponential stability.

1 Introduction

Thermoelastic plates of Kirchhoff type modeled by

$$u_{tt} + b\Delta^2 u + d\Delta\theta = 0, \tag{1.1}$$

$$\theta_t + \operatorname{div} q - d\Delta u_t = 0, \tag{1.2}$$

$$\tau q_t + q + \kappa\nabla\theta = 0, \tag{1.3}$$

for $(u, \theta, q) = (u, \theta, q)(t, x)$ denoting the displacement, the temperature and heat flux in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $t \geq 0$, $x \in \Omega$, with $b, d, \kappa > 0, \tau \geq 0$, have been discussed in recent years with respect to well-posedness and asymptotic behavior in time (also for unbounded domains) and both for $\tau = 0$ and for $\tau > 0$.

So-called non-simple materials are modeled by two temperatures, the thermodynamic temperature θ and the conductive temperature φ, related to each other in the following way, see [2, 3, 4, 32],

$$\theta = \varphi - a\Delta\varphi \tag{1.4}$$

with a constant $a \geq 0$. The corresponding extension of the classical thermoelastic plate model (1.1)–(1.3) then reads as

$$u_{tt} + b\Delta^2 u + d\Delta\theta = 0, \tag{1.5}$$

$$\theta_t + \operatorname{div} q - d\,\Delta u_t = 0, \tag{1.6}$$

$$\tau q_t + q + \kappa\nabla\varphi = 0, \tag{1.7}$$

$$\theta - \varphi + a\Delta\varphi = 0. \tag{1.8}$$

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Boundary conditions will be given for u, φ and/or q below.

For $a = 0$ we recover (1.1)–(1.3). In this case, in particular, it is known that one has, for appropriate boundary conditions, exponential stability for the Fourier type heat conduction given by $\tau = 0$, while it is not exponentially stable for the Cattaneo (Maxwell) type given for a positive relaxation constant $\tau > 0$, see [26, 27, 8]. Also the Cauchy problem ($\Omega = \mathbb{R}^n$) has been investigated in [30], where the loss of exponential stability in bounded domains is reflected in a regularity loss in the description of polynomial decay of solutions.

Here, we shall investigate initial boundary value problems for the case $a > 0$. We are first interested in the well-posedness both for $\tau = 0$ and for $\tau > 0$, which is more delicate in comparison to the case $a = 0$, since there will be no regularity gain in the temperature triggered by the differential equations. The second main topic will be to investigate exponential stability for $\tau = 0$ resp. non-exponential stability for $\tau > 0$. In particular for the case $\tau = 0$, we will consider several boundary conditions, using different methods.

This way we also contribute a further example where the different heat conduction models, one by Fourier ($\tau = 0$), one by Cattaneo ($\tau > 0$) predict different qualitative behavior, see [9] for the thermoelastic Timoshenko model, and [26, 27, 8] for the classical thermoelastic model ($a = 0$). As was indicated in [29], these might not be exceptions, that is, the change from Fourier's to Cattaneo's "law" is likely to lead to a loss of exponential stability.

Some related papers are given as follows:

Case $a = 0$: For bounded domains and for $\tau = 0$, there are many results in particular on exponential stability, see for example [1, 13, 14, 15, 16, 17, 19, 21, 22]. For results for the Cauchy problem or in general exterior domains see for example [5, 6, 7, 21, 22, 30]. For $\tau > 0$, exponential stability in bounded domains is lost [26, 27, 8], for the Cauchy problem we encounter a regularity loss [30].

Case $a > 0$: Here only the *second-order* system

$$
\begin{aligned}
u_{tt} - bu_{xx} + d\theta_x &= 0, && (1.9)\\
\theta_t + q_x + du_{tx} &= 0, && (1.10)\\
\tau q_t + q + \kappa\varphi_x &= 0, && (1.11)\\
\theta - \varphi + a\varphi_{xx} &= 0, && (1.12)
\end{aligned}
$$

in one space dimension in a bounded interval has been studied with respect to exponential stability for $a = 0$ [25], the well-posedness was obtained in any space dimension [24]. The non-exponential stability for $\tau > 0$ was proved in in [18]. We carry over considerations to the fourth-order thermoelastic plate, which exhibits more complex difficulties, cp., for example, Section 7.

Our main new contributions are

- First discussion of the fourth-order thermoelastic plate system with two temperatures.

- Proof of well-posedness for rather weak regular solutions, both for $\tau = 0$ and for $\tau > 0$.

- Proof of exponential stability ($\tau = 0$) resp. the lack thereof ($\tau > 0$) for different boundary conditions with different methods; this way also providing another example for the problem of "right" modeling with Fourier or Cattaneo type laws.

- Providing the rigorous proof of exponential stability for the *second*-order system.

The paper is organized as follows. In Section 2, the well-posedness for the thermoelastic plate model (1.5)–(1.8) (with appropriate boundary conditions and initial conditions) is investigated for $\tau = 0$, while the corresponding results for $\tau > 0$ are given in Section 3. The exponential stability for (1.5)–(1.8) is shown for $\tau = 0$ for the boundary conditions (2.5) and (2.7) in Section 4 with semigroup methods, while Section 5 shows the general non-exponential stability for $\tau > 0$. The exponential stability for the *second-order* thermoelastic model (1.9)–(1.12) is proved for $\tau = 0$ in Section 6. The exponential stability for (1.5)–(1.8) is proved for $\tau = 0$ in Section 7 for Robin type boundary conditions on the temperature with energy and lifting methods.

We use standard notation, in particular the Sobolev spaces $L^p = L^p(\Omega)$, $p \geq 1$, and $H^s = W^{s,2}(\Omega)$, $s \in \mathbf{N}_0$, with their associated norms $\|\cdot\|_{L^p}$ resp. $\|\cdot\|_{H^s}$. The inner product in a Hilbert space X is given by $\langle \cdot, \cdot \rangle_X$, and $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2}$. By Id we denote the identity on some given space.

2 Well-posedness for $\tau = 0$

Here we extend the work on the second-order system in [25] to the fourth-order system.

We start proving the well-posedness of the system (1.5)–(1.8) with $\tau = 0$, i.e. for

$$u_{tt} + b\Delta^2 u + d\Delta\theta = 0 \quad \text{in } [0,\infty) \times \Omega, \tag{2.1}$$

$$\theta_t - \kappa\Delta\varphi - d\,\Delta u_t = 0 \quad \text{in } [0,\infty) \times \Omega, \tag{2.2}$$

$$\theta - \varphi + a\Delta\varphi = 0 \quad \text{in } [0,\infty) \times \Omega, \tag{2.3}$$

recalling that $a > 0$. Initial conditions are given by

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0 \qquad \text{in } \Omega, \tag{2.4}$$

while boundary conditions are prescribed either by

$$u(t,\cdot) = \frac{\partial u}{\partial \nu}(t,\cdot) = 0, \qquad \varphi(t,\cdot) = 0 \qquad \text{in } [0,\infty) \times \partial\Omega, \tag{2.5}$$

or by

$$u(t,\cdot) = \frac{\partial u}{\partial \nu}(t,\cdot) = 0, \qquad \frac{\partial \varphi}{\partial \nu}(t,\cdot) = 0 \qquad \text{in } [0,\infty) \times \partial\Omega, \tag{2.6}$$

or by

$$u(t,\cdot) = (b\Delta u + d\theta)(t,\cdot) = 0, \qquad \varphi(t,\cdot) = 0 \qquad \text{in } [0,\infty) \times \partial\Omega, \tag{2.7}$$

where ν denotes the exterior normal at the boundary. We write

$$\varphi = (\mathrm{Id} - a\Delta)^{-1}\,\theta. \tag{2.8}$$

Here $(\mathrm{Id} - a\Delta)^{-1}$ denotes the homeomorphism from L^2 onto $H^2 \cap H_0^1$ in case of the boundary conditions (2.5) or (2.7), and from L^2_* onto $H^2 \cap \{\varphi \,|\, \frac{\partial\varphi}{\partial\nu} = 0\} \cap L^2_*$ in case of the boundary conditions (2.6), where

$$L^2_* := \{\, f \in L^2 \,|\, \int_\Omega f(x)dx = 0\,\}.$$

The boundary condition (2.7) will be treated at the end of this section. Then we obtain

$$u_{tt} + \Delta(b\Delta u + d\theta) \;=\; 0 \qquad \text{in } [0,\infty) \times \Omega, \tag{2.9}$$

$$\theta_t - B\theta - d\Delta u_t \;=\; 0 \qquad \text{in } [0,\infty) \times \Omega, \tag{2.10}$$

with

$$B : \begin{cases} L^2 \to L^2 & \text{for (2.5) or (2.7)}, \\ L^2_* \to L^2_* & \text{for (2.6)}, \end{cases} \tag{2.11}$$

$$B\theta := \kappa\Delta\,(\mathrm{Id} - a\Delta)^{-1}\,\theta,$$

B being (just) a bounded operator. By (2.3), we have for the boundary conditions (2.6)

$$\int_\Omega \theta\,dx = \int_\Omega \varphi\,dx,$$

and this is an invariant, i.e.

$$\int_\Omega \theta_0(x)dx = 0 \Longrightarrow \forall\, t \geq 0 : \int_\Omega \theta(t,x)(x)dx = 0,$$

which can be obtained by integrating (2.10).

Remark 2.1. *The equation (2.10) for, essentially, θ does not trigger any regularity for θ, in contrast to the situation where $a = 0$ (only one temperature $\theta = \varphi$). For $a = 0$ we would have the classical operator $B = \kappa\Delta$ on its usual domain. On the other hand, in equation (2.9) one needs, yet formally, $\Delta\theta$. This lack of regularity will be reflected in a lack of separate for regulariy for u and θ. We shall have a connected regularity, see below.*

The operator B satisfies for $\theta = \varphi - a\Delta\varphi$

$$\langle B\theta, \theta\rangle = -\kappa\|\nabla\varphi\|^2_{L^2} - \kappa a\|\Delta\varphi\|^2_{L^2} \leq 0. \tag{2.12}$$

We transform the system (2.9), (2.10) into a system of first order in time for $V := (u, u_t, \theta)'$, where $'$ denotes the transposed matrix:

$$V_t = \begin{pmatrix} 0 & 1 & 0 \\ -b\Delta^2 & 0 & -d\Delta \\ 0 & d\Delta & B \end{pmatrix} V \equiv A_f V, \qquad V(0,\cdot) = V^0 := (u_0, u_1, \theta_0)'. \tag{2.13}$$

This formal system with the formal differential symbol A_f will be considered as an evolution equation in an associated Hilbert space

$$\mathcal{H} := \begin{cases} H_0^2 \times L^2 \times L^2 & \text{for (2.5)}, \\ H_0^2 \times L^2 \times L^2_* & \text{for (2.6)}, \\ \left(H^2 \cap H_0^1\right) \times L^2 \times L^2 & \text{for (2.7)}, \end{cases} \tag{2.14}$$

with inner product

$$\langle V,\tilde V\rangle_{\mathcal H} := b\langle \Delta u,\Delta\tilde u\rangle + \langle v,\tilde v\rangle + \langle\theta,\tilde\theta\rangle.$$

Then

$$V_t = AV,\quad V(t=0)=V^0, \tag{2.15}$$

where

$$A: D(A)\subset\mathcal H\to\mathcal H,\qquad AV := A_f V, \tag{2.16}$$

for $V\in D(A)$ with

$$D(A) := \begin{cases} \{\, V=(u,v,\theta)'\in\mathcal H \mid v\in H_0^2,\ \Delta(b\Delta u+d\theta)\in L^2\,\} & \text{for (2.5), (2.6)},\\ \{\, V=(u,v,\theta)'\in\mathcal H \mid (b\Delta u+d\theta)\in H^2\cap H_0^1\,\} & \text{for (2.7)}.\end{cases} \tag{2.17}$$

In the definition of $D(A)$, the problem of the missing (separate) regularity for θ is reflected. cp. Remark 2.1. One just has the combined regularity $\Delta(b\Delta u+d\theta)\in L^2$, not writing $\Delta^2 u$, $\Delta\theta\in L^2$, and this way $A_f V$ has to be interpreted.

As usual, $\Delta(b\Delta u+d\theta)\in L^2$ means

$$\exists\, h\in L^2\ \forall\,\psi\in C_0^\infty : \langle b\Delta u+d\theta,\Delta\psi\rangle = \langle h,\psi\rangle. \tag{2.18}$$

We will show that A generates a contraction semigroup.

Lemma 2.2. *$D(A)$ is dense in $\mathcal H$, and for $V=(u,v,\theta)'\in D(A)$, with $\theta=\varphi-a\Delta\varphi$, we have the dissipativity of A,*

$$\mathrm{Re}\,\langle AV,V\rangle_{\mathcal H} = \langle B\theta,\theta\rangle = -\kappa\|\nabla\varphi\|_{L^2}^2 - \kappa a\|\Delta\varphi\|_{L^2}^2 \le 0.$$

Proof: $(C_0^\infty)^3$ is contained in $D(A)$ and dense in $\mathcal H$.

$$\begin{aligned}\mathrm{Re}\,\langle AV,V\rangle_{\mathcal H} &= \mathrm{Re}\,\left(b\langle\Delta v,\Delta u\rangle + \langle\Delta(-b\Delta u-d\theta),v\rangle + \langle d\Delta v+B\theta,\theta\rangle\right)\\ &= \langle B\theta,\theta\rangle \qquad\qquad (\text{using } v\in H_0^2 \text{ for (2.5), (2.6)})\\ &= -\kappa\|\nabla\varphi\|_{L^2}^2 - \kappa a\|\Delta\varphi\|_{L^2}^2 \le 0 \qquad (\text{by (2.12)})\end{aligned}$$

□

Lemma 2.3. *The range of* Id $-A$ *equals $\mathcal H$.*

Proof: $(\mathrm{Id}-A)V=F$ is, for given $F\in\mathcal H$, equivalent to finding $V\in D(A)$ solving

$$\left.\begin{aligned} u-v &= F^1,\\ v+\Delta(b\Delta u+d\theta) &= F^2,\\ \theta-d\Delta v-B\theta &= F^3.\end{aligned}\right\}$$

$v:=u-F^1$ will be given if (u,θ) satisfy

$$\left.\begin{aligned} u+\Delta(b\Delta u+d\theta) &= F^2+F^1,\\ \theta-d\Delta u-B\theta &= F^3-d\Delta F^1.\end{aligned}\right\} \tag{2.19}$$

Consider the sesquilinear form $\beta : \mathcal{K} \to \mathbf{C}$, where

$$\mathcal{K} := \begin{cases} H_0^2 \times L^2 & \text{for (2.5)}, \\ H_0^2 \times L_*^2 & \text{for (2.6)}, \\ \left(H^2 \cap H_0^1\right) \times L^2 & \text{for (2.7)}, \end{cases}$$

$$\beta\left((u_1,\theta_1),(u_2,\theta_2)\right) := \langle u_1, u_2\rangle + \langle b\Delta u_1 + d\theta_1, \Delta u_2\rangle + \langle \theta_1, \theta_2\rangle - \langle d\Delta u_1, \theta_2\rangle - \langle B\theta_1, \theta_2\rangle,$$

and the associated variational problem to find, for any $(h,g) \in L^2 \times L^2_{(*)}$, a unique $(u,\theta) \in \mathcal{K}$ satisfying

$$\forall\,(\psi_1,\psi_2) \in \mathcal{K}:\ \beta\left((u,\theta),(\psi_1,\psi_2)\right) = \langle h, \psi_1\rangle + \langle g, \psi_2\rangle. \tag{2.20}$$

Solving (2.20) with

$$h := F^1 + F^2, \qquad g := F^3 - d\Delta F^1$$

gives the solution (u,θ) to (2.19). Here we use for the case of the boundary conditions (2.7) the self-adjointness of the Dirichlet-Laplace operator in L^2 with domain $H_0^2 \cap H_0^1$.

The solvability of the variational problem follows from the Theorem of Lax and Milgram, observing

$$|\beta\left((u_1,\theta_1),(u_2,\theta_2)\right)| \le c\|(u_1,\theta_1)\|_{H^2\times L^2}\|(u_2,\theta_2)\|_{H^2\times L^2},$$

with some positive constant $c > 0$, and, using the boundary conditions,

$$\begin{aligned} \operatorname{Re}\beta\left((u,\theta),(u,\theta)\right) &= \|u\|_{L^2}^2 + b\|\Delta u\|_{L^2}^2 + \|\theta\|_{L^2}^2 - \langle B\theta, \theta\rangle \\ &\ge \|u\|_{L^2}^2 + b\|\Delta u\|_{L^2}^2 + \|\theta\|_{L^2}^2 \\ &\ge c\|(u,\theta)\|_{H^2\times L^2}^2 \end{aligned}$$

with some positive constant $c > 0$ by elliptic regularity.

□

By the Lumer-Phillips theorem we conclude

Theorem 2.4. *A generates a contraction semigroup, and, for any $V^0 \in D(A)$, there is a unique solution V to (2.15) satisfying*

$$V \in C^1\left([0,\infty),\mathcal{H}\right) \cap C^0\left([0,\infty), D(A)\right).$$

By the contractivity we obtain the following stability estimate for the solution V to (2.15), for $t \ge 0$,

$$\|u(t,\cdot)\|_{H^2} + \|u_t(t,\cdot)\|_{L^2} + \|\theta(t,\cdot)\|_{L^2} \le k\left(\|u_0\|_{H^2} + \|u_1\|_{L^2} + \|\theta_0\|_{L^2}\right), \tag{2.21}$$

with a constant $k > 0$ not depending on t or on the data. More precisely, we have for the associated energy

$$E(t) := b\|\Delta u(t,\cdot)\|_{L^2}^2 + \|u_t(t,\cdot)\|_{L^2}^2 + \|\theta(t,\cdot)\|_{L^2}^2$$

the relation

$$E(t) = E(0) + 2\int_0^t \langle B\theta, \theta\rangle(r)dr \tag{2.22}$$

for any $V^0 \in D(A)$ and $t \ge 0$. This can be easily seen multiplying (2.9) by u_t (in L^2), and (2.10) by θ, adding, and using $u_t(t,\cdot) \in H_0^2$.

3 Well-posedness for $\tau > 0$

The model (1.5)–(1.8) for thermoelastic plates of Kirchhoff type with two temperatures under the Cattaneo law, i.e. for $\tau > 0$,

$$\begin{aligned} u_{tt} + b\Delta^2 u + d\Delta\theta &= 0, && (3.1)\\ \theta_t + \operatorname{div} q - d\,\Delta u_t &= 0, && (3.2)\\ \tau q_t + q + \kappa\nabla\varphi &= 0, && (3.3)\\ \theta - \varphi + a\Delta\varphi &= 0. && (3.4) \end{aligned}$$

will be shown to be well-posed for two different boundary conditions..

The well-posedness requires the choice of suitable representations of the solutions and corresponding phase spaces. The regularity issue is even more complicated due to the fact that the heat flux is not immediately of the same regularity as the gradient of the temperature φ, as it was in the case of the Fourier model discussed in the previous section.

The issue of only combined regularity for (u, θ, q) only, in contrast to separate regularity for each of u, θ, q, comes up again requiring the right spaces and domains of operators.

Initial conditions are given by

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0, \quad q(0,\cdot) = q_0 \qquad \text{in } \Omega, \tag{3.5}$$

while boundary conditions are prescribed either as before in (2.5) by

$$u(t,\cdot) = \frac{\partial u}{\partial \nu}(t,\cdot) = 0, \qquad \varphi(t,\cdot) = 0 \qquad \text{in } [0,\infty) \times \partial\Omega, \tag{3.6}$$

or as before in (2.7) by

$$u(t,\cdot) = (b\Delta u + d\theta)(t,\cdot) = 0, \qquad \varphi(t,\cdot) = 0 \qquad \text{in } [0,\infty) \times \partial\Omega. \tag{3.7}$$

Defining $w := u_t$, we obtain from (3.1)–(3.4)

$$\begin{aligned} w_{tt} + \Delta(b\Delta w + d\theta_t) &= 0, && (3.8)\\ \tau\theta_{tt} + \theta_t - B\theta - d\Delta w - \tau d\Delta w_t &= 0, && (3.9) \end{aligned}$$

where $B = \kappa\Delta\,(\mathrm{Id} - a\Delta)^{-1}$ denotes the operator defined in (2.11).

Solving (3.8), (3.9) with inital conditions

$$\begin{aligned} w(0,\cdot) = u_1 =: w_0, \quad w_t(0,\cdot) = -\Delta(b\Delta w + d\Delta\theta_0) =: w_1,\\ \theta(0,\cdot) = \theta_0, \quad \theta_t(0,\cdot) = -\mathrm{div} q_0 + d\Delta u_1 =: \theta_1 \end{aligned} \tag{3.10}$$

and boundary conditions on $\partial\Omega$, either

$$w(t,\cdot) = \frac{\partial w}{\partial \nu}(t,\cdot) = 0, \qquad \varphi(t,\cdot) = 0, \tag{3.11}$$

or

$$w(t,\cdot) = (b\Delta w + d\theta_t)(t,\cdot) = 0, \qquad \varphi(t,\cdot) = 0 \qquad \text{in } [0,\infty)\times\partial\Omega, \tag{3.12}$$

one gets a solution (u,θ,q) to (3.1)–(3.6) by integration. For the solvability for (w,θ), we transform it to a first-order system $W := (w, w_t, \theta, \theta_t)'$,

$$W_t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -b\Delta^2 & 0 & 0 & -d\Delta \\ 0 & 0 & 0 & 1 \\ \frac{1}{\tau}d\Delta & d\Delta & \frac{1}{\tau}B & -\frac{1}{\tau} \end{pmatrix} W \equiv A_{1,f}W, \qquad W(0,\cdot) = W^0 := (w_0, w_1, \theta_0, \theta_1)'. \tag{3.13}$$

System (3.13) together with the boundary conditions (3.11) or (3.12) will be considered as an evolution equation in the associated Hilbert space

$$\mathcal{H}_1 := \begin{cases} H_0^2 \times L^2 \times L^2 \times L^2 & \text{for (3.11)}, \\ \left(H^2 \cap H_0^1\right) \times L^2 \times L^2 \times L^2 & \text{for (3.12)}, \end{cases} \tag{3.14}$$

with inner product

$$\langle V, \tilde{V}\rangle_{\mathcal{H}} := b\langle \Delta w, \Delta\tilde{w}\rangle + \langle z, \tilde{z}\rangle + \langle \theta, \tilde{\theta}\rangle + \langle y, \tilde{y}\rangle.$$

Then

$$W_t = A_1 W, \quad W(t=0) = W^0, \tag{3.15}$$

where

$$A_1 : D(A_1) \subset \mathcal{H}_1 \to \mathcal{H}_1, \qquad A_1 W := A_{1,f}W, \tag{3.16}$$

for $W \in D(A_1)$ with

$$D(A_1) := \begin{cases} \{\, W = (w,z,\theta,y)' \in \mathcal{H}_1 \,|\, z \in H_0^2,\, \Delta(b\Delta w + dy) \in L^2 \,\} & \text{for (3.11)}, \\ \{\, W = (w,z,\theta,y)' \in \mathcal{H}_1 \,|\, z,\, (b\Delta w + dy) \in H^2 \cap H_0^1 \,\} & \text{for (3.12)}. \end{cases} \tag{3.17}$$

In the definition of $D(A_1)$, the problem of the missing (separate) regularity for $\Delta w, y$ is reflected again, cp. Section 2.

As in Section 2 one can show that A_1 generates a C_0-semigroup. For this purpose we write

$$\begin{aligned} A_1 W &= \begin{pmatrix} z \\ -\Delta(b\Delta w + d\Delta y) \\ y \\ \frac{1}{\tau}d\Delta w + d\Delta z + \frac{1}{\tau}B\theta - \frac{1}{\tau}y \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} z \\ -\Delta(b\Delta w + d\Delta y) \\ 0 \\ d\Delta z + \frac{1}{\tau}B\theta \end{pmatrix}}_{\equiv A_{11}W} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ y \\ \frac{1}{\tau}d\Delta w + \frac{1}{\tau}B\theta - \frac{1}{\tau}y \end{pmatrix}}_{\equiv A_{12}W} \end{aligned}$$

The operator

$$A_{12}: \mathcal{H}_1 \to \mathcal{H}_1$$

is bounded, and for

$$A_{11}: D(A_{11}) := D(A_1) \subset \mathcal{H}_1 \to \mathcal{H}_1$$

we have

Lemma 3.1. *(i) $D(A_{11})$ is dense in $\mathcal{H}_1$, and A_{11} is dissipative,*

$$\operatorname{Re}\langle A_{11}W, W\rangle_{\mathcal{H}_1} = 0.$$

(ii) The range of $\mathrm{Id} - A_{11}$ *equals $\mathcal{H}_1$.*

Proof: (i) is easy, and to solve $(\mathrm{Id} - A_{11})W = F$ is, for given $F \in \mathcal{H}_1$, we may argue as in the proof of Lemma 2.3 using the bilinear form $\beta_{11}: \mathcal{K}_1 \to \mathbf{C}$ with

$$\mathcal{K}_1 := \begin{cases} H_0^2 \times L^2 & \text{for (3.11)}, \\ \left(H^2 \cap H_0^1\right) \times L^2 & \text{for (3.12)}, \end{cases}$$

$$\beta_{11}\left((w_1, y_1), (w_2, y_2)\right) := \langle w_1, w_2\rangle + \langle b\Delta w_1 + dy_1, \Delta w_2\rangle + \langle y_1, y_2\rangle - \langle d\Delta w_1, y_2\rangle$$

and the Theorem of Lax and Milgram.
□
As a consequence we obtain the well-posedness of (3.15), corresponding to the boundary conditions (3.11) or (3.12),

Theorem 3.2. *A_1 generates a C_0-semigroup, and, for any $W^0 \in D(A_1)$, there is a unique solution W to (3.15) satisfying*

$$W \in C^1\left([0,\infty), \mathcal{H}_1\right) \cap C^0\left([0,\infty), D(A_1)\right).$$

In Section 5 we will demonstrate for the hinged boundary conditions (3.12) that there is *not* exponential decay. That is, we will have another example of a thermoelastic system for which the modeling with he Fourier law leads to an exponentially stable system (see Section 4), while the modeling with the Cattaneo law does not yield exponential stability (see Section 5).

4 Exponential stability for $\tau = 0$

Here we will show the exponential stability of the thermoelastic plate with two temperatures for $\tau = 0$, and for both boundary conditions (2.5) or (2.7) using semigroup methods. We use the following characterization of exponential stability given in [20] going back to Gearhart [10], Huang [11] and Prüß [23].

Theorem 4.1. *Let $\{e^{t\mathcal{A}_*}\}_{t\geq 0}$ be a C_0-semigroup of contractions generated by the operator $\mathcal{A}_*$ in the Hilbert space $\mathcal{H}_*$. Then the semigroup is exponentially stable if and only if $i\mathbb{R} \subseteq \varrho(\mathcal{A}_*)$ (resolvent set) and*

$$\overline{\lim}_{|\beta|\to\infty}||(i\beta I - \mathcal{A}_*)^{-1}|| < \infty, \qquad \beta \in \mathbb{R}. \tag{4.1}$$

The three steps to obtain exponential stability for $\{e^{tA}\}_{t\geq 0}$, with A from Section 2, are:

Step 1: Prove $0 \in \varrho(A)$. Step 2: Prove $i\mathbb{R} \subset \varrho(A)$. Step 3: Prove (4.1) for $\mathcal{A}_* := A$.

For *Step 1* we solve $A(u,v,\theta)' = F$ for $F = (F^1, F^2, F^3)' \in \mathcal{H}$, which is equivalent to solving

$$\left.\begin{array}{rcl} v &=& F^1, \\ -\Delta(b\Delta u + d\theta) &=& F^2, \\ d\Delta v + B\theta &=& F^3. \end{array}\right\}$$

Choosing $v := F^1$, having the desired regularity in $D(A)$, we solve

$$B\theta = F^3 + d\Delta F^1$$

by

$$\theta := B^{-1}(F^3 + d\Delta F^1) = \frac{1}{\kappa}\left(\Delta^{-1} - a\mathrm{Id}\right)(F^3 + d\Delta F^1).$$

For the boundary condition (2.5), we solve

$$b\Delta^2 u = -F^2 - \Delta\theta \in H^{-2} = (H_0^2)' \quad \text{(dual space)}$$

with a unique $u \in H_0^2$. By construction we have

$$(u,v,\theta)' \in D(A), \qquad \|(u,v,\theta)\|_{\mathcal{H}} \leq \|F\|_{\mathcal{H}}. \tag{4.2}$$

For the boundary condition (2.7), we first solve

$$-\Delta w = F^2, \qquad w \in H^2 \cap H_0^1,$$

and then

$$b\Delta u = w - d\theta, \quad u \in H^2 \cap H_0^1.$$

Again we have (4.2), thus for both boundary condition $0 \in \varrho(A)$.

To prove *Step 2* we assume that the imaginary axis is not contained in the resolvent set. Following standard arguments ([20, p.25]), there exists a real number $\omega \neq 0$ with $\|A^{-1}\|^{-1} \leq |\omega| < \infty$ such that the set $\{i\lambda \subset i\mathbf{R} \,|\, |\lambda| < |\omega|\} \subset \varrho(A)$ and $\sup\{\|(i\lambda - A)^{-1}\| \,|\, |\lambda| < |\omega|\} = \infty$. This implies the existence of sequences $(\lambda_n)_n \subset \mathbf{R}$ and $(W_n)_n \subset D(A)$ with

$$\lambda_n \to \omega, \quad \|W_n\|_{\mathcal{H}} = 1, \quad (i\lambda_n - A)W_n =: F_n \to 0, \qquad \text{as} \quad n \to \infty.$$

With the notation $W_n = (u_n, v_n, \theta_n)'$ and $F_n = (F_n^1, F_n^2, F_n^3)'$ we conclude

$$i\lambda_n u_n - v_n = F_n^1, \tag{4.3}$$

$$i\lambda_n v_n + \Delta(b\Delta u_n + d\theta_n) = F_n^2, \tag{4.4}$$

$$i\lambda_n \theta_n - B\theta_n - d\Delta v_n = F_n^3. \tag{4.5}$$

The dissipativity of A from Lemma 2.2 gives for $\theta_n = \varphi_n - a\Delta\varphi_n$

$$\kappa\|\nabla\varphi_n\|^2 + a\kappa\|\Delta\varphi_n\|^2 = -\mathrm{Re}\,\langle AW_n, W_n\rangle_{\mathcal{H}} \to 0,$$

implying

$$\varphi_n \to 0 \quad \text{in } H^2, \qquad \theta_n \to 0, \qquad B\theta_n \to 0. \tag{4.6}$$

From (4.5) we conclude

$$\frac{\Delta v_n}{\lambda_n} \to 0. \tag{4.7}$$

This implies together with (4.3)

$$\Delta u_n \to 0,$$

hence

$$u_n \to 0 \quad \text{in } H^2. \tag{4.8}$$

Multiplying (4.4) by u_n in L^2 yields

$$i\langle v_n, \lambda_n u_n\rangle + \langle \Delta(b\Delta u_n + d\theta_n), u_n\rangle = \langle F_n^2, u_n\rangle \to 0.$$

Using

$$\langle \Delta(b\Delta u_n + d\theta_n), u_n\rangle = \langle b\Delta u_n + d\theta_n, \Delta u_n\rangle$$

which holds under both boundary conditions (2.5) or (2.7), we conclude

$$-\|v_n\|^2 - \langle v_n, F_n^1\rangle + b\|\Delta u_n\|^2 + d\langle \theta_n, \Delta u_n\rangle \to 0,$$

implying

$$v_n \to 0. \tag{4.9}$$

With (4.6), (4.8), (4.9) we conclude $\|W_n\|_{\mathcal{H}} \to 0$ which is a contradiction to $\|W_n\|_{\mathcal{H}} = 1$. Hence we have proved $i\mathbf{R} \subset \varrho A$.

Finally, to prove *Step 3* we can argue by assuming that (4.1) does not hold and conclude a contradiction in the same way as in Step 2, the only difference being that $|\lambda_n| \to \infty$ instead of $\lambda_n \to \omega$.

Altogether we have proved the exponential stability.

Theorem 4.2. *Solutions to the thermoelastic plate equation with two temperatures, both for the boundary conditions (2.5) and for the boundary conditions (2.7), tend to zero exponentially uniformly, i.e. the associated semigroup* $\{e^{tA}\}_{t\geq 0}$ *is exponentially stable.*

5 Non-exponential stability for $\tau > 0$

Here we discuss the thermoelastic plate with two temperatures and $\tau > 0$, (1.5)–(1.8), in space dimensions $n \geq 1$,

$$u_{tt} + b\Delta^2 u + d\Delta\theta = 0, \tag{5.1}$$

$$\theta_t + \operatorname{div} q - d\,\Delta u_t = 0, \tag{5.2}$$

$$\tau q_t + q + \kappa\nabla\varphi = 0, \tag{5.3}$$

$$\theta - \varphi + a\Delta\varphi = 0 \tag{5.4}$$

with hinged boundary conditions,

$$u(t,\cdot) = (b\Delta u + d\theta)(t,\cdot) = 0, \qquad \varphi(t,\cdot) = 0. \tag{5.5}$$

We will prove the existence of slowly decaying solutions giving the non-exponential stability. That is we have another example where Fourier models gives exponential stability (according to Section 4; cp. also Section 6), while the Cattaneo model is not exponentially stable.

Theorem 5.1. *The system (5.1)–(5.5) is not exponentially stable.*

Proof: Let $(\chi_j)_j$ denote the eigenfunctions of the Laplace operator for Dirichlet boundary conditions,

$$-\Delta\chi_j = \lambda_j\chi_j, \quad \chi_j = 0 \text{ on } \partial\Omega,$$

with eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to \infty$ as $j \to \infty$.

We make the ansatz

$$u_j(t,x) = a_j(t)\chi_j(x), \qquad q_j(t,x) = d_j(t,x)\nabla\chi_j(x),$$

$$\varphi_j(t,x) = b_j(t)\chi_j(x), \qquad \theta_j(t,x) = (1+a\lambda_j)b_j(t)\chi_j(x).$$

This ansatz is compatible with the differential equations (5.1)–(5.4) and with the boundary conditions (5.5). It gives a solution $(u_j, q_j, \varphi_j, \theta_j)$ if the coefficients (a_j, b_j, d_j) satisfy the following system of ODEs, where a prime $'$ denotes here differentiation with respect to time t,

$$\left.\begin{array}{rcl} a_j'' + b\lambda_j^2 a_j - d\lambda_j(1+a\lambda_j)b_j &=& 0, \\ (1+a\lambda_j)b_j' - \lambda_j d_j + d\lambda_j a_j' &=& 0, \\ \tau d_j' + d_j + \kappa b_j &=& 0. \end{array}\right\} \tag{5.6}$$

System (5.6) is equivalent to a first-order system for the column vector $V_j := (a_j, a_j', b_j, q_j)$,

$$V_j' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -b\lambda_j^2 & 0 & d\lambda_j(1+a\lambda_j) & 0 \\ 0 & -\frac{d\lambda_j}{1+a\lambda_j} & 0 & \frac{\lambda_j}{1+a\lambda_j} \\ 0 & 0 & -\frac{\kappa}{\tau} & -\frac{1}{\tau} \end{pmatrix} V_j \equiv \mathcal{A}_j V_j. \tag{5.7}$$

We are looking for solutions to (5.7) of type $V_j(t) = \mathrm{e}^{\omega_j t}V_j^0$. In other words, ω_j has to be an eigenvalue of $\mathcal{A}_j$ with eigenvector V_j^0 as initial data. It is the aim to demonstrate that, for any given small $\varepsilon > 0$, we have some j and some eigenvalue ω_j such that the real part $\mathrm{Re}\,\omega_j$ of ω_j is larger than $-\varepsilon$. This will contradict the exponential stability. We have

$$\begin{aligned} \det(\mathcal{A}_j - \omega\,\mathrm{Id}) &= \frac{1}{\tau(1+a\lambda_j)}\left\{\tau(1+a\lambda_j)\omega^4 + (1+a\lambda_j)\omega^3 + \right. \\ &\quad \left[\tau d^2\lambda_j^2(1+a\lambda_j) + \kappa\lambda_j + \tau b\lambda_j^2(1+a\lambda_j)\right]\omega^2 \\ &\quad \left. + \left[d^2\lambda_j^2(1+a\lambda_j) + b\lambda_j^2(1+a\lambda_j)\right]\omega + b\kappa\lambda_j^3\right\} \\ &\equiv \frac{1}{\tau(1+a\lambda_j)}\,P_j(\omega). \end{aligned}$$

To show that

$$\forall \varepsilon > 0 \;\; \exists j \;\; \exists \omega_j, \; P_j(\omega_j) = 0 : \;\; \operatorname{Re} \omega_j \geq -\varepsilon,$$

we introduce, for small $\varepsilon > 0$,

$$z := \omega + \varepsilon, \qquad P_{j,\varepsilon}(z) := P_j(z - \varepsilon).$$

That is, we have to show

$$\forall \; 0 < \varepsilon \ll 1 \;\; \exists j \;\; \exists z_j, \; P_{j,\varepsilon}(z_j) = 0 : \;\; \operatorname{Re} z_j \geq 0. \tag{5.8}$$

To prove (5.8) we start with computing

$$P_{j,\varepsilon}(z) = q_4 z^4 + q_3 z^3 + q_2 z^2 + q_1 z + q_0$$

where

$$\begin{aligned}
q_4 &= \tau(1 + a\lambda_j), \\
q_3 &= (1 - 4\varepsilon\tau)(1 + a\lambda_j), \\
q_2 &= 6\tau\varepsilon^2 - 3\varepsilon + (6\tau\varepsilon^2 a - 3\varepsilon a\kappa)\lambda_j, \\
q_1 &= -4\tau\varepsilon^3(1 + a\lambda_j) + 3\varepsilon^2(1 + a\lambda_j) - 2\varepsilon\left(\tau(b + d^2)\lambda_j^2 + \tau a(b + d^2)\lambda_j^3\right) \\
&\quad + b\lambda_j^2 + ab\lambda_j^3 + d^2\lambda_j^2 + ad^2\lambda_j^3, \\
q_0 &= \tau\varepsilon^4(1 + a\lambda_j) - \varepsilon^3(1 + a\lambda_j) + \varepsilon^2\left((\tau b + \tau d^2)\lambda_j^2 + \tau a(b + d^2)\lambda_j^3 + \kappa\lambda_j\right) \\
&\quad - \varepsilon\left(b^2\lambda_j^2 + ab\lambda_j^3 + d^2\lambda_j^2 + ad^2\lambda_j^3\right) + b\kappa\lambda_j^3.
\end{aligned}$$

The coefficients $q_0, \ldots, q_4$ are positive for sufficiently small ε and large j, since $\lambda_j \to \infty$. We use the Hurwitz criterion [31]: Let

$$\mathbb{H} := \begin{pmatrix} q_3 & q_4 & 0 & 0 \\ q_1 & q_2 & q_3 & q_4 \\ 0 & q_0 & q_1 & q_2 \\ 0 & 0 & 0 & q_0 \end{pmatrix}$$

denote the Hurwitz matrix associated to the polynomial $P_{j,\varepsilon}$. Then (5.8) is fulfilled if we find, for given small $\varepsilon > 0$, a (sufficiently large) index j such that one of the principal minors of $\mathbb{H}^j$ is not positive. The principal minors are given by the determinants $\det D_m^j$ of the matrices D_m^j, for $m = 1, 2, 3, 4$, where D_m^j denotes the upper left square submatrix of $\mathbb{H}^j$ consisting of the elements $\mathbb{H}_{11}^j, \ldots, \mathbb{H}_{mm}^j$.

We compute

$$\det D_{j,2} = q_3 q_2 - q_1 q_4 = -2\varepsilon\tau^2 a^2 (b + d^2)\lambda_j^4 + \mathcal{O}(\lambda_j^3) < 0$$

as $j \to \infty$.

□

6 Exponential stability for the second-order system

The *second-order* thermoelastic system with two temperatures (1.9)–(1.12) is for $\tau = 0$ given by

$$u_{tt} - bu_{xx} + d\theta_x = 0 \quad \text{in } [0,\infty) \times (0,1), \tag{6.1}$$

$$\theta_t + q_x + du_{tx} = 0 \quad \text{in } [0,\infty) \times (0,1), \tag{6.2}$$

$$q + \kappa\varphi_x = 0 \quad \text{in } [0,\infty) \times (0,1), \tag{6.3}$$

$$\theta - \varphi + a\varphi_{xx} = 0 \quad \text{in } [0,\infty) \times (0,1), \tag{6.4}$$

together with initial conditions,

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0 \qquad \text{in } (0,1), \tag{6.5}$$

and boundary conditions

$$u(t,0) = u(t,1) = 0, \quad \varphi_x(t,0) = \varphi_x(t,1) = 0, \qquad \text{for } t \geq 0. \tag{6.6}$$

Assuming

$$\int_0^1 \theta_0(x)dx = 0 \tag{6.7}$$

we avoid trivial non-decaying solutions as $(u \equiv 0, \theta \equiv 1)$. We notice that by equation (6.2) and the boundary conditions (6.6) the property of mean value zero for θ is invariant in time. i.e. assuming (6.7), we have for any $t \geq 0$

$$\int_0^1 \theta(t,x)(x)dx = 0. \tag{6.8}$$

In [24] Quintanilla investigated time-harmonic solutions showing that the real parts of the possible frequencies are strictly less than zero indicating exponential stability of the semigroup. Here we strictly prove the exponential stability by energy estimates, on one hand in order to complete [24], and on the other hand, as first demonstration of possible considerations for proving exponential stability with energy methods in the case of two temperatures. The latter will be considered for Robin type boundary conditions in the temperature for the fourth-order system in Section 7.

The well-posedness of (6.1)–(6.6) has been studied (for Dirichlet type boundary conditions instead of Neumann type boundary conditions for φ) by Quintanilla in [25] for any space dimension $n = 1, 2, 3$.

As regularity of the solutions we have

$$u \in C^1([0,\infty), H_0^1),\ u_{tt} \in C^0(([0,\infty), L^2),\ \theta \in C^1([0,\infty), L^2),\ (bu_x - d\theta) \in C^0([0,\infty), H^1).$$

Let $(\mathrm{Id} - a\partial_{xx})^{-1}$ denote the homeomorphism from $L^2 \cap \{\, f \in L^2 \mid \int_0^1 f(x)dx = 0 \,\}$ onto $H^2 \cap \{\, \varphi \in L^2 \mid \int_0^1 \varphi(x)dx = 0,\ \ \varphi_x(0) = \varphi_x(1) = 0 \,\}$. Defining

$$B_1 : L^2 \to L^2, \qquad B_1\theta := \kappa\partial_{xx} (\mathrm{Id} - a\partial_{xx})^{-1} \theta, \tag{6.9}$$

we rewrite (6.1)–(6.4) as

$$\begin{aligned} u_{tt} - bu_{xx} + d\theta_x &= 0, & (6.10)\\ \theta_t - B_1\theta + du_{tx} &= 0. & (6.11)\end{aligned}$$

Defining the associated energy

$$E_1(t) := b\|u_x(t,\cdot)\|^2_{L^2} + \|u_t(t,\cdot)\|^2_{L^2} + \|\theta(t,\cdot)\|^2_{L^2}$$

we have, cp. (2.22), (2.12),

$$\frac{d}{dt}E_1(t) = -2\langle B_1\theta,\theta\rangle = -\kappa\|\varphi_x\|^2_{L^2} - \kappa a\|\varphi_{xx}\|^2_{L^2}.$$

Observing

$$\theta = \varphi - a\varphi_{xx}$$

we conclude

$$\frac{d}{dt}E_1 \leq -c_1\|\theta\|^2_{L^2}, \tag{6.12}$$

where c_1 will denote a positive constant in this section, possibly varying from line to line, and not depending on t or the initial data.

Multiplying (6.10) by u in L^2, we obtain

$$\begin{aligned} \frac{d}{dt}\frac{d}{2}\langle u_t,u\rangle &= -\frac{db}{2}\|u_x\|^2_{L^2} + \frac{d}{2}\|u_t\|^2_{L^2} + \frac{d^2}{2}\langle\theta,u_x\rangle \\ &\leq -\frac{db}{4}\|u_x\|^2_{L^2} + \frac{d}{2}\|u_t\|^2_{L^2} + \frac{d^3}{4b}\|\theta\|^2_{L^2}. & (6.13)\end{aligned}$$

Because of the given regularity, not providing higher regularity for u and θ separately, we cannot use some multipliers typically used in thermoelasticity, cp. [28]. Therefore, we lift the regularity by integrating in space. For this purpose let

$$\eta(t,x) := \int_0^x \theta(t,y)dy. \tag{6.14}$$

Integrating (6.11) we get $\eta(t,\cdot) \in H^1_0$ and

$$\eta_t - \int_0^x B_1\theta dy + u_t = 0. \tag{6.15}$$

Let P denote the operator given by

$$(Pw)(x) := \int_0^x w(y)dy.$$

Remark 6.1. *Let B denote the operator defined in (2.11) in Section 2 again, now for $\Omega = (0,1)$ Then we have*

$$P \circ B_1 = B \circ P \qquad \text{on } L^2.$$

This is equivalent to proving $p = q_x$ for p and q satisfying

$$p - ap_{xx} = \theta,\ p_x(0) = p_x(1) = 0, \qquad q - aq_{xx} = P\theta,\ q(0) = q(1) = 0,$$

which easily follows using the boundary conditions and uniqueness for the boundary value problems.

Multiplying (6.15) by u_t, we obtain

$$\begin{aligned}\frac{d}{dt}\langle \eta, u_t\rangle &= \langle PB_1\theta, u_t\rangle - d\|u_t\|_{L^2}^2 + \langle \eta, u_{tt}\rangle \\ &= \langle PB_1\theta, u_t\rangle - d\|u_t\|_{L^2}^2 + \langle \eta, (bu_x - d\theta)_x\rangle \\ &= \langle PB_1\theta, u_t\rangle - d\|u_t\|_{L^2}^2 - b\langle \theta, u_x\rangle + d\|\theta\|_{L^2}^2 \\ &\le \frac{1}{d}\|PB_1\theta\|_{L^2}^2 - \frac{3d}{4}\|u_t\|_{L^2}^2 + \frac{db}{8}\|u_x\|_{L^2}^2 + \left(d + \frac{2}{db}\right)\|\theta\|_{L^2}^2. \end{aligned} \tag{6.16}$$

Since

$$\|PB_1\theta\|_{L^2}^2 \le c\|\theta\|_{L^2}^2$$

for some (generic) $c > 0$, we conclude from (6.16)

$$\frac{d}{dt}\langle \eta, u_t\rangle \le -\frac{3d}{4}\|u_t\|_{L^2}^2 + \frac{db}{8}\|u_x\|_{L^2}^2 + c\|\theta\|_{L^2}^2. \tag{6.17}$$

Combining (6.13) and (6.17) we get

$$\frac{d}{dt}\mathrm{Re}\,\left(\langle u_t, u\rangle + \langle \eta, u_t\rangle\right) \le -\frac{d}{4}\|u_t\|_{L^2}^2 - \frac{db}{8}\|u_x\|_{L^2}^2 + c\|\theta\|_{L^2}^2. \tag{6.18}$$

Defining the Lyapunov functional L_1 for $\varepsilon > 0$ by

$$L_1(t) := E_1(t) + \varepsilon\left\{\mathrm{Re}\,\left(\langle u_t, u\rangle + \langle \eta, u_t\rangle\right)\right\},$$

and choosing ε small enough, we obtain from (6.12) and (6.18)

$$\frac{d}{dt}L_1(t) \le -\alpha_1 E_1(t), \tag{6.19}$$

with some $\alpha_1 > 0$ depending on ε and on c_1 (i.e. the coefficients of the differential equations). Observing

$$|\varepsilon\left\{\mathrm{Re}\,\left(\langle u_t, u\rangle + \langle \eta, u_t\rangle\right)\right\}| \le \frac{1}{2}E_1(t)$$

for sufficiently small ε, we have, for any $t \ge 0$,

$$\frac{1}{2}E_1(t) \le L_1(t) \le \frac{3}{2}E_1(t) \tag{6.20}$$

and then, by (6.8),

$$\frac{d}{dt}L_1(t) \le -\underbrace{\frac{2}{3}\alpha_1}_{=:\alpha} L_1(t),$$

and, finally, obtain

Theorem 6.2. *The energy E_1 for the system (6.10), (6.11) decays exponentially. For any $t \ge 0$ we have*

$$E_1(t) \le 3E_1(0)\mathrm{e}^{-\alpha t}.$$

with a positive constant α which does not depend on t or on the data. In terms of the associated semigroup, we have exponential stability.

The statement on the exponential stabilty of the semigroup follows, as usual, from the fact, that after transforming to a first-order system for $V = (u, u_t, \theta)$ as in Section 2, the square of the norm of the solution is equivalent to E_1.

The discussion in this section prepares the one of the next section for the fourth-order operator where the (low) regularity of the solution also will require a (more complicated) lifting in space, here reflected by the definition of η above.

7 Exponential stability for the fourth-order system for $\tau = 0$ for Robin type boundary conditions

We prove the exponential stability for the thermoelastic plate equation with two temperatures and $\tau = 0$, (2.1)–(2.4), (2.6), in one space dimension,

$$\begin{aligned} u_{tt} + bu_{xxxx} + d\theta_{xx} &= 0 \qquad \text{in } [0,\infty)\times(0,1), && (7.1)\\ \theta_t + q_x - du_{txx} &= 0 \qquad \text{in } [0,\infty)\times(0,1), && (7.2)\\ q + \kappa\varphi_x &= 0 \qquad \text{in } [0,\infty)\times(0,1), && (7.3)\\ \theta - \varphi + a\varphi_{xx} &= 0 \qquad \text{in } [0,\infty)\times(0,1), && (7.4) \end{aligned}$$

$$u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1, \quad \theta(0,\cdot) = \theta_0 \qquad \text{in } (0,1), \tag{7.5}$$

with boundary conditions, for $t \geq 0$,

$$u(t,0) = u(t,1) = u_x(t,0) = u_x(t,1) = 0, \tag{7.6}$$

$$R_1[\varphi(t,\cdot)] := \varphi_x(t,1) - \varphi_x(t,0) = 0, \quad R_2[\varphi(t,\cdot)] := \varphi_x(t,1) - (\varphi(t,1) - \varphi(t,0)) = 0. \tag{7.7}$$

These mixed periodic-Robin-type boundary conditions for φ are different from those considered in Section 2, so we will first give the arguments for the existence of solutions. The reason for considering these boundary conditions is mainly technical: the Dirichlet resp. Neumann boundary conditions for φ, studied in (2.5) resp. (2.6) in Section 2 and discussed in Section 4 in any space dimention, do not allow to carry over the energy approach below. The choice of the function h below has to assure $h \in H_0^2$ and simultaneously to be compatible with different aspects of the equations. Lifting the regularity, expressed in the definition of h and corresponding to the lifting given for the second-order problem in the previous section in the definition of η in (6.14), seems appropriate.

For the existence of solutions in appropriate space we first consider the operator

$$\Delta_m : D(\Delta_m) := \{\, \varphi \in H^2 \,|\, R_1[\varphi] = R_2[\varphi] = 0 \,\} \subset L^2 \to L^2,$$

$$\Delta_m \varphi := \varphi_{xx}.$$

Lemma 7.1. $(\mathrm{Id} - a\Delta_m)^{-1}$ *exists on* L^2 *and is a homeomorphism onto* $D(\Delta_m)$ *(the latter equipped with the* H^2*-norm).*

PROOF: By the theory of boundary value problems for ODEs, it is sufficient to check that the homogeneous problem

$$\varphi - a\varphi_{xx} = 0, \quad R_1[\varphi] = R_2[\varphi] = 0 \tag{7.8}$$

only has the trivial solution. Let φ solve (7.8). Then

$$\varphi(x) = \alpha\, \mathrm{e}^{\frac{1}{\sqrt{a}}\, x} + \beta\, \mathrm{e}^{-\frac{1}{\sqrt{a}}\, x}$$

with constants $\alpha, \beta \in \mathbf{R}$. Since $R_1[\varphi] = 0$ we get

$$\beta = \alpha \frac{\mathrm{e}^{\frac{1}{\sqrt{a}}} - 1}{\mathrm{e}^{-\frac{1}{\sqrt{a}}} - 1}.$$

Using $R_2[\varphi] = 0$ we conclude

$$\alpha \left(\frac{1}{\sqrt{a}} + 1 - \mathrm{e}^{\frac{1}{\sqrt{a}}} \right) = \alpha \frac{\mathrm{e}^{\frac{1}{\sqrt{a}}} - 1}{\mathrm{e}^{-\frac{1}{\sqrt{a}}} - 1} \left(\frac{1}{\sqrt{a}} - 1 + \mathrm{e}^{-\frac{1}{\sqrt{a}}} \right). \tag{7.9}$$

Equation (7.9) is never satisfied for $0 < a < \infty$ unless $\alpha = 0$ which implies $\varphi = 0$. The argument for this is, for example, to look, for $\gamma := \frac{1}{\sqrt{a}} \in (0, \infty)$, at the function

$$f(\gamma) := (2 - \gamma)\, \mathrm{e}^{\gamma} + (2 + \gamma)\, \mathrm{e}^{-\gamma} - 4.$$

Equation (7.9) is equivalent to

$$\alpha f(\gamma) = 0,$$

and it suffices to prove, for all $\gamma \in (0, \infty)$,

$$f(\gamma) < 0.$$

The latter follows easily for $\gamma \geq 0$, and for $0 < \gamma < 2$ it suffices to look at

$$f'(\gamma) = (1 - \gamma)\, \mathrm{e}^{\gamma} - (1 + \gamma)\, \mathrm{e}^{-\gamma},$$

observing $f'(\gamma) < 0$ for $\gamma \geq 1$. For $0 < \gamma < 1$ we notice $f'(0) < 0$, and f' does not have any zero in $(0, 1)$ since

$$\mathrm{e}^{2\gamma} < \frac{1 + \gamma}{1 - \gamma}$$

for $\gamma \in (0, 1)$ (by considering $g(z) := z - \ln(2 + z) - \ln(2 - z)$ for $z \in (0, 2)$ and proving $g' < 0$ there).

□

We rewrite the differential equations as in Section 2 in the form

$$u_{tt} + \partial_{xx}(bu_{xx} + d\theta) = 0, \tag{7.10}$$

$$\theta_t - B_m\theta - du_{txx} = 0, \tag{7.11}$$

where

$$B_m : L^2 \to L^2, \qquad B_m\theta := \kappa\Delta\left(\mathrm{Id} - a\Delta_m\right)^{-1}\theta. \tag{7.12}$$

The transformation to a first-order system for $V := (u, u_t, \theta)'$ reads as

$$V_t = A_m V, \quad V(t=0) = V^0 := (u_0, u_1, \theta_0)', \tag{7.13}$$

where

$$A_m : D(A_m) \subset \mathcal{H}_m \to \mathcal{H}_m, \qquad A_m V := \begin{pmatrix} 0 & 1 & 0 \\ -b\partial_{xxxx} & 0 & -d\partial_{xx} \\ 0 & d\partial_{xx} & B_m \end{pmatrix} V,$$

for $V \in D(A_m)$ with

$$\mathcal{H}_m := H_0^2 \times L^2 \times L^2,$$

with inner product

$$\langle V, \tilde{V}\rangle_{\mathcal{H}_m} := b\langle u_{xx}, \tilde{u}_{xx}\rangle + \langle v, \tilde{v}\rangle + \langle \theta, \tilde{\theta}\rangle.$$

$$D(A_m) := \{\, V = (u, v, \theta)' \in \mathcal{H}_m \,|\, v \in H_0^2,\ \partial_{xx}(bu_{xx} + d\theta) \in L^2 \,\}.$$

The unique solvability is obtained as in Section 2 giving

Theorem 7.2. *A_m generates a contraction semigroup, and, for any $V^0 \in D(A_m)$, there is a unique solution V to (7.13) satisfying*

$$V \in C^1\left([0,\infty), \mathcal{H}_m\right) \cap C^0\left([0,\infty), D(A_m)\right).$$

The exponential stability is of course only on the orthogonal complement of the null space of A_m.

Lemma 7.3. *(i)* $N(A_m) = \{0\} \times \{0\} \times \{\, x \mapsto \alpha x + \beta \,|\, \alpha, \beta \in \mathbf{R}\,\}$.

(ii) $N(A_m)^\perp = H_0^2 \times L^2 \times \{\, f \in L^2 \,|\, \int_0^1 f(x)dx = \int_0^1 x f(x)dx = 0\,\}$.

Proof: $A_m V = 0$ implies

$$v = 0, \quad \partial_{xx}(b\partial_{xx}u + d\theta) = 0, \quad B_m\theta = 0.$$

From $B_m\theta = 0$ we conclude $\varphi_{xx} = 0$, hence $\varphi(x) = \alpha x + \beta$. But all these functions are in $D(\Delta_m)$ since

$$R_1[x \mapsto \alpha x + \beta] = R_2[x \mapsto \alpha x + \beta] = 0.$$

Then

$$\theta(x) = \varphi(x) - a\varphi_{xx}(x) = \varphi(x) = \alpha x + \beta.$$

Finally, $\partial_{xxxx}u = 0$ and $u \in H_0^2$ imply $u = 0$.
This proves (i) and hence (ii).
□
For the following Lemma, the special boundary conditions for φ are important.

Lemma 7.4. *$N(A_m)^\perp$ is invariant under A_m.*

PROOF: Let $V = (u,v,\theta)' \in N(A_m)^\perp \cap D(A_m)$, i.e. $V \in D(A_m)$ and $\int_0^1 \theta(x)dx = \int_0^1 x\theta(x)dx = 0$. We have to show

$$\int_0^1 dv_{xx}(x) + B_m\theta(x)dx = \int_0^1 x\left(dv_{xx}(x) + B_m\theta(x)\right)dx = 0.$$

Since $v \in H_0^2$, we have

$$\int_0^1 v_{xx}(x)dx = v_x(1) - v_x(0) = 0, \qquad \int_0^1 xv_{xx}dx = v_x(1) - (v(1) - v(0)) = 0.$$

Moreover,

$$\int_0^1 B_m\theta(x)dx = \kappa\int_0^1 \varphi_{xx}(x)dx = \kappa\left(\varphi_x(1) - \varphi_x(0)\right) = \kappa R_1[\phi] = 0$$

and

$$\int_0^1 xB_m\theta(x)dx = \kappa\int_0^1 x\varphi_{xx}(x)dx = \kappa\left(\varphi_x(1) - (\varphi(1) - \varphi(0))\right) = \kappa R_2[\varphi] = 0.$$

□

Lemma 7.5. *For $\theta = \varphi - a\varphi_{xx}$ with $\int_0^1 \varphi(x)dx = 0$ and $\int_0^1 x\varphi(x)dx = 0$, we have*

$$\int_0^1 \theta(x)dx = 0, \qquad \int_0^1 x\theta(x)dx = 0.$$

PROOF: Using $R_1[\varphi] = R_2[\varphi] = 0$, we have

$$\int_0^1 \theta(x)dx = \int_0^1 \varphi(x)dx, \qquad \int_0^1 x\theta(x)dx = \int_0^1 x\varphi(x)dx.$$

□

Defining the associated energy

$$E_m(t) := b\|u_{xx}(t,\cdot)\|_{L^2}^2 + \|u_t(t,\cdot)\|_{L^2}^2 + \|\theta\|_{L^2}^2,$$

we have

$$\frac{d}{dt}E_m = \langle B_m\theta, \theta\rangle \leq -\kappa a\|\varphi_{xx}\|_{L^2}^2 \leq -c_1\|\theta\|_{L^2}^2, \tag{7.14}$$

where c_1 will denote a positive constant in this section, possibly varying from line to line, and not depending on t or the initial data. (7.14) follows, using $R_1[\varphi] = R_2[\varphi] = 0$, from

$$\begin{aligned}
\langle B_m\theta, \theta\rangle &= \kappa\langle\varphi_{xx}, \varphi - a\varphi_{xx}\rangle = \kappa\left(\varphi_x(1)\varphi(1) - \varphi_x(0)\varphi(0) - (\|\varphi_x\|_{L^2}^2 + a\|\varphi_{xx}\|_{L^2}^2)\right)\\
&= \kappa\left((\varphi(1) - \varphi(0))^2 - (\|\varphi_x\|_{L^2}^2 + a\|\varphi_{xx}\|_{L^2}^2)\right)\\
&= \kappa\left(\left(\int_0^1 \varphi_x(x)dx\right)^2 - (\|\varphi_x\|_{L^2}^2 + a\|\varphi_{xx}\|_{L^2}^2)\right)\\
&\leq -\kappa a\|\varphi_{xx}\|_{L^2}^2.
\end{aligned}$$

Multiplying (7.10) by u in L^2, we obtain

$$\begin{aligned}
\frac{d}{dt}\frac{d}{2}\langle u_t, u\rangle &= -\frac{db}{2}\|u_{xx}\|_{L^2}^2 + \frac{d}{2}\|u_t\|_{L^2}^2 + \frac{d^2}{2}\langle \theta, u_x\rangle \\
&\leq -\frac{db}{4}\|u_{xx}\|_{L^2}^2 + \frac{d}{2}\|u_t\|_{L^2}^2 + \frac{d^3}{4b}\|\theta\|_{L^2}^2. \qquad (7.15)
\end{aligned}$$

The regularity lifting is done as follows. Let

$$\eta(t,x) := \int_0^x \theta(t,y)dy. \qquad (7.16)$$

By Lemma 7.5 we have $\eta \in H_0^1$. Let

$$h(t,x) := \int_0^x \eta(t,y)dy = \int_0^x \int_0^y \theta(t,z)dzdy. \qquad (7.17)$$

Then

$$h \in H_0^2, \qquad h_{xx} = \theta.$$

The boundary conditions for h follow from Lemma 7.5 since

$$h_x(t,1) = \int_0^1 \theta(t,x)dx = 0. \qquad h(t,1) = \int_0^1 \theta(t,x)dx - \int_0^1 x\theta(t,x)dx = 0.$$

That is, the reason for looking at the special boundary conditions was to, finally, achieve $h \in H_0^2$, which will allow a kind of partial integration in (7.19).

Integrating (7.11) twice, using $u \in H_0^2$, we obtain

$$h_t - P_1 B_m \theta - du_t = 0, \qquad (7.18)$$

where

$$(P_1 w)(x) := \int_0^x \int_0^y w(z)dzdy$$

defines a bounded operator in L^2. Multiplying (7.18) by u_t in L^2, we obtain

$$\begin{aligned}
-\frac{d}{dt}\langle h, u_t\rangle &= -\langle P_1 B_m \theta, u_t\rangle + d\|u_t\|_{L^2}^2 - \langle h, u_{tt}\rangle \\
&= -\langle P_1 B_m \theta, u_t\rangle - d\|u_t\|_{L^2}^2 + \langle h, (bu_{xx} + d\theta)_{xx}\rangle \\
&= -\langle P_1 B_m \theta, u_t\rangle - d\|u_t\|_{L^2}^2 - b\langle \theta, u_{xx}\rangle - d\|\theta\|_{L^2}^2. \qquad (7.19)
\end{aligned}$$

The last equality was obtained because $h \in H_0^2$. Thus

$$\begin{aligned}
-\frac{d}{dt}\langle h, u_t\rangle &\leq \frac{1}{d}\|P_1 B_m \theta\|_{L^2}^2 - \frac{3d}{4}\|u_t\|_{L^2}^2 + \frac{db}{8}\|u_{xx}\|_{L^2}^2 + \left(d + \frac{2}{db}\right)\|\theta\|_{L^2}^2 \\
&\leq -\frac{3d}{4}\|u_t\|_{L^2}^2 + \frac{db}{8}\|u_{xx}\|_{L^2}^2 + c\|\theta\|_{L^2}^2, \qquad (7.20)
\end{aligned}$$

with some (generic) constant $c > 0$. Combining (7.15) and (7.20) we get

$$\frac{d}{dt}\mathrm{Re}\ (\langle u_t, u\rangle - \langle h, u_t\rangle) \leq -\frac{d}{4}\|u_t\|_{L^2}^2 - \frac{db}{8}\|u_{xx}\|_{L^2}^2 + c\|\theta\|_{L^2}^2. \qquad (7.21)$$

Defining the Lyapunov functional L_m for $\varepsilon > 0$ by

$$L_m(t) := E_m(t) + \varepsilon \left\{ \text{Re} \left(\langle u_t, u \rangle - \langle h, u_t \rangle \right) \right\},$$

and choosing ε small enough, we obtain from (6.12) and (6.18)

$$\frac{d}{dt} L_m \le -\alpha_2 E_m,$$

with some $\alpha_2 > 0$. As in Section 4 we conclude

$$\frac{d}{dt} L_m \le -\underbrace{\frac{2}{3}\alpha_1}_{=:\alpha} L_m,$$

and, finally, obtain

Theorem 7.6. *The energy E_m for the system (7.10), (7.11) decays exponentially. For any $t \ge 0$ we have*

$$E_m(t) \le 3E_m(0)\mathrm{e}^{-\alpha t}.$$

with a positive constant α which does not depend on t or on the data. In terms of the associated semigroup, we have exponential stability.

The statement on the exponential stability of the semigroup follows again from the fact, that after transforming to a first-order system for $V = (u, u_t, \theta)$, the square of the norm of the solution is equivalent to E_m.

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