A REMARK ON DESCENDING 
SUMS OF SQUARES REPRESENTATIONS 

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Let \( f = f(x_1, \ldots, x_n) \) be a polynomial with rational coefficients which is a sum of squares (sos) of polynomials with real coefficients. Sturmfels asked whether \( f \) is necessarily a sum of squares of polynomials with rational coefficients. From the assumption it follows easily that \( f \) is a sum of squares of polynomials with real algebraic coefficients. Hence there exists a real number field \( K \) such that \( f \) is sos in \( K[x_1, \ldots, x_n] \).

In [1], Hillar showed that the answer to Sturmfels’s question is positive if the real number field \( K \) can be chosen to be Galois over \( \mathbb{Q} \). Moreover, he provided bounds for the number of squares needed to write \( f \) over \( \mathbb{Q} \), in terms of the number of squares needed over \( K \) and the degree \([K : \mathbb{Q}]\). In fact, the polynomial ring \( \mathbb{Q}[x_1, \ldots, x_n] \) could be replaced for these results by any commutative \( \mathbb{Q} \)-algebra \( A \) (and accordingly \( K[x_1, \ldots, x_n] \) by \( A \otimes_{\mathbb{Q}} K \)).

The purpose of this note is to give a very short proof of a generalization of this result. This proof yields a significantly smaller bound for the number of squares necessary to express \( f \) as a sum of squares over \( \mathbb{Q} \). While in [1] this bound is exponential in the degree \([K : \mathbb{Q}]\), our bound is linear.

We prefer to work over an arbitrary real base field \( k \), since there is no difference in the proof.

Given a finite extension \( K/k \) of real fields, consider the associated trace quadratic form. This is the quadratic form \( \tau: K \to k, y \mapsto \text{tr}_{K/k}(y^2) \) over \( k \). It has the following well-known basic property: For any ordering \( P \) of \( k \), the Sylvester signature of \( \tau \) at \( P \) is equal to the number of extensions of the ordering \( P \) to \( K \). See [2] (Lemma 3.2.7 or Theorem 3.4.5), for example.

For any commutative ring \( A \) denote by \( \Sigma A^2 \) the set of sums of squares of \( A \). Assume that every ordering of \( k \) has \( d := [K : k] \) different extensions to \( K \). (It is easy to see that this is equivalent to the condition that every ordering of \( k \) extends to the Galois hull of \( K/k \).) Then \( \tau \) is positive definite with respect to every ordering of \( k \). Diagonalizing \( \tau \) therefore gives \( a_1, \ldots, a_d \in \Sigma k^2 \), together with a \( k \)-linear basis \( y_1, \ldots, y_d \) of \( K \), such that

\[
\text{tr}_{K/k}\left( \left( \sum_{i=1}^{d} x_iy_i \right)^2 \right) = \sum_{i=1}^{d} a_ix_i^2
\]  

(1)

holds for all \( x_1, \ldots, x_d \in k \). More generally, if \( A \) is an arbitrary (commutative) \( k \)-Algebra and \( A_K := A \otimes_k K \), then

\[
\text{tr}_{A_K/A}\left( \left( \sum_{i=1}^{d} x_i \otimes y_i \right)^2 \right) = \sum_{i=1}^{d} a_ix_i^2
\]

(2)

holds for all \( x_1, \ldots, x_n \in A \).

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Proposition 1. Let $K/k$ be an extension of real fields of finite degree $d = [K : k]$, and assume that every ordering of $k$ extends to $d$ different orderings of $K$. Then there exist $c_1, \ldots, c_d \in \Sigma k^2$ with the following property:

For every $k$-algebra $A$ and every $f \in A$ which is a sum of $m$ squares in $A_K = A \otimes_k K$, there are $f_1, \ldots, f_d \in A$ such that each $f_i$ is a sum of $m$ squares in $A$, and such that

$$f = \sum_{i=1}^{d} c_i f_i.$$ 

In particular, $f$ is a sum of $dm \cdot p(k)$ squares in $A$.

Here $p(k)$ denotes the Pythagoras number of $k$, i.e., the smallest number $p$ such that every sum of squares in $k$ is a sum of $p$ squares in $k$. (If no such number $p$ exists one puts $p(k) = \infty$.)

Proof. Choose $a_i \in \Sigma k^2$ and $y_i \in K$ ($i = 1, \ldots, d$) as before. It suffices to take $c_i = a_i^d$ for $i = 1, \ldots, d$. Indeed, assuming $f = g_1^2 + \cdots + g_m^2$ with $g_1, \ldots, g_m \in A_K$, we get

$$df = \text{tr}_{A_K/A}(f) = \sum_{j=1}^{m} \text{tr}_{A_K/A}(g_j^2) = \sum_{j=1}^{m} \sum_{i=1}^{d} a_i x_{ij}^2,$$

where the $x_{ij} \in A$ are determined by $g_j = \sum_{i=1}^{d} x_{ij} \otimes y_i$ ($j = 1, \ldots, m$). So we can put $f_i = \sum_{j=1}^{m} x_{ij}^2$ ($i = 1, \ldots, d$). \[\square\]

Remark. In [1] it was shown (for $k = \mathbb{Q}$) that if $K/\mathbb{Q}$ is a totally real number field with Galois hull $L/\mathbb{Q}$, if $A$ is a $\mathbb{Q}$-algebra and $f \in A$ is a sum of $m$ squares in $A \otimes_\mathbb{Q} K$, then $f$ is a sum of

$$4m \cdot 2^{e+1} \left( \frac{e+1}{2} \right) = 2^{e+2} (e+1) \cdot m$$

squares in $A$, with $e := [L : \mathbb{Q}]$.

The qualitative part of the above result extends immediately to the following more general situation. Let $K/k$ be an extension as in the proposition, and let $A$ be a $k$-algebra. Fix elements $h_1, \ldots, h_r \in A$ and consider the so-called (pseudo-) quadratic module

$$M := \left\{ \sum_{i=1}^{r} s_i h_i : s_1, \ldots, s_r \in \Sigma A^2 \right\}$$

generated by the $h_i$. Similarly, let

$$M_K = \left\{ \sum_{i=1}^{r} t_i h_i : t_1, \ldots, t_r \in \Sigma A_K^2 \right\}$$

be the (pseudo-) quadratic module generated by $M$ in $A_K$. Then we have:

Proposition 2. $A \cap M_K = M$. \[\square\]

References


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