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Exercises for Real Algebraic Geometry II

Sheet 1

Please upload your solutions on ILIAS by Monday 27 April 2020 at 11:45

Exercise 1
Let $\mathbb{R}[x]$ be the ring of polynomials in one variable. Decide which of the following quadratic modules in $\mathbb{R}[x]$ are preorders:

(a) $M = QM(x)$;
(b) $M = QM(x, 1-x)$;
(c) $M = QM(x+1,\, x^2-x)$.

Exercise 2
Write every polynomial $0 \neq f \in \mathbb{R}[x,y]$ as

$$f \;=\; \sum_{i\geq 0} f_i(x)\, y^i$$

where $f_i \in \mathbb{R}[x]$ are univariate polynomials. Let $n = \deg_y(f)$ and $m = \deg_x(f_n)$, let $a_{mn} \in \mathbb{R}$ be the coefficient of $x^m y^n$ in f. Let $M \subseteq \mathbb{R}[x,y]$ be the set of all polynomials $f \neq 0$ for which $(-1)^{mn} a_{mn} > 0$, together with $f = 0$.

(a) M is a semiorder of $\mathbb{R}[x,y]$.
(b) M is not contained in any positive cone of $\mathbb{R}[x,y]$.

Exercise 3
Let $n \in \mathbb{N}$, we consider the tuples of variables $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$. Consider the involutive ring automorphism $f \mapsto f^*$ of the polynomial ring $\mathbb{C}[z,w]$ defined by $z_j^* = w_j$ $(j = 1, \ldots, n)$ and $c^* = \overline{c}$ for $c \in \mathbb{C}$ (complex conjugate), see II.2.15.

(a) The fix ring of $*$ is $A := \mathbb{R}[x,y]$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $x_j = \frac{1}{2}(z_j + w_j)$, $y_j = \frac{1}{2i}(z_j - w_j)$ $(j = 1, \ldots, n)$.
(b) Given $f \in \mathbb{C}[z,w]$, let $|f|^2 := ff^*$. The set $\{\sum_{j=1}^r |f_j(z,w)|^2 \colon r \in \mathbb{N}, f_j \in \mathbb{C}[z,w]\ (j = 1, \ldots, r)\}$ is equal to ΣA^2.
(c) Let $\Sigma_h A^2 := \{\sum_{j=1}^r |f_j(z)|^2 \colon r \in \mathbb{N},\ f_j \in \mathbb{C}[z]\ (j = 1, \ldots, r)\}$. Prove that $\Sigma_h A^2$ is a generating semiring in A.
(d) Show that $\Sigma_h A^2 \neq \Sigma A^2$. (*Hint*: Consider polynomials of degree 2.)

Exercise 4
Using the notation of Exercise 3 let $\Sigma_h := \Sigma_h A^2$, and let I be an ideal of $A = \mathbb{R}[x,y]$. The semiring $\Sigma_h + I$ of A is archimedean if and only if I contains an element of the form

$$c + q + \sum_{j=1}^{n} |z_j|^2$$

with $c \in \mathbb{R}$ and $q \in \Sigma_h$.