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Exercises for Real Algebraic Geometry II

Sheet 2

Please upload your solutions on ILIAS by Monday 4 May 2020 at 11:45

Exercise 5

Let $\mathbb{R}[x]$ be the polynomial ring in one variable, let $S \subseteq \mathbb{R}[x]$ be the semiring generated by $\mathbb{R}_+$ and x, and let $M = S + S(1-x)$.

(a) S is a generating semiring of $\mathbb{R}[x]$, and M is an archimedean S-module.

(b) If $0 < c < 1$ and $f = c + (1-x^2)^2$ then $f > 0$ on X_M, but $f \notin M$.

Hence the archimedean Positivstellensatz is usually false for archimedean modules over generating semirings.

Exercise 6

The dehomogenized version of Pólya's theorem is false: Find a polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ (necessarily inhomogeneous) with $f > 0$ on $C = \{\xi \in \mathbb{R}^n \colon \xi_1 \geq 0, \dots, \xi_n \geq 0\}$ such that $(1 + x_1 + \cdots + x_n)^N \cdot f$ has a negative coefficient for each $N \geq 0$.

Exercise 7

Let c be a positive real number, and let

$$f(x,y) \;=\; (x+y)^2 + c(x-y)^2.$$

Show that if $n \in \mathbb{N}$ is even with $n < c - 1$, then the form $(x+y)^n \cdot f(x,y)$ has a negative coefficient.

Exercise 8

Use the notation from Exercise 4. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$, let $f(z,w) \in \mathbb{C}[z,w]$ such that $f = f^*$ and $f(u, \overline{u}) > 0$ for every $u \in \mathbb{C}^n$ with $|u| = 1$. Prove Quillen's theorem: There exist finitely many polynomials $p_1, \dots, p_r \in \mathbb{C}[z]$ mit

$$f(u, \overline{u}) \;=\; \sum_{j=1}^{r} |p_j(u)|^2$$

for every $u \in \mathbb{C}^n$ with $|u| = 1$.