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Exercises for Real Algebraic Geometry II

Sheet 3

Please upload your solutions on ILIAS by Monday 11 May 2020 at 11:45

Exercise 9

Let $S \subseteq \mathbb{R}[x]$ be the semiring generated by $1 - x^2$ and all fourth powers in $\mathbb{R}[x]$. Show that S is not archimedean.

Remark: If one does the same with 6th powers instead of 4th powers, the semiring becomes archimedean. However I do not know any easily accessible proof.

Exercise 10

Let R be a non-archimedean real closed field, and let $\varepsilon \in R$ with $0 < n\varepsilon < 1$ for every $n \in \mathbb{N}$. Let $T = PO(x^3(1-x))$ in $R[x]$. The linear polynomial $f := x + \varepsilon$ satisfies $f > 0$ on $X_T = [0, 1]$, but $f \notin T$. (*Hint:* Use the convex hull of $\mathbb{Z}$ in R.)

Exercise 11

If $0 \neq f \in \mathbb{R}[\mathtt{x}] = \mathbb{R}[x_1, \dots, x_n]$, let $\tilde{f}$ denote the leading form (highest degree subform) of f. Let $f_1, \dots, f_r \in \mathbb{R}[\mathtt{x}]$ be such that $S(\tilde{f}_1, \dots, \tilde{f}_r) = \{0\}$. Then the basic closed set $S(f_1, \dots, f_r)$ is compact.

Exercise 12

Let A be a finitely generated $\mathbb{R}$-algebra. Use Schmüdgen's theorem to show the equivalence of the following two conditions:

(i) The topological space $\mathrm{Hom}(A, \mathbb{R})$ is compact;

(ii) for a suitable integer $n \geq 1$, there exists a surjective homomorphism of $\mathbb{R}$-algebras $\mathbb{R}[x_1, \dots, x_n]/\langle 1 - x_1^2 - \cdots - x_n^2 \rangle \to A$.

In other words, if V is any affine $\mathbb{R}$-variety for which $V(\mathbb{R})$ is compact, then V is isomorphic to a closed subvariety of a sphere.