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Exercises for Real Algebraic Geometry II

Sheet 4

Please upload your solutions on ILIAS by Monday 18 May 2020 at 11:45

Exercise 13
Let A be a ring containing $\mathbb{Q}$, and let M be a quadratic module in A. If (A, M) has an order unit u then M is archimedean. (*Hint:* Show $\varphi(1) > 0$ for every pure state φ of (A, M, u).)

Exercise 14
Let A be a ring and $S \subseteq A$ an archimedean semiring, let $M \subseteq A$ be an S-pseudomodule, and let $f \in M$. Show that $I := \mathrm{supp}(M + Af)$ is an ideal of A and that f is an order unit of $(I, M \cap I)$.

Exercise 15
Let A be a ring with $\mathbb{R} \subseteq A$, let $I \subseteq A$ be an ideal, and let $M \subseteq I$ be a quadratic pseudomodule with order unit $u \in M$ and with $x^2 \in M$ for every $x \in I$. The goal of this exercise and the next is to give a proof of the following result (compare Theorem 8.12):

> **Theorem:** *Every monic state φ of (I, M, u) satisfying $\varphi(ab) = \varphi(au)\varphi(b)$ for all $a \in A$, $b \in I$ and $\varphi(u^2) \neq 0$ is a pure state.*

Towards this end, prove the following:

(a) For real numbers $a, b \in [0, 1]$ one has $\sqrt{ab} + \sqrt{(1-a)(1-b)} \leq 1$, with equality only if $a = b$.
(b) Let $S(I, M) = \{\psi \in \mathrm{Hom}(I, \mathbb{R})\colon \psi|_M \geq 0\}$, the vector space of additive maps $I \to \mathbb{R}$ which are nonnegative on M. Every $\psi \in S(I, M)$ is $\mathbb{R}$-linear.
(c) Let $\psi \in S(I, M)$. For all $f, g \in I$ show $\psi(fg)^2 \leq \psi(f^2)\psi(g^2)$ (1). For every $f \in M$ show $\psi(f^2)^2 \leq \psi(f)\psi(f^3)$ (2).

Hint for (c): To show (1), consider $\psi((tf - g)^2)$ for $t \in \mathbb{R}$. Use a similar argument to show (2).

Exercise 16
(Continuation) With notation and hypotheses as in Exercise 15, let $\varphi \in S(I, M)$ as in the theorem, and let $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1, \varphi_2 \in S(I, M)$. Put $\lambda = \varphi(u^2)$.

(d) Show $\lambda > 0$.
(e) Let $f \in M$ with $\varphi(f) = 1$, put $a := \varphi_1(f)$ and $b := \frac{\varphi_1(f^3)}{\lambda^2}$. Calculate $\varphi(f^2)$, and use (c)(2) and (a) to conclude: $\varphi_1(f^2) = \lambda\varphi_1(f)$.
(f) Let $f, g \in M$ with $\varphi(f) = \varphi(g) = 1$. Use (c)(1) to show $\varphi_1(f) = \varphi_1(g)$.
(g) Conclude that $\varphi_1 = \varphi_1(u) \cdot \varphi$, and use this to prove the theorem.

Hint for (f): Consider $\varphi((f - g)^2)$.