CONVEXITY ESTIMATES FOR FLOWS BY POWERS OF THE MEAN CURVATURE

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appendix with Oliver C. Schnürer

Abstract. We study the evolution of a closed, convex hypersurface in $\mathbb{R}^{n+1}$ in direction of its normal vector, where the speed equals a power $k \geqslant 1$ of the mean curvature. We show that if initially the ratio of the biggest and smallest principal curvature at every point is close enough to 1, depending only on k and n, then this is maintained under the flow. As a consequence we obtain that, when rescaling appropriately as the flow contracts to a point, the evolving surfaces converge to the unit sphere.

1. Introduction

In this paper we investigate the following problem. Let M^n be a smooth, compact manifold without boundary, and $F_0 : M^n \to \mathbb{R}^{n+1}$ a smooth immersion which is convex. We study smooth families of immersions $F(\cdot,t) : M^n \times [0,T) \to \mathbb{R}^{n+1}$, which satisfy

$$(\star) \qquad \begin{cases} F(\cdot,0) = F_0(\cdot) \\ \dfrac{dF}{dt}(\cdot,t) = -H^k(\cdot,t)\nu(\cdot,t)\ . \end{cases}$$

where $k > 0$, H is the mean curvature and ν is the outer unit normal, such that $-H\nu = \vec{H}$ is the mean curvature vector. Throughout the paper we will call such a flow an H^k-flow. In [9] we were able to show:

Theorem 1.1. *Let $F_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion with positive mean curvature, that is $H(F_0(M^n)) > 0$. Then there exists a unique, smooth solution to the initial value problem $(\star)$ on a maximal, finite time interval $[0,T)$. In the case that*

i) *$F_0(M^n)$ is strictly convex for $0 < k < 1$,*
ii) *$F_0(M^n)$ is weakly convex for $k \geqslant 1$,*

then the surfaces $F(M^n,t)$ are strictly convex for all $t > 0$ and they contract for $t \to T$ to a point in $\mathbb{R}^{n+1}$.

Here 'weakly convex' and 'strictly convex', resp. are defined as all the eigenvalues of the second fundamental form being nonnegative, and positive, resp. In this paper we want to present the following extension of the previous result. Let K denote the

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Gauss curvature. To control the pinching of the initial hypersurface we use that by the arithmetic-geometric mean inequality we have

$$0 \leqslant K/H^n \leqslant 1/n^n,$$

with equality on the right side if and only if all eigenvalues are equal.

Theorem 1.2. *For $k \geqslant 1$ there exists a nonnegative constant $C(n,k) < 1/n^n$ such that the following holds: If the initial hypersurface is pinched in the sense that*

$$\frac{K(p)}{H^n(p)} > C(n,k) \text{ for all } p \in M,$$

then this is preserved under the H^k-flow. The constant $C(n,k)$ is increasing in k, $\lim_{k\searrow 1} C(n,k) = 0$ and $\lim_{k\to\infty} C(n,k) = 1/n^n$. Furthermore the rescaled embeddings

$$\tilde{F}(\tau,p) := ((k+1)n^k(T-t))^{-1/(k+1)} \left(F(\tau,p) - x_0\right)$$

converge for $\tau \to \infty$ exponentially in the C^∞-topology to the unit sphere. Here $\tau := -(k+1)^{-1}n^{-k}\log(1-t/T)$, where T is the maximal time of existence of the unrescaled flow and x_0 is the point in $\mathbb{R}^{n+1}$ where the surfaces contract to.

Together with O. Schnürer we give in the appendix a further extension in the case of 2-dimensional hypersurfaces in $\mathbb{R}^3$. For $1 \leqslant k \leqslant 5$ we show that no initial pinching condition is needed to ensure that the rescaled embeddings as described above converge to the unit sphere.

For $k = 1$ the flow considered in ($\star$) is the mean curvature flow. In that case the statements of both Theorems are implied by the results of G. Huisken in [7], who showed that convex surfaces remain convex under the flow and contract to a 'round' point in finite time. Similar results were obtained by B. Chow for the nth root of the Gauss curvature in [4] and for the square root of the scalar curvature in [5]. B. Andrews extended these results in [1] to a whole class of normal velocities which are homogeneous of degree one in the principal curvatures. For normal velocities which have a degree of homogeneity greater than one B. Chow proved a result similar to Theorem 1.2 for powers of the Gauss curvature in [4]. The behavior of the initial pinching condition in Chow's result in the degree of homogeneity is analogous to our result here.

Evolving 2-dimensional hypersurfaces more is known. In [2], B. Andrews shows that convex surfaces moving by Gauss curvature converge to round points without any initial pinching condition. O. Schnürer in [8] obtained similar results for a list of different normal velocities with homogeneity greater than one, including the cases H^2, H^3, H^4. For general speeds of higher homogeneity, B. Andrews showed in [3] that an initial pinching condition, depending only on the degree of homogeneity, ensures the convergence to a round point.

The rest of the paper is organized as follows. In section 2 we compute the evolution equation of the quantity K/H^n, and applying the maximum principle we can deduce that if the surface is initially pinched good enough it remains so under the flow. Further investigation shows that that the pinching improves if the mean curvature explodes. Interestingly this follows again by an application of the maximum principle and doesn't need integral estimates as in the case of the mean curvature flow.

In section 3 we define a natural rescaling of the flow to show that the surfaces become more and more spherical as they contract to a point. Since the rescaled flow might not be anymore uniformly parabolic, we cannot directly apply estimates of Krylov-Safanov type to deduce smooth convergence to a sphere. But since the evolution equation for the mean curvature can be written in the form of a porous medium equation, we can circumvent this using Hölder estimates for this type of equations.
In the appendix we present the result in the 2-dimensional case.

2. Pinching Estimates

We first state the evolution equations for geometric quantities like the induced metric g_{ij}, the induced measure $d\mu$, the second fundamental form h_{ij} or equivalently the Weingarten map $W_p = \{h^i{}_j\} : T_pM \to T_pM$. Further important quantities are the mean curvature $H = g^{ij}h_{ij}$, the norm of the of the second fundamental form squared $|A|^2 = h^{ij}h_{ij}$ and the Gauss curvature $K = \det(h^i{}_j)$. In the following we will always assume that our evolving surfaces are strictly convex, such that $H, K > 0$ and the inverse of the Weingarten map $(b^i_j) := (h^i_j)^{-1}$ is well-defined. In the case $k \geqslant 1$, Theorem 1.1 yields that the evolving surfaces are always strictly convex for positive times.

Lemma 2.1. *The following evolution equations hold.*

(i) $\dfrac{\partial}{\partial t}\, g_{ij} = -2H^k h_{ij}\ ,$

(ii) $\dfrac{\partial}{\partial t}\, d\mu = -H^{k+1}\, d\mu\ ,$

(iii)
$$\frac{\partial}{\partial t}\, h^i_j = kH^{k-1}\Delta h^i_j + k(k-1)H^{k-2}\nabla^i H \nabla_j H - (k-1)H^k h^i_l h^l_j + kH^{k-1}|A|^2 h^i_j\ ,$$

(iv)
$$\frac{\partial}{\partial t}\, H^l = kH^{k-1}\Delta H^l + lk(k-l)H^{k+l-3}|\nabla H|^2 + l|A|^2 H^{l+k-1}, \quad l \in \mathbb{R}\ ,$$

(v)
$$\frac{\partial}{\partial t}\, |A|^2 = kH^{k-1}\Delta|A|^2 - 2kH^{k-1}|\nabla A|^2 + 2k(k-1)H^{k-2}h_i{}^j\nabla^i H \nabla_j H - 2(k-1)H^k\, tr(A^3) + 2kH^{k-1}|A|^4\ ,$$

(vi)
$$\begin{aligned}\frac{\partial}{\partial t}\, K &= kH^{k-1}\Delta K - kH^{k-1}K^{-1}|\nabla K|^2 - kH^{k-1}K\nabla^i b^{lm}\nabla_i h_{lm} \\ &\quad + k(k-1)H^{k-2}Kb^{lm}\nabla_l H\nabla_m H - (k-1)H^{k+1}K \\ &\quad + knH^{k-1}|A|^2K \\ &= kH^{k-1}\Delta K - \frac{k(n-1)}{n}H^{k-1}\frac{|\nabla K|^2}{K} - \frac{k}{n}H^{2n+k-1}\frac{|\nabla(KH^{-n})|^2}{K} \\ &\quad + k(k-1)H^{k-2}Kb^{lm}\nabla_l H\nabla_m H + kH^{k-3}K|H\nabla_i h_{mn} - h_{mn}\nabla_i H|^2_{g,b} \\ &\quad - (k-1)H^{k+1}K + knH^{k-1}|A|^2K\ ,\end{aligned}$$

where

$$|H\nabla_i h_{mn} - h_{mn}\nabla_i H|^2_{g,b} := g^{ij}b^{km}b^{ln}(H\nabla_i h_{mn} - h_{mn}\nabla_i H)(H\nabla_j h_{kl} - h_{kl}\nabla_j H)\ .$$

Proof. $(i)-(v)$ follows from a direct calculation as for example in [1]. The first two lines in (vi) are again a direct calculation. The second equality then follows from the identity

$$\frac{1}{nK}|\nabla K|^2 + K\nabla^i b^{lm}\nabla_i h_{lm} = -\frac{K}{H^2}|H\nabla_i h_{mn} - h_{mn}\nabla_i H|^2_{g,b} + \frac{H^{2n}}{nK}|\nabla(KH^{-n})|^2 \ .$$

□

An appropriate test function to control the pinching of the principal curvatures along the flow turns out to be K/H^n.

Lemma 2.2. *The quantity K/H^n satisfies the evolution equation*

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\frac{K}{H^n}\Big) = kH^{k-1}\bigg(&\Delta\Big(\frac{K}{H^n}\Big) + \frac{(n+1)}{nH^n}\nabla_i\Big(\frac{K}{H^n}\Big)\nabla_i H^n - \frac{(n-1)}{nK}\nabla_i\Big(\frac{K}{H^n}\Big)\nabla_i K\\ &- \frac{H^n}{nK}\Big|\nabla\Big(\frac{K}{H^n}\Big)\Big|^2 + \frac{K}{H^{n+2}}\big|H\nabla_i h_{mn} - h_{mn}\nabla_i H\big|^2_{g,b}\\ &+ (k-1)\frac{K}{H^{n+1}}\big(b^{ij} - \frac{n}{H}g^{ij}\big)\nabla_i H\nabla_j H + \frac{(k-1)}{k}\frac{K}{H^n}\big(n|A|^2 - H^2\big)\bigg) \ .\end{aligned}$$

Proof. This follows from computing

$$\frac{\partial}{\partial t}\Big(\frac{K}{H^n}\Big) = \frac{1}{H^n}\frac{\partial}{\partial t}K - \frac{K}{H^{2n}}\frac{\partial}{\partial t}H^n$$

and the evolution equations of K and H^n above. Note that we can write

$$\frac{1}{H^n}\Delta K - \frac{K}{H^{2n}}\Delta H^n - \frac{(n-1)}{n}\frac{1}{KH^n}|\nabla K|^2 =$$

$$\Delta\Big(\frac{K}{H^n}\Big) + \frac{(n+1)}{nH^n}\nabla_i\Big(\frac{K}{H^n}\Big)\nabla_i H^n - \frac{(n-1)}{nK}\nabla_i\Big(\frac{K}{H^n}\Big)\nabla_i K - n(n-1)\frac{K}{H^{n+2}}|\nabla H|^2 \ .$$

□

We now aim to apply the maximum principle and show that that $\min_{p\in M}\frac{K}{H^n}(p,t)$ is non-decreasing in t. We argue as in [4]. Since always $|A|^2 \geqslant \frac{1}{n}H^2$, the lowest order terms have the right sign. From [7] we know that if $h_{ij} \geqslant \varepsilon H g_{ij}$, for some $\varepsilon > 0$, then

$$|H\nabla_i h_{kl} - \nabla_i H h_{kl}|^2_g \geqslant \varepsilon^2\frac{n-1}{2}H^2|\nabla H|^2 \ .$$

So by estimating $b^{ij} \geqslant g^{ij}/H$ we obtain

$$\frac{K}{H^{n+2}}\big|H\nabla_i h_{mn} - h_{mn}\nabla_i H\big|^2_{g,b} \geqslant \varepsilon^2\frac{n-1}{2}\frac{K}{H^{n+2}}|\nabla H|^2 \ .$$

Let $C(n,k)$ denote the minimal constant such that $0 \leqslant C(n,k) < 1/n^n$ and for an $\varepsilon \geqslant 0$ it holds

$$\frac{K}{H^n} \geqslant C(n,k) \Longrightarrow h_{ij} \geqslant \varepsilon H g_{ij} \ \text{ and } \ |b^{ij} - \frac{n}{H}g^{ij}| \leqslant \frac{\varepsilon^2}{2(k-1)H} \ .$$

It then follows that

$$(k-1)\frac{K}{H^{n+1}}\big|\big(b^{ij} - \frac{n}{H}g^{ij}\big)\nabla_i H\nabla_j H\big| \leqslant \varepsilon^2\frac{n-1}{2}\frac{K}{H^{n+2}}|\nabla H|^2 \ ,$$

which yields:

Corollary 2.3. *For $k \geqslant 1$ there exists a constant $0 \leqslant C(n,k) < 1/n^n$ such that the following holds: If the initial hypersurface is pinched in the sense that*

$$\frac{K(p)}{H^n(p)} \geqslant C(n,k) \qquad \text{for all } p \in M,$$

then this is preserved under the H^k-flow. The constant $C(n,k)$ is increasing in k, $\lim_{k\searrow 1} C(n,k) = 0$ and $\lim_{k\to\infty} C(n,k) = 1/n^n$.

A next step is to show that at points where the mean curvature is big, the principal curvatures approach each other. To do this we define

$$f := \frac{1}{n^n} - \frac{K}{H^n} \quad \text{and} \quad f_\sigma := H^\sigma f\ .$$

Note that $0 \leqslant f \leqslant 1/n^n$ and $f(p,t) = 0$ iff the principal curvatures at (p,t) are all equal.

Lemma 2.4. *The quantity f_σ has the evolution equation*

$$\begin{aligned}\frac{\partial f_\sigma}{\partial t} = kH^{k-1}\Big(&\Delta f_\sigma + 2\Big(1-\sigma\Big(1+\frac{H^n}{K}f\Big)\Big)\langle\nabla f_\sigma, \frac{\nabla H}{H}\rangle + \frac{H^{n-\sigma}}{K}|\nabla f_\sigma|^2 \\ &+ H^\sigma\Big(\sigma\Big(k-2+\sigma\Big(1+f\frac{H^n}{K}\Big)\Big)f\frac{|\nabla H|^2}{H^2} - \frac{K}{H^{n+2}}\big|H\nabla_i h_{mn} - h_{mn}\nabla_i H\big|^2_{g,b} \\ &- \frac{(k-1)K}{H^{n+1}}\big(b^{ij} + \frac{n}{H}g^{ij}\big)\nabla_i H\nabla_j H - \frac{(k-1)K}{kH^n}\big(n|A|^2 - H^2\big) + \frac{\sigma}{k}f|A|^2\Big)\Big)\ .\end{aligned}$$

Proof. From the preceding Lemma and Lemma 2.1 we have

$$\begin{aligned}\frac{\partial f_\sigma}{\partial t} &= H^\sigma\frac{\partial f}{\partial t} + \sigma H^{\sigma-1}f\frac{\partial H}{\partial t} \\ &= kH^{k-1}\Big(H^\sigma\Delta f + \sigma H^{\sigma-1}f\Delta H + H^\sigma\Big(\frac{n-1}{H}\langle\nabla f,\nabla H\rangle \\ &\qquad - \frac{n-1}{nK}\langle\nabla f,\nabla K\rangle + \frac{K}{H^{n+2}}\big|H\nabla_i h_{mn} - h_{mn}\nabla_i H\big|^2_{g,b} + \frac{H^n}{nK}|\nabla f|^2 \\ &\qquad + (k-1)\sigma f\frac{|\nabla H|^2}{H^2} - \frac{(k-1)K}{H^{n+1}}\big(b^{ij} - \frac{n}{H}g^{ij}\big)\nabla_i H\nabla_j H \\ &\qquad - \frac{(k-1)K}{kH^n}\big(n|A|^2 - H^2\big) + \frac{\sigma}{k}f|A|^2\Big)\Big)\ .\end{aligned} \tag{1}$$

We then compute

$$\begin{aligned}\nabla_i f_\sigma &= H^\sigma\nabla_i f + \sigma\frac{f_\sigma}{H}\nabla_i H \\ \nabla_i K &= -H^{n-\sigma}\nabla_i f_\sigma + \Big(n\frac{K}{H} + \sigma H^{n-1}f\Big)\nabla_i H \\ H^\sigma\langle\nabla f,\nabla H\rangle &= \langle\nabla f_\sigma,\nabla H\rangle - \sigma\frac{f_\sigma}{H}|\nabla H|^2 \\ \langle\nabla K,\nabla f\rangle &= -H^{n-2\sigma}|\nabla f_\sigma|^2 + H^{-\sigma}\Big(n\frac{K}{H} + 2\sigma H^{n-1}f\Big)\langle\nabla H,\nabla f_\sigma\rangle \\ &\quad - \sigma(nK + \sigma H^n f)f\frac{|\nabla H|^2}{H^2}\end{aligned}$$

$$|\nabla f|^2 = H^{-2\sigma}\Big(|\nabla f_\sigma|^2 - 2\sigma\frac{f_\sigma}{H}\langle\nabla f_\sigma,\nabla H\rangle + \sigma^2 f_\sigma^2\frac{|\nabla H|^2}{H^2}\Big)$$

$$\begin{aligned}\Delta f_\sigma &= H^\sigma\Delta f + 2\sigma H^{\sigma-1}\langle\nabla f,\nabla H\rangle + \sigma(\sigma-1)f\frac{|\nabla H|^2}{H^{2-\sigma}} + \sigma H^{\sigma-1}f\Delta H\\ &= H^\sigma\Delta f + \sigma H^{\sigma-1}f\Delta H + \frac{2\sigma}{H}\langle\nabla f_\sigma,\nabla H\rangle - \sigma(\sigma+1)f_\sigma\frac{|\nabla H|^2}{H^2}\ .\end{aligned}$$

Inserting these computations in (1) and collecting terms gives the stated evolution equation. □

The next lemma will be needed to show that for small σ the lowest order terms have the right sign.

Lemma 2.5. *Assume that $\lambda_i \geqslant \varepsilon H > 0$ for some $\varepsilon > 0$ and for all $i = 1\dots n$. Then there exists a $\delta > 0$ such that*

$$\frac{n|A|^2 - H^2}{H^2} \geqslant \delta\Big(\frac{1}{n^n} - \frac{K}{H^n}\Big)\ .$$

Proof. Normalize the eigenvalues by defining

$$\tilde\lambda_i := \frac{\lambda_i}{H}\ .$$

Then $\varepsilon \leqslant \tilde\lambda_i \leqslant 1,\ i = 1\dots n$ and $\sum_{i=1}^n \tilde\lambda_i = 1$. Since $n|A|^2 - H^2 = \sum_{i<j}(\lambda_i - \lambda_j)^2$ we can write the desired inequality as

$$\sum_{i<j}(\tilde\lambda_i - \tilde\lambda_j)^2 \geqslant \delta\big(\frac{1}{n^n} - K(\tilde\lambda)\big)\ .$$

Both sides are positive and they are zero if and only if $(\tilde\lambda_1,\dots,\tilde\lambda_n) = (1/n,\dots,1/n)$. We do a Taylor expansion of both sides around $(1/n,\dots,1/n)$ on the hyperplane $\{\sum_{i=1}^n \tilde\lambda_i = 1\}$. Since the left hand side is a polynomial of second order, its hessian has to be strictly positive and the claim follows by the compactness of the set $\{\sum_{i=1}^n \tilde\lambda_i = 1,\ \tilde\lambda_j \geqslant \varepsilon,\ j = 1,\dots,n\}$. □

Applying the maximum principle leads to the following theorem.

Theorem 2.6. *If $k > 1$ and the initial hypersurface is pinched in the sense that*

$$\frac{K(p)}{H^n(p)} > C(n,k) \qquad \forall\, p \in M, \tag{2}$$

then there exists a $\sigma > 0$ such that

$$f_\sigma(p,t) \leqslant \max_{p\in M} f_\sigma(p,0) \quad \forall\,(p,t) \in M\times[0,T)\ . \tag{3}$$

Proof. To apply the maximum principle to the evolution equation of f_σ, we have to show that

$$\begin{aligned}&\sigma\Big(k - 2 + \sigma\Big(1 + f\frac{H^n}{K}\Big)\Big)f\frac{|\nabla H|^2}{H^2} - \frac{K}{H^{n+2}}\big|H\nabla_i h_{mn} - h_{mn}\nabla_i H\big|^2_{g,b}\\ &-\frac{(k-1)K}{H^{n+1}}\big(b^{ij} + \frac{n}{H}g^{ij}\big)\nabla_i H\nabla_j H - \frac{(k-1)K}{kH^n}\big(n|A|^2 - H^2\big) + \frac{\sigma}{k}f|A|^2 \leqslant 0\ .\end{aligned} \tag{4}$$

By Lemma 2.3 the lower bound

$$\frac{K(p)}{H^n(p)} \geqslant C_0 \tag{5}$$

is preserved, so as in the proof of Lemma there exists a $\eta > 0$ such that

$$-\frac{K}{H^{n+2}}\big|H\nabla_i h_{mn} - h_{mn}\nabla_i H\big|^2_{g,b} - \frac{(k-1)K}{H^{n+1}}\big(b^{ij} + \frac{n}{H}g^{ij}\big)\nabla_i H\nabla_j H \leqslant -\eta\frac{|\nabla H|^2}{H^2}\ .$$

Choosing σ small enough the terms involving ∇H in (4) have the right sign. Since $|A^2| \leqslant H^2$ and (5) implies that $\lambda_i \geqslant \varepsilon H$ for some $\varepsilon > 0$, we conclude by Lemma 2.5 that also

$$-\frac{(k-1)K}{kH^n}\big(n|A|^2 - H^2\big) + \frac{\sigma}{k}f|A|^2 \leqslant 0$$

for σ maybe chosen even smaller. □

3. Rescaling and Convergence

In this chapter we consider a natural rescaling of the evolution equation ($\star$). It agrees with the rescaling for the mean curvature flow of convex surfaces in [7] and for the general class of flows treated in [1]. To be able to apply Theorem 2.6 we will henceforth assume that $k > 1$ and that the initial surface is pinched in the sense of (2). For $k = 1$, i.e. the mean curvature flow, the results are known by the work of Huisken [7]. The next section follows closely Chapter 7 of [1].

For the evolution of a sphere with initial radius R_0 we can compute that $R(t) = ((k+1)n^k(T-t))^{1/(k+1)}$, where $T = R_0^{k+1}/(n^k(k+1))$ is the maximal time of existence. This suggests the following general rescaling. Assume we have a general convex solution of ($\star$) on a maximal time interval $[0, T)$. By Theorem 1.1 we know that the surfaces contract to a point x_0 in $\mathbb{R}^{n+1}$ as $t \to T$. Define $\alpha := (k+1)n^k$, and a new time parameter τ by $\tau := -\frac{1}{\alpha}\log(1 - \frac{t}{T})$. The rescaled immersions $\tilde{F}_\tau := (\alpha(T-t))^{-1/(k+1)}(F_t - x_0)$ then satisfy the evolution equation

$$\frac{\partial \tilde{F}}{\partial \tau}(p,\tau) = -\tilde{H}^k(p,\tau)\tilde{\nu}(p,\tau) + n^k\tilde{F}(p,\tau) \quad \text{for } \tau \in [0,\infty)\ . \tag{6}$$

In following we will distinguish geometric quantities associated with the rescaled immersions by a tilde.

Lemma 3.1. *The evolution equations for the rescaled metric $\tilde{g}_{ij}$, the induced measure $d\tilde{\mu}$ and the rescaled mean curvature $\tilde{H}$ are*

$$\frac{\partial}{\partial \tau}\,\tilde{g}_{ij} = -2\tilde{H}^k\tilde{h}_{ij} + 2n^k\tilde{g}_{ij} \tag{i}$$

$$\frac{\partial}{\partial \tau}\,d\tilde{\mu} = -\tilde{H}^{k+1}d\tilde{\mu} + n^{k+1}d\tilde{\mu} \tag{ii}$$

$$\frac{\partial}{\partial \tau}\,\tilde{H} = \Delta\tilde{H}^k + |\tilde{A}|^2\tilde{H}^k - n^k\tilde{H}\ . \tag{iii}$$

To control the convergence we relate the principal curvatures to the inner radius ρ_- and the outer radius ρ_+, which are defined as

$$\begin{aligned}\rho_+(t) &:= \inf\{r : B_r(y) \text{ encloses } F_t(M) \text{ for some } y \in \mathbb{R}^{n+1}\}\\ \rho_-(t) &:= \sup\{r : B_r(y) \text{ is enclosed by } F_t(M) \text{ for some } y \in \mathbb{R}^{n+1}\}\end{aligned}$$

where $B_r(y)$ is the ball with radius r centered at y. By Corollary 2.3 we know that there is a constant C_1 such that

$$\lambda_{\max}(p,t) \leqslant C_1\lambda_{\min}(p,t) \quad \forall\,(p,t) \in M \times [0,T)\ .$$

Lemma 5.4 in [1] then guarantees a constant C_2 such that

$$\rho_+ \leqslant C_2 \rho_- \ .$$

Lemma 3.2.

$$\frac{1}{C_2} \leqslant \tilde\rho_- \leqslant 1 \leqslant \tilde\rho_+ \leqslant C_2 \ ,$$

for all $\tau \geqslant 0$.

The proof of this Lemma is nearly identical to the proof of Lemma 7.2 in [1], just the rescaling is different.

Now Lemma 4.5 in [9] gives that

$$H(p,t) \leqslant \frac{3}{2}\frac{n(k+1)}{k}\frac{1}{\delta} \quad \text{for } (p,t) \in M \times [0,T']$$

if all surfaces $M_t, t \in [0,T']$ enclose a fixed Ball $B_\delta(y_0)$ for some $y_0 \in \mathbb{R}^{n+1}$, δ small enough. This gives

$$\sup_{(p,t)\in M\times[0,T']} H(p,t) \leqslant \frac{3}{2}\frac{n(k+1)}{k}\frac{1}{\rho_-(T')}$$

and we obtain

$$\tilde H(\tau) \leqslant C_3 \quad \forall\, \tau \in [0,\infty) \ , \tag{7}$$

i.e. we have a uniform bound on $|\tilde A|^2$. To obtain $C^{2,\alpha}$-estimates we cannot apply the results of Krylov, since we don't know of a suitable Harnack-inequality for this flow which guarantees a positive lower bound for $\tilde H$ and so ensures uniform parabolicity. We circumvent this by noting that the evolution equation for $\tilde H$ can be written in the form of a porous medium equation, see Lemma 3.1 (*iii*). In [6] E. DiBenedetto and A. Friedman have established interior Hölder-estimates for such equations, which we aim to apply.

Lemma 3.3. *For a rescaled flow, there is a constant* $C_4 = C(n,k,C_3)$ *such that*

$$\int_{\tau_1}^{\tau_2} |\nabla \tilde H^k|^2 \, d\tilde\mu \, d\tau \leqslant C_4(1+\tau_2-\tau_1)$$

for $0 \leqslant \tau_1 < \tau_2 < \infty$.

Proof. Computing

$$\frac{\partial}{\partial \tau}\int \tilde H^{k+1} \, d\tilde\mu$$

together with integration by parts leads to the inequality

$$\begin{aligned}\int |\nabla \tilde H^k|^2 \, d\tilde\mu \leqslant \frac{-1}{k+1}\frac{\partial}{\partial\tau}\int \tilde H^{k+1}\, d\tilde\mu + \frac{k}{k+1}\int \tilde H^{2k+2}\, d\tilde\mu \\ + n^k \frac{n-k-1}{k+1}\int \tilde H^{k+1}\, d\tilde\mu \ .\end{aligned}$$

Then integrate this inequality. By equation (7) the mean curvature $\tilde H$ is uniformly bounded, and since $\tilde M_\tau$ is convex it holds that

$$\tilde\mu(\tilde M_\tau) \leqslant n\omega_{n+1}\tilde\rho_+^n \ .$$

$\square$

Lemma 3.4. *There are constants $C_5, \eta > 0$ and $0 < \alpha < 1$ such that for every $(p,\tau) \in M \times (\eta, \infty)$ the α-Hölder norm in space-time of $\tilde{H}$ on $B_\eta(p) \times (\tau - \eta, \tau + \eta)$ is bounded by C_5.*

Proof. Pick a $(p_0, \tau_0) \in M \times [0, \infty)$. By Lemma 3.2 there is a $y \in \mathbb{R}^{n+1}$ such that $B_{1/C_2}(y)$ is enclosed by $\tilde{M}_{\tau_0}$. Since the speed of the rescaled evolution is uniformly bounded, there is a $\zeta > 0$, not depending on (p_0, τ_0), such that $\tilde{M}_\tau$ encloses $B_{1/(2C_2)}(y)$ for all $\tau \in [\tau_0 - \zeta, \tau_0 + \zeta]$. Since all the $\tilde{M}_\tau$ are convex, we have that

$$\langle \tilde{\nu}, \tilde{F} - y \rangle (p, \tau) \geqslant \frac{1}{2C_2} \quad \forall\, (p,\tau) \in M \times [\tau_0 - \zeta, \tau_0 + \zeta] \ .$$

Because also $\tilde{\rho}_+$ is uniformly bounded, there is a uniform constant $\eta > 0$, $2\eta \leqslant \zeta$, such that the evolving surfaces $\tilde{M}_\tau \cap ((B_{2\eta}(p_0) \cap T_{p_0}\tilde{M}_{\tau_0}) \times \mathbb{R} \cdot \tilde{\nu}(p_0, \tau_0))$ can be written as a graph of a function $u(x, \tau)$ on $B_{2\eta}(p_0) \cap T_{p_0}\tilde{M}_{\tau_0}$, which is uniformly bounded in C^2. The evolution equation of $\tilde{H}$ in this coordinates can then be written in the form

$$\frac{\partial}{\partial \tau}\tilde{H}(x,\tau) = D_i\Big(\big(\delta^{ij} - \frac{D^i u D^j u}{1 + |Du|^2}\big) D_j \tilde{H}^k\Big) + b^i D_i \tilde{H}^k + c \ ,$$

where $b^i = b^i(x, u, Du, D^2u)$ and $c = c(x, u, Du, D^2u)$, which are uniformly bounded. From Lemma 3.3 we have

$$\int_{\tau_0 - 2\eta}^{\tau_0 + 2\eta} \int_{B_{2\eta}(p_0) \cap T_{p_0}\tilde{M}_{\tau_0}} |D\tilde{H}^k|^2 \, dx \leqslant C \ .$$

These are all requirements to apply Theorem 1.2 in [6], which gives the claimed interior Hölder estimates. ☐

We now take a sequence of times (τ_j) with $\tau_j \to \infty$. Since the second fundamental form is uniformly bounded we can extract a subsequence of times, again denoted by (τ_j), such that

$$\tilde{M}_{\tau_j} \to M_\infty \ ,$$

in $C^{1,\alpha}$ and M_∞ is a convex $C^{1,1}$-surface. Because $\tilde{\rho}_+ \leqslant C_2$ there exist points p_j, such that

$$\tilde{H}(p_j, \tau_j) \geqslant \frac{n}{C_2} \ .$$

Assume $\tilde{F}(p_j, \tau_j) \to p_\infty \in M_\infty$. By Lemma 3.4 there is a $\delta > 0$ such that

$$\tilde{H}\big|_{B_\delta(p_j) \cap \tilde{M}_{\tau_j}} \geqslant \frac{n}{2C_2} \ .$$

Using parabolic Schauder estimates we obtain locally uniform C^∞-estimates as well as the convergence of

$$\tilde{M}_{\tau_j} \cap B_\delta(p_\infty) \to M_\infty \cap B_\delta(p_\infty)$$

in C^∞. By Theorem 2.6 it follows that $M_\infty \cap B_\delta(p_\infty)$ is totally umbilic, and by this part of a sphere with mean curvature grater or equal to n/C_2. Repeating this argument with a limit point $q_\infty \in B_\delta(p_\infty) \setminus B_{\delta/2}(p_\infty)$ we can extend the region where M_∞ is known to be spherical, and after finitely many steps we obtain that M_∞ is a sphere. As in Corollary 7.15 in [1] we obtain that M_∞ is a sphere with radius 1, centered at the origin. So the limit surface is independent of the approximating sequence and the whole flow converges in C^∞.

Theorem 3.5. *The immersion $\tilde{F}_\tau$ converges exponentially to a limit immersion $\tilde{F}_\infty$ with image equal to the unit sphere. Where under exponential convergence it is understood that there exist constants $\delta_i, C_i > 0$, $i \geqslant 0$ such that*

$$\begin{aligned}
\Big|\log\Big(\frac{\tilde{g}_\infty(\xi,\xi)}{\tilde{g}_\tau(\xi,\xi)}\Big)\Big| &\leqslant C_0 e^{-\delta_0\tau} \quad \text{for every non-zero tangent vector } \xi,\\
\big|\nabla^{(i)}(\tilde{A}_\tau - \tilde{A}_\infty)\big| &\leqslant C_i e^{-\delta_i\tau}\ ,\\
|\tilde{H}_\tau - n| &\leqslant C_0 e^{-\delta_0\tau}\ .
\end{aligned}$$

Proof. Let again $\tilde{f} := 1/n^n - \tilde{K}/\tilde{H}^n$. By Lemma 2.4 and Lemma 2.5 we have that

$$\frac{\partial}{\partial\tau}\tilde{f}(\tau) \leqslant k\tilde{H}^{k-1}\Big(\Delta\tilde{f} + \frac{2}{\tilde{H}}\langle\nabla\tilde{f},\nabla\tilde{H}\rangle + \frac{\tilde{H}^n}{\tilde{K}}|\nabla\tilde{f}|^2 - \delta\tilde{H}^2\tilde{f}\Big)$$

for $\delta > 0$ small enough. Since $\tilde{H}^2 \geqslant 1/(4n^2)$ for τ big enough, there exists a $\delta' > 0$ and a constant C, such that

$$\tilde{f}(\tau) \leqslant Ce^{-\delta'\tau}\ .$$

With a similar argument as in Lemma 2.5 we can deduce that this implies

$$|\tilde{\lambda}_{\max} - \tilde{\lambda}_{\min}|(p,\tau) \leqslant Ce^{-\delta'\tau} \quad \forall\, p \in M.$$

Now write $w_{ij} := \tilde{h}_{ij} - \frac{\tilde{H}}{n}\tilde{g}_{ij}$ and observe that

$$|w|^2 = w^{ij}w_{ij} = |\tilde{A}|^2 - \frac{1}{n}\tilde{H}^2 = \frac{1}{n}\sum_{i\neq j}(\tilde{\lambda}_i - \tilde{\lambda}_j)^2 \leqslant Ce^{-2\delta'\tau}\ .$$

This gives by interpolation that

$$|\nabla_k w_{ij}|\ \leqslant Ce^{-\delta''\tau} \qquad i,j,k = 1,\ldots,n\ ,$$

for some $0 < \delta'' \leqslant \delta'$. Assume we have $\tilde{g}_{ij} = \delta_{ij}$ at a point p, then

$$\nabla_k\tilde{H} = \sum_i \nabla_k\tilde{h}_{ii} = \sum_i \nabla_k\tilde{h}_{ki} = \frac{\nabla_k\tilde{H}}{n} + \sum_i \nabla_i w_{ki}\ ,$$

which gives

$$|\nabla_k\tilde{H}| \leqslant \frac{n}{n-1}\sum_i |\nabla_i w_{ki}| \leqslant Ce^{-\delta''\tau}\ .$$

Similarly one obtains

$$|\nabla\tilde{A}| \leqslant Ce^{-\delta'''\tau}\ ,$$

and all higher derivatives by interpolation. The remaining estimates on curvature follow by writing the converging surfaces $\tilde{M}_\tau$ as normal graphs over the unit sphere and using interpolation again. The estimate on the metric and the convergence of the immersions is again as in [1]. □

Appendix A. The 2-dimensional Case

with Oliver C. Schnürer

In this appendix we want to show that in the case of 2-dimensional surfaces in $\mathbb{R}^3$, for $1 \leqslant k \leqslant 5$, one can drop the initial pinching condition in Theorem 1.2. In concise form the theorem is stated as follows.

Theorem A.1. *A smooth closed convex surface in $\mathbb{R}^3$, contracting with normal velocity H^k, where $1 \leqslant k \leqslant 5$, contracts to a round point in finite time.*

The case $k = 1$ is again the well-known mean curvature flow, see [7], and the cases $k = 2, 3, 4$ were already treated in [8]. The proof of this Theorem depends crucially on the following pinching estimate.

Lemma A.2. *For a family of smooth closed strictly convex surfaces M_t in $\mathbb{R}^3$, flowing with normal velocity H^k, where $1 \leqslant k \leqslant 5$, the quantity*

$$\max_{M_t} \frac{(\lambda_1 + \lambda_2)^{2k}(\lambda_1 - \lambda_2)^2}{4\lambda_1^2\lambda_2^2} \tag{8}$$

is non-increasing in time.

Here λ_1, λ_2 denote the principal curvatures of M_t. This monotone quantity was found by O. Schnürer using a sieve algorithm implemented into a computer program. The algorithm uses random numbers for the test, whether the "right-hand side" of the evolution equation of such a quantity is non-positive and thus the maximum principle can be applied to prove monotonicity. For a detailed discussion of this algorithm see [8], where this method is also applied to find monotone quantities for a whole list of different normal velocities. The proof of Lemma A.2 follows closely the proofs in the paper cited above.

Proof of Lemma A.2. Set

$$w := \frac{H^{2k}(2|A|^2 - H^2)}{(H^2 - |A|^2)^2} = \frac{(\lambda_1 + \lambda_2)^{2k}(\lambda_1 - \lambda_2)^2}{4\lambda_1^2\lambda_2^2} \ .$$

For the rest of the proof we consider a point, where $w|_{M_t}$ attains a positive maximum for some $t > 0$. By the maximum principle, it suffices to show that $\tilde{w} := \log w$ is non-increasing in such a point for our Theorem to follow. Now rewrite

$$\begin{aligned}\tilde{w} &= 2\log(H^k) + \log(2|A|^2 - H^2) - 2\log(H^2 - |A|^2)\\ &\equiv 2\log A + \log B - 2\log C \ .\end{aligned}$$

In a critical point of $\tilde{w}$, we obtain

$$\begin{aligned}\Big(\frac{d}{dt} - kH^{k-1}\Delta\Big)\tilde{w} = {}& \frac{2}{A}\Big(\frac{d}{dt} - kH^{k-1}\Delta\Big)A + \frac{1}{B}\Big(\frac{d}{dt} - kH^{k-1}\Delta\Big)B\\ &- \frac{2}{C}\Big(\frac{d}{dt} - kH^{k-1}\Delta\Big)C + \frac{k}{2}H^{k-1}\frac{1}{B^2}|\nabla B|^2\\ &- \frac{2k}{AB}H^{k-1}\langle\nabla A, \nabla B\rangle\end{aligned}$$

and for $i = 1, 2$

$$2AB\,\nabla_i C = 2CB\,\nabla_i A + AC\,\nabla_i B \ ,$$

which yields

$$\nabla_i h_{22} = \frac{\lambda_2^2 H + k\lambda_1\lambda_2(\lambda_1 - \lambda_2)}{\lambda_1^2 H - k\lambda_1\lambda_2(\lambda_1 - \lambda_2)}\nabla_i h_{11} =: \frac{\alpha}{\beta}\nabla_i h_{11} \ , \tag{9}$$

where we assume for the moment that $\alpha, \beta \neq 0$. Here we have chosen normal coordinates such that the second fundamental form is diagonal, i.e. $h_{11} = \lambda_1,\ h_{22} =$

$\lambda_2,\ h_{12} = h_{21} = 0$. We now insert the evolution equations from Lemma 2.1 and combine the above results to obtain

$$\begin{aligned}
\Big(\frac{d}{dt} - kH^{k-1}\Delta\Big)\tilde{w} &= \frac{2}{A}\Big(\frac{d}{dt} - kH^{k-1}\Delta\Big)H^k + \Big(\frac{2}{B} + \frac{2}{C}\Big)\Big(\frac{d}{dt} - kH^{k-1}\Delta\Big)|A|^2 \\
&\quad - \Big(\frac{1}{B} + \frac{2}{C}\Big)\Big(\frac{d}{dt} - kH^{k-1}\Delta\Big)H^2 + \frac{k}{2}\frac{H^{k-1}}{B^2}|\nabla(h_{11} - h_{22})^2|^2 \\
&\quad - 2k\frac{H^{k-1}}{AB}\langle\nabla H^k, \nabla(h_{11} - h_{22})^2\rangle \\
&= 2kH^{k-1}|A|^2 + \frac{2|A|^2}{(2|A|^2 - H^2)(H^2 - |A|^2)}\Big(- 2kH^{k-1}|\nabla A|^2 \\
&\quad + 2k(k-1)H^{k-2}h^{ij}\nabla_i H\nabla_j H - 2(k-1)H^k\mathrm{tr}(A^3) \\
&\quad + 2kH^{k-1}|A|^4\Big) - \frac{3|A|^2 - H^2}{(2|A|^2 - H^2)(H^2 - |A|^2)}\Big(2H^{k+1}|A|^2 \\
&\quad + 2k(k-2)H^{k-1}|\nabla H|^2\Big) + 2k\frac{H^{k-1}}{(\lambda_1 - \lambda_2)^2}|\nabla(h_{11} - h_{22})|^2 \\
&\quad - \frac{4k^2}{(\lambda_1 - \lambda_2)}H^{k-2}\langle\nabla H, \nabla(h_{11} - h_{22})\rangle\ .
\end{aligned}$$

The terms not containing derivatives of h_{ij} cancel. We multiply the RHS by $(2|A|^2 - H^2)(H^2 - |A|^2)H^{2-k}$ and use (9):

$$\begin{aligned}
RHS\cdot\frac{2(\lambda_1 - \lambda_2)^2\lambda_1\lambda_2}{H^{k-2}} =&\ - 4kH|A|^2|\nabla A|^2 + 4k(k-1)|A|^2h^{ij}\nabla_i H\nabla_j H \\
&\ - 2k(k-2)H(3|A|^2 - H^2)|\nabla H|^2 \\
&\ + 4k\lambda_1\lambda_2 H|\nabla(h_{11} - h_{22})|^2 \\
&\ - 8k^2\lambda_1\lambda_2(\lambda_1 - \lambda_2)\langle\nabla H, \nabla(h_{11} - h_{22})\rangle \\
=&\ \Big(- 4kH|A|^2\big(\beta^2 + 3\alpha^2\big) + 4k(k-1)|A|^2\big(\lambda_1(\alpha + \beta)^2\big) \\
&\ - 2k(k-2)H(3|A|^2 - H^2)(\alpha + \beta)^2 + 4k\lambda_1\lambda_2 H(\alpha - \beta)^2 \\
&\ + 8k^2\lambda_1\lambda_2(\lambda_1 - \lambda_2)(\alpha^2 - \beta^2)\Big)\frac{(\nabla_1 h_{11})^2}{\beta^2} \\
&\ + \Big(\cdots\Big)\frac{(\nabla_2 h_{22})^2}{\alpha^2} \\
=&\ - 4\lambda_2^2 H\Big((3k-1)\lambda_1^6 + 4k(1-k)\lambda_1^5\lambda_2 \\
&\ + (8k^2 - 9k + 7)\lambda_1^4\lambda_2^2 + 4(1-k^2)\lambda_1^3\lambda_2^3 \\
&\ + (5k+1)\lambda_1^2\lambda_2^4 + 4(1-k)\lambda_1\lambda_2^5 \\
&\ + (k+1)\lambda_2^6\Big)\frac{(\nabla_1 h_{11})^2}{\beta^2} + \Big(\cdots\Big)\frac{(\nabla_2 h_{22})^2}{\alpha^2}
\end{aligned}$$

where the term in brackets in front of $(\nabla_2 h_{22})^2/\alpha^2$ is similar to the one before, just with α and β as well as λ_1 and λ_2 interchanged. To complete the proof of the lemma it remains to check that the polynomial in front of $(\nabla_1 h_{11})^2/\beta^2$ is non-positive. Since it is homogeneous in λ_1, λ_2 we can choose $\lambda_2 = 1$ and it remains to show that

$$(3k-1)\lambda_1^6 + 4k(1-k)\lambda_1^5 + (8k^2-9k+7)\lambda_1^4 + 4(1-k^2)\lambda_1^3 \\ + (5k+1)\lambda_1^2 + 4(1-k)\lambda_1 + (k+1) \geqslant 0 \tag{10}$$

for $\lambda_1 \geqslant 0$ and $1 \leqslant k \leqslant 5$. The inequality for $k = 1$ is trivial, in the case $k = 5$ we use a computer algebra system and Sturm's theorem. One can check that this even extends to $k = 5.17...$. To see that the inequality holds also for $1 < k < 5$ we reorder the polynomial in powers of k:

$$-(4\lambda_1^3(\lambda_1-1)^2)k^2+(3\lambda_1^6+4\lambda_1^5-9\lambda_1^4+5\lambda_1^2-4\lambda_1+1)k-\lambda_1^6+7\lambda_1^4+4\lambda_1^3+\lambda_1^2+4\lambda_1+1\,.$$

Thus for fixed $\lambda_1 \geqslant 0$ this is a concave function in k, which proves (10).
In the case that $\beta = 0$ it follows that $\alpha \neq 0$ since the evolving surfaces are strictly convex. Equation (9) then yields $\nabla_1 h_{11} = \nabla_2 h_{11} = 0$. In a similar manner as before we obtain now polynomials in λ_1, λ_2 multiplying $(\nabla_1 h_{22})^2$ and $(\nabla_2 h_{22})^2$. The term in front of $(\nabla_2 h_{22})^2$ is the same as above and the one in front of $(\nabla_1 h_{22})^2$ is given by

$$-12kH|A|^2+4k(k-1)\lambda_1|A|^2-2k(k-2)H(3|A|^2-H^2)+4k\lambda_1\lambda_2 H+8k^2\lambda_1\lambda_2(\lambda_1-\lambda_2)\,.$$

Choosing $\lambda_2 = 1$ it remains again to check that

$$-8k\lambda_1^3 + 8k(k-1)\lambda_1^2 - 4k(k+3)\lambda_1 - 4k(k+1) \leqslant 0$$

for all $\lambda_1 \geqslant 0$ and $1 \leqslant k \leqslant 5$. This can be done as before. The case $\alpha = 0$ is a mirror image of the case $\beta = 0$. □

Proof of Theorem A.1. By Theorem 1.1 the surfaces M_t become immediately strictly convex for $t > 0$. Now choose a sufficiently small $0 < \varepsilon < T$ such that the flow is smooth and strictly convex on the interval (ε, T), where T is the maximal time of existence. Thus the quantity w, as defined above, is well defined on this interval, and bounded from above by Lemma A.2. Note that this implies

$$1 \leqslant \frac{\lambda_{\max}(p,t)}{\lambda_{\min}(p,t)} \leqslant 1 + \frac{C}{H^{k-1}}$$

for all $(p,t) \in M^2 \times (\varepsilon, T)$. This replaces the estimate (3) for $k > 1$ and the proof follows analogously to the proof of Theorem 1.2.
An alternative way of proving Theorem A.1 would be to follow the proofs in [1] or [8]. □

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