TRANSLATING SOLUTIONS FOR GAUSS CURVATURE FLOWS WITH NEUMANN BOUNDARY CONDITION

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Abstract. We consider the evolution of strictly convex hypersurfaces driven by the non-parametric logarithmic Gauß curvature flow subject to a Neumann boundary condition. In particular, solutions exist globally and converge smoothly to translating solutions.

1. Introduction

In this paper we consider the evolution of non-parametric strictly convex hypersurfaces in a cylinder $\Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}$ driven by the non-parametric logarithmic Gauß curvature flow subject to a Neumann boundary condition. For this flow equation we show that solutions to the corresponding initial value problem exist for all time and converge smoothly to a solution that moves by translation.

To be more precisely, we address to the following slightly more general problem: Let $\Omega \subset \mathbb{R}^n$ be a smooth strictly convex bounded domain. (In this paper we use strictly convex provided the respective principal curvatures are all positive.) Let $f : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ be a smooth positive function. At the boundary we have a given smooth function $\varphi : \partial\Omega \to \mathbb{R}$. Assume that $u_0 : \overline{\Omega} \to \mathbb{R}$ is a smooth strictly convex function that satisfies the boundary condition

$$(u_0)_\nu(x) \equiv D_\nu u_0(x) = \varphi(x) \quad \forall\, x \in \partial\Omega,$$

where ν is the inner unit normal to Ω and indices will here and later denote partial derivatives. We will prove the following

Theorem 1.1. *For Ω, f, φ, ν, and u_0 as introduced above, there exists a global strictly convex solution $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ such that*

$$\begin{cases} \dot{u} = \log\det D^2 u - \log f(x, Du) & \text{in } \Omega \times [0,\infty) \\ u_\nu = \varphi(x) & \text{on } \partial\Omega \times [0,\infty) \\ u|_{t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where u is smooth for all times t and approaches u_0 in $C^2\left(\overline{\Omega}\right)$ as $t \downarrow 0$. Moreover, $u(\cdot, t)$ converges smoothly to a translating solution, i. e. to a solution with constant time derivative.

For the problem mentioned above, where the flow is driven by the Gauß curvature, the flow equation takes the form

$$\dot{u} = \log\det D^2u - \tfrac{n+2}{2}\log\left(1+|Du|^2\right). \tag{1.2}$$

Similar as in [6] it seems possible to us, to use a more general flow equation. This generalization of the theorem above, however, seems to us to be only of technical interest and furthermore it seems easy to combine the techniques of the mentioned paper and those used here to obtain this generalization. For the proof of our Main Theorem 1.1, however, we assume that we have the more general boundary condition

$$u_\beta = \varphi(x),$$

where β is a unit vectorfield which is C^1 close to ν, i. e., there exists a small positive constant $c_\beta > 0$ such that $\|\nu - \beta\|_{C^1} \le c_\beta$. Hence, our theorem remains true for such an oblique boundary condition. This generalization of a Neumann boundary condition is also studied in [7].

Translating solutions can also be considered as elliptic solutions. Thus we obtain (for a proof with elliptic methods see Section 4)

Corollary 1.2. *For Ω, f, ν, and φ as introduced above, there exists a strictly convex solution $(v, u) \in \mathbb{R}_+ \times C^\infty\left(\overline{\Omega}\right)$ to the boundary value problem*

$$\begin{cases} \det D^2u = v\cdot f(x, Du) & \text{in } \Omega, \\ u_\nu = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

provided that there exists a smooth strictly convex function u_0 that fulfills the boundary condition $(u_0)_\nu = \varphi(x)$.

A similar situation to that of Theorem 1.1 is considered for the mean curvature flow in [2]. Hypersurfaces of prescribed Gauß curvature subject to a Neumann boundary condition are found in the pioneering paper [4]. This is extended in [7] to the situation of an oblique boundary value condition. A flow approach is used in [6] to solve these kind of equations for Neumann and second boundary value problems. For a second boundary value problem translating solutions for curvature flows are considered in [5].

It is already mentioned in [5] that solutions to the second boundary value problem converge in a similar situation as in Theorem 1.1 to translating solutions. For the non-parametric logarithmic Gauß curvature flow this follows by adapting methods of [6]. For the Neumann boundary value problem, however, a new proof is needed to show that the oscillation of the function, whose graph is the hypersurface, remains bounded. Therefore we establish in Section 3 a generalization of the (spatial) C^1-estimates of [4]. The main difference is that in our situation the C^0-bounds and even the oscillation estimates are not uniform before we use this new estimate. There is another essential difference compared to the second boundary value problem. From the maximum principle we get that for both the Neumann and for the second boundary value problem, smooth translating solutions are unique up to translation. Especially the final "vertical" velocity is independent of the initial value u_0. However, it is not clear to the authors how this final velocity can be determined easily from the data of the problem in the case of a Neumann boundary condition. For the C^2-estimates in the presence of a Neumann boundary condition, we need an auxiliary barrier function. To guarantee that these C^2-estimates are uniform in time, we need to shift this auxiliary barrier function so that the vertical difference to our solution remains bounded. This can be obtained easily, when the final velocity is known. We need time-independent C^2-estimates to prove smooth convergence to a translating solution.

We proceed as follows. As explained in [6] we can use standard parabolic theory to obtain shorttime existence. Therefore we may assume without loss of generality that we have a smooth solution (up to $t = 0$) for a short time interval. Within Section 2 and 3 we prove time-independent lower order estimates. In Section 4 we show the existence of a unique (up to an additive constant) translating solution using elliptic methods. As soon as we know the velocity of a translating solution, we can prove time-independent a priori C^2-estimates in Section 5. The estimates of Krylov, Safonov and Evans and Schauder theory give uniform control for higher derivatives. Finally we prove smooth convergence to a translating solution.

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2. $\dot{u}$-estimates

Notation 2.1. We use indices to denote partial derivatives, f_{p_i} denotes a derivative with respect to the gradient. For a vector ξ we define $u_\xi \equiv \xi^i u_i$. For the logarithm of f we use $\hat{f} \equiv \log f$. We use the Einstein summation convention and sum over repeated upper and lower indices. The inverse of the Hessian of u is denoted by $\left(u^{ij}\right) = (u_{ij})^{-1}$. We remark that – besides in the case u^{ij} – indices are lifted with respect to the Euclidean metric. The letter c denotes a positive constant and may have a new value at each occurrence. Furthermore we may assume that $0 \in \Omega$.

Lemma 2.2. *As long as a smooth convex solution of (1.1) exists, we obtain the estimate*

$$\min\left\{\min_{t=0}\dot u, 0\right\} \le \dot u \le \max\left\{\max_{t=0}\dot u, 0\right\}.$$

Proof. Similar to [6] we consider

$$r := (\dot u)^2.$$

We get the evolution equation

$$\dot r = u^{ij} r_{ij} - 2u^{ij}\dot u_i \dot u_j - \hat f_{p_i} r_i. \tag{2.1}$$

From the evolution equation (2.1) we see that $\dot r \le 0$ in a maximum of r in $\Omega \times [0,t]$. Assume that a maximum of r restricted to $\overline{\Omega} \times [0,t]$ occurs in (t_0, x_0) with $t_0 > 0$ and $x_0 \in \partial\Omega$. If r is constant, then u is a translating solution and our lemma holds. Otherwise we use the Hopf boundary point lemma and get $r_\beta(x_0) < 0$. This is impossible as we get in this case at x_0

$$0 > r_\beta = \left((\dot u)^2\right)_\beta = 2\dot u \dot u_\beta = 2\dot u \frac{d}{dt}\varphi(x) = 0.$$

In this formula we interchanged time derivatives and spatial derivatives. To do so, we need that u is sufficiently smooth. This is fulfilled since by assumption $t_0 > 0$. □

Integrating the obtained estimate yields

Corollary 2.3. *As long as a smooth convex solution of (1.1) exists, we obtain the estimate*

$$|u(x,t)| \le \sup_\Omega |u_0| + \sup_\Omega |\dot u(\cdot, 0)| \cdot t.$$

3. Ice-cream cone estimate

The name for the next theorem comes from the fact that its proof uses cones and balls similarly to their occurrence when ice-cream is in a cone of waffle. The result generalizes the C^1-estimates of [4] and is essential for our proof; namely, the C^0-estimates obtained from integrating $\dot u$ are much to weak and stronger oscillation estimates are needed.

We wish to mention oscillation estimates of Urbas [8] that can be combined with the C^1-estimates of [4] to prove the following Theorem provided that $f(x, Du)$ has the special structure as in the Gauß curvature flow (1.2). More precisely, $f(x, Du)$ must have an appropriate growth in Du. It is interesting to observe that the oscillation estimate of Urbas uses the convexity of u and the equation of prescribed Gauß curvature, whereas Theorem 3.1 combines convexity and the boundary condition.

Theorem 3.1 (Ice-cream cone estimate)**.** *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $u : \overline{\Omega} \to \mathbb{R}$ a smooth strictly convex function with $|u_\beta|$ uniformly bounded on $\partial\Omega$, where β is a unit vector field on $\partial\Omega$ such that $\langle \beta, \nu\rangle \ge \tilde{c}_\beta$ for a positive constant $\tilde{c}_\beta > 0$ (recall that ν is the inner unit normal to $\partial\Omega$). Then there is a uniform bound for* $\sup|Du|$ *independent of* $\sup|u|$.

In view of Lemma 2.2 the result above concludes an estimate for the full C^1-norm of solutions u to (1.1). Note that only the estimate for the derivatives of u is uniform in time.

Proof. We argue by contradiction. Assume that there exists a point x_0, where $|Du|$ is maximal and equals M which is assumed to be sufficiently large. We will show that this leads to a contradiction if M is larger than a suitably chosen constant M_0. As u is strictly convex we see that $x_0 \in \partial\Omega$. At x_0 we find a tangential direction ξ_0 such that $\langle Du(x_0), \xi_0\rangle$ is maximal compared to all other tangential directions there. Here and later we denote unit vectors as directions. We wish to prove a lower estimate for this quantity in terms of M. Let ξ_1 be a direction such that $\langle Du(x_0), \xi_1\rangle = |Du|$. Similar to [7] we decompose a direction ξ using β and a tangential vector $\tau(\xi)$ as

$$\xi = \tau(\xi) + \frac{\langle \nu, \xi\rangle}{\langle \beta, \nu\rangle}\beta, \tag{3.1}$$

where

$$\tau(\xi) = \xi - \langle \nu, \xi\rangle\nu - \frac{\langle \nu, \xi\rangle}{\langle \beta, \nu\rangle}\beta^T, \quad \beta^T = \beta - \langle \beta, \nu\rangle\nu.$$

Note that $|\tau(\xi)|$ is a priori bounded. Decomposing ξ_1 we get

$$\begin{aligned} M &= \langle Du, \xi_1\rangle = \langle Du, \tau(\xi_1)\rangle + \frac{\langle \nu, \xi_1\rangle}{\langle \beta, \nu\rangle}\langle Du, \beta\rangle \\ &\le |\tau(\xi_1)| \cdot \max_{\substack{\tau \in T_{x_0}\partial\Omega \\ |\tau|=1}} \langle Du, \tau\rangle + c \\ &= |\tau(\xi_1)| \cdot \langle Du(x_0), \xi_0\rangle + c \end{aligned}$$

and deduce that $\langle Du(x_0), \xi_0\rangle \ge \frac{M}{c}$ for $M \ge M_0$, provided that M_0 is sufficiently large. For a direction ξ near ξ_0, say $|\xi - \xi_0| < \varepsilon = \frac{1}{2c} < 1$, we obtain

$$\langle Du(x_0), \xi\rangle = \langle Du, \xi_0\rangle + \langle Du, \xi - \xi_0\rangle \ge \tfrac{M}{c} - M|\xi - \xi_0| \ge \varepsilon M. \tag{3.2}$$

From the convexity of u we deduce that $\langle Du(y), \xi\rangle \ge \varepsilon M$ for all points $y \in \overline{\Omega}$ of the form $y = x_0 + \lambda \cdot \xi$. Here $\lambda > 0$ and ξ are chosen such that $|\xi - \xi_0| \le \varepsilon$ and $x_0 + t \cdot \lambda \cdot \xi \in \overline{\Omega}$ for all $t \in [0, 1]$.

We deduce from the smoothness of Ω and the uniform boundedness of the principal curvatures of $\partial\Omega \subset \mathbb{R}^{n+1}$ that there exists $R > 0$ and $x_1 \in \partial\Omega$ with $|x_0 - x_1| > 2R$ such that especially $|Du| \ge \varepsilon M$ in $\partial\Omega \cap B_R(x_1)$. Assume moreover that any $x \in B_R(x_1) \cap \partial\Omega$ is of the form $x_0 + \lambda \cdot \xi$ as described above with $|\xi - \xi_0| \le \varepsilon$. Due to our construction we have

$$\min_{x \in B_R(x_1) \cap \partial\Omega} u(x) > u(x_0). \tag{3.3}$$

Because of the shape of the set of directions ξ satisfying (3.2) together with the ball mentioned above, we call our theorem the ice-cream cone estimate.

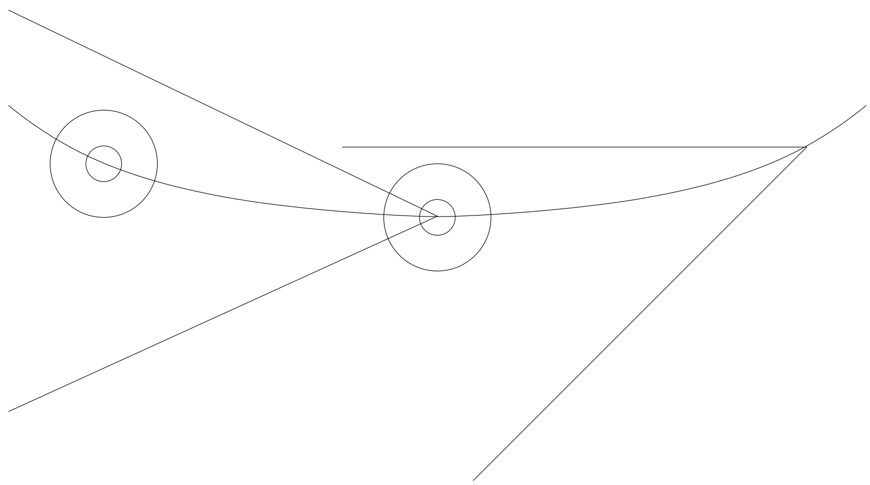


Figure 1. Ice-cream cone estimate

Figure 1 shows part of $\partial\Omega$ and two cones corresponding to the directions ξ as well as two pairs of concentric balls. The larger ones are the balls B_R mentioned above, the smaller ones are introduced in the following.

Now we may proceed iteratively. Note that R and ε can be chosen as fixed constants independent of the point x_0. As long as $|Du(x_i)| \ge M\varepsilon^i \ge M_0$ we can find a further point x_{i+1} in the same way as we found x_1 starting from x_0. Thus for $M = \sup |Du|$ sufficiently large we can construct a sequence of points $\{x_i\}_{i=0,\ldots,N}$ of arbitrarily large length N satisfying for all $i \ge 1$

$$|Du| \ge M\varepsilon^i \quad \text{on } \partial\Omega \cap B_R(x_i)$$

and

$$\min_{x \in B_R(x_i) \cap \partial\Omega} u(x) > u(x_{i-1}).$$

Since $\partial\Omega$ has finite measure and bounded principal curvatures there is an upper bound $N_0(\rho)$ on the number of pairwise disjoint restricted balls $B_\rho(y_j) \cap \partial\Omega$ for fixed $\rho > 0$ and $y_j \in \partial\Omega$.

Hence, if $M = \sup |Du| > M_0\varepsilon^{-N_0\left(\frac{R}{3}\right)}$ there will be two points x_{i_0}, x_{j_0} with $i_0 > j_0 > 0$ such that

$$B_{\frac{R}{3}}(x_{i_0}) \cap B_{\frac{R}{3}}(x_{j_0}) \cap \partial\Omega \ne \emptyset.$$

From $x_{j_0} \in B_R(x_{i_0})$ we deduce a contradiction since

$$u(x_{j_0}) < u(x_{j_0+1}) < \ldots < u(x_{i_0-1}) < \min_{x \in B_R(x_{i_0}) \cap \partial\Omega} u(x) \leq u(x_{j_0})$$

is clearly impossible. □

4. Existence of translating solutions

This section is devoted to finding solutions to the elliptic equation

$$\begin{cases} v = \log\det D^2u - \log f(x, Du) & \text{in } \Omega, \\ u_\beta = \varphi & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The absence of any monotonicity property (wrt. u) both in f as well as in the boundary condition φ limits seriously the existence of solutions.

Step 1: Here we show that for given $\varepsilon > 0$ and $v \in \mathbb{R}$ there is a unique solution $u_{\varepsilon,v}$ of

$$\begin{cases} \det D^2u = e^v f(x, Du)e^{\varepsilon u} & \text{in } \Omega, \\ u_\beta = \varphi & \text{on } \partial\Omega. \end{cases} \tag{$*_{\varepsilon,v}$}$$

Note that the dependence on v is continuous and strictly decreasing. In fact we have the explicit relation

$$u_{\varepsilon,v} = u_{\varepsilon,0} - v/\varepsilon.$$

To show the unique existence of $u_{\varepsilon,v}$ we will derive below an a priori sup bound, then the ice-cream cone estimate gives the C^1-bound. Having the full C^1-norm controlled we can estimate the C^2-norm exactly as in Urbas [7], since the strict monotonicity assumption on the boundary condition therein is not used in this part. A detailed proof for this is given in the next section where we extend the argument to the parabolic case. Bounds for higher C^k-norms follow via the estimates due to Krylov, Safonov, Evans, and from Schauder theory.

To get the sup bound we define suitable barriers for $(*_{\varepsilon,0})$. Recall that u_0 is a convex function satisfying $(u_0)_\beta = \varphi$ on $\partial\Omega$. Define $u_\varepsilon^\pm = u_0 \pm M/\varepsilon$, where $M > 0$ will be chosen later. Then

$$\frac{\det D^2 u_\varepsilon^\pm}{f(x, Du_\varepsilon^\pm)e^{\varepsilon u_\varepsilon^\pm}} = \frac{\det D^2 u_0}{f(x, Du_0)e^{\varepsilon u_0 \pm M}} = g(x)e^{-\varepsilon u_0}e^{\mp M},$$

with $c^{-1} < g(x) < c$. Hence for $\varepsilon < 1$ there exists a large constant $M > 0$ such that u_ε^+ is a strict supersolution and u_ε^- is a strict subsolution of $(*_{\varepsilon,0})$. This implies, that

$$u_\varepsilon^- < u_{\varepsilon,0} < u_\varepsilon^+, \tag{4.2}$$

or equivalently,

$$\left|u_{\varepsilon,v} - (u_0 - \tfrac{v}{\varepsilon})\right| < \frac{M}{\varepsilon}.$$

Step 2: Now we consider the limit $\varepsilon \to 0$. In general, we cannot expect that the sup bounds for $u_{\varepsilon,v}$ can be obtained uniformly in ε. In fact it follows from the maximum principle that only for a unique v there is a solution to $(*_{\varepsilon,v})$ with $\varepsilon = 0$. Observe that (4.2) implies that

$$u_{\varepsilon,+M} < u_0 < u_{\varepsilon,-M}.$$

Therefore we can find for every $\varepsilon > 0$ a unique $v_\varepsilon \in (-M, M)$ such that $u_{\varepsilon,v_\varepsilon}(0) = u_0(0)$. Note that (4.2) does not suffice to control the oscillation of $u_{\varepsilon,v_\varepsilon}$ uniformly in ε. We employ the ice-cream cone estimate to bound $u_{\varepsilon,v_\varepsilon}$ uniformly in C^1. Again, uniform C^1-bounds imply uniform higher C^k-bounds.

Now we choose a sequence $\varepsilon_i \to 0$ as $i \to \infty$. Since v_{ε_i} is bounded, there is a subsequence (relabeled) such that $v_{\varepsilon_i} \to v^\infty$ and $u_{\varepsilon_i,v_{\varepsilon_i}} \to u^\infty_{\text{ell}}$ in any C^k-norm.

The extension $u^\infty(x,t) := u^\infty_{\text{ell}}(x) + v^\infty t$ is a translating solution as by construction u^∞ satisfies

$$\begin{cases} v^\infty = \dot{u}^\infty = \log\det D^2 u^\infty - \log f(x, Du^\infty) & \text{in } \Omega, \\ \qquad u^\infty_\beta = \varphi(x) & \text{on } \partial\Omega. \end{cases}$$

5. Parabolic C^2-estimates

We remark that this proof is an adaption of the respective proofs in [4, 6, 7]. It seems important to us for the proof of the a priori C^2-estimates to have a translating solution u^∞ and especially to know its velocity v^∞.

We will construct an auxiliary function for these estimates. Assume that $u^\infty_{\text{ell}} > u_0$. We define

$$\tilde{\varphi}(x,z) = \varphi(x) + (z - u^\infty_{\text{ell}}).$$

Due to the uniform estimates for the norm of the gradient of u we can find μ_0 such that

$$\min\{f(x, Du),\ f(x, Du^\infty_{\text{ell}})\} \ge \mu_0.$$

Consider the following boundary value problem for $0 < \rho < 1$

$$\begin{cases} v^\infty = \log\det D^2\psi - \log\frac{\mu_0}{2} & \text{in } \Omega, \\ \psi_\beta = \tilde{\varphi}\left(x, \psi + \rho\cdot|x|^2\right) - 2\rho\,\langle x, \beta\rangle & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

We wish to show uniform a priori C^2-estimates for ψ. Lemma 3.1 gives estimates for the gradient and estimates for the second derivatives follow from C^1-estimates and [7]. Thus it remains to prove uniform C^0-estimates. As $\tilde{\varphi}(x,z) \to \infty$ uniformly as $z \to \infty$, we get an upper bound for ψ, as any strictly convex solution has to fulfill $\psi_\beta < 0$ somewhere on $\partial\Omega$. The lower bound follows using the maximum principle as follows. For $\psi - u^\infty_{\text{ell}}$ we get the differential inequality

$$\begin{cases} 0 & > & \log\det D^2\psi - \log\det D^2 u^\infty_{\text{ell}} & \text{in } \Omega, \\ (\psi - u^\infty_{\text{ell}})_\beta & = & \psi - u^\infty_{\text{ell}} + \rho\cdot|x|^2 - 2\rho\,\langle x,\beta\rangle & \text{on } \partial\Omega. \end{cases}$$

Thus $\psi - u^\infty$ cannot attain an interior minimum. If a minimum occurs on $\partial\Omega$, we get

$$0 \le (\psi - u^\infty)_\beta = \psi - u^\infty + \rho\cdot|x|^2 - 2\rho\,\langle x,\beta\rangle.$$

Thus ψ is uniformly bounded below and we can solve (5.1). Due to the uniform C^2-estimates, we can fix $\lambda > 0$ such that

$$\psi_{ij} \ge \lambda\delta_{ij} \tag{5.2}$$

in the matrix sense. Further on, these estimates allow to fix $\rho > 0$ such that $\overline{\psi} := \psi + \rho\cdot|x|^2$ satisfies

$$\begin{cases} v^\infty & > & \log\det D^2\overline{\psi} - \log\mu_0 & \text{in } \Omega, \\ \overline{\psi}_\beta & = & \tilde{\varphi}\left(x,\overline{\psi}\right) & \text{on } \partial\Omega. \end{cases}$$

Now we apply the maximum principle and get $u^\infty_{\text{ell}} \le \overline{\psi}$. We extend ψ and $\overline{\psi}$ by setting $\psi(x,t) := \psi(x) + t\cdot v^\infty$ and $\overline{\psi}(x,t) := \overline{\psi}(x) + t\cdot v^\infty$ and see that $u \le u^\infty \le \overline{\psi}$, where u^∞ is defined as in Section 4. So we get

$$\overline{\psi}_\beta - u_\beta = \overline{\psi}_\beta(\cdot,0) - \varphi(x) = \left(\overline{\psi} - u^\infty\right)(\cdot,0) = \overline{\psi} - u^\infty \ge 0$$

and furthermore for a sufficiently small $\delta_0 > 0$

$$(\psi - u)_\beta = \left(\overline{\psi} - \rho\cdot|x|^2 - u\right)_\beta \ge -2\rho\,\langle x,\beta\rangle \ge \delta_0 > 0, \tag{5.3}$$

provided that β is C^0-close to ν. Here we used that $0 \in \Omega$ and thus we have $\langle x,\nu\rangle < 0$. Using these preparations, we are able to prove C^2-a priori estimates similar to [6, 7]. For the reader's convenience, we repeat the arguments.

5.1. **Preliminary results.** We use τ for a direction tangential to $\partial\Omega$.

Lemma 5.1 (Mixed C^2-estimates at the boundary)**.** *Let u be a solution of our flow equation (1.1). Then the absolute value of $u_{\tau\beta}$ remains a priori bounded on $\partial\Omega$ during the evolution.*

Proof. We represent $\partial\Omega$ locally as graph ω over its tangent plane at a fixed point $x_0 \in \partial\Omega$ such that locally $\Omega = \{(\hat{x}, x^n) : x^n > \omega(\hat{x})\}$. We extend β smoothly and differentiate the oblique boundary condition

$$\beta^i(\hat{x}) u_i(\hat{x},\omega(\hat{x})) = \varphi(\hat{x},\omega(\hat{x})), \quad \hat{x} \in \mathbb{R}^{n-1},$$

with respect to $\hat{x}^j$, $1 \le j \le n-1$,

$$\beta^i_j u_i + \beta^i u_{ij} + \beta^i u_{in}\omega_j = \varphi_j + \varphi_n\omega_j$$

and obtain at $x_0 \equiv (\hat{x}_0, \omega(\hat{x}_0)) \in \partial\Omega$ a bound for $\beta^i u_{ij}$ in view of the gradient estimates and $D\omega(\hat{x}_0) = 0$. Multiplying with τ^j gives the result. We remark that it is only possible to multiply the equation with a tangential vector as the differentiation with respect to $\hat{x}^j$ and so also j correspond to tangential directions. $\square$

Lemma 5.2 (Double oblique C^2-estimates at the boundary)**.** *For any solution of the flow equation (1.1) the absolute value of $u_{\beta\beta}$ is a priori bounded from above on $\partial\Omega$. ($u_{\beta\beta} > 0$ also follows from the strict convexity of a solution.)*

Proof. We assume the same geometric situation as in the proof of Lemma 5.1 with $x_0 \in \partial\Omega$. From (1.1) we obtain

$$\dot{u}_k = u^{ij} u_{ijk} - (\hat{f}_k + \hat{f}_{p_i} u_{ik})$$

and define therefore

$$Lw := \dot{w} - u^{ij} w_{ij} + \hat{f}_{p_i} w_i,$$

where we evaluate the terms by using the function u. From the definition of L it is easy to see that for appropriate extensions of β and φ

$$\left| L\left(\beta^k u_k - \varphi(x)\right)\right| \le c \cdot \left(1 + \operatorname{tr} u^{ij}\right).$$

We define $\Omega_\delta := \Omega \cap B_\delta(x_0)$ for $\delta > 0$ sufficiently small and set

$$\vartheta := d - \mu d^2$$

for $\mu \gg 1$ sufficiently large where d denotes the distance from $\partial\Omega$. We will show that $L\vartheta \ge \frac{\varepsilon}{3} \operatorname{tr} u^{ij}$ for a small constant $\varepsilon > 0$ (depending only on a positive lower bound for the principal curvatures of $\partial\Omega$) in Ω_δ.

$$\begin{aligned} L\vartheta &= -u^{ij} d_{ij} + 2\mu u^{ij} d_i d_j + 2\mu u^{ij} d d_{ij} + \hat{f}_{p_i}(d_i - 2\mu d d_i) \\ &\ge -u^{ij} d_{ij} + 2\mu u^{ij} d_i d_j - c\mu d\left(1 + \operatorname{tr} u^{ij}\right) - c. \end{aligned}$$

We use the strict convexity of $\partial\Omega$, $d_i \approx \delta_{in}$, $|u^{kl}| \le \operatorname{tr} u^{ij}$, $1 \le k,\, l \le n$, and the inequality for arithmetic and geometric means

$$\begin{aligned} L\vartheta &\ge \varepsilon \operatorname{tr} u^{ij} + \mu u^{nn} - c\mu\delta\left(1 + \operatorname{tr} u^{ij}\right) - c \\ &\ge \frac{n}{3}\left(\det u^{ij}\right)^{\frac{1}{n}} \cdot \varepsilon^{\frac{n-1}{n}} \cdot \mu^{\frac{1}{n}} + \frac{2}{3}\varepsilon \operatorname{tr} u^{ij} \\ &\quad - c\mu\delta\left(1 + \operatorname{tr} u^{ij}\right) - c. \end{aligned} \tag{5.4}$$

Let us explain more precisely how we used the inequality for arithmetic and geometric means. We get immediately from the inequality for arithmetic and geometric means

$$\frac{\varepsilon \operatorname{tr} u^{ij} + \mu u^{nn}}{3} \ge \frac{n}{3} \varepsilon^{\frac{n-1}{n}} \mu^{\frac{1}{n}} \prod_{i=1}^{n} u^{ii}. \tag{5.5}$$

We may assume that $(u^{ij})_{i,j<n}$ is diagonal and combine (5.5) with

$$\begin{aligned}\det u^{ij} &= \det\begin{pmatrix} u^{11} & 0 & \cdots & 0 & u^{1n} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & u^{n-1\,n-1} & u^{n-1\,n} \\ u^{1n} & \cdots & \cdots & u^{n-1\,n} & u^{nn} \end{pmatrix} \\ &= \prod_{i=1}^{n} u^{ii} - \sum_{i<n} |u^{ni}|^2 \prod_{\substack{j\neq i \\ j<n}} u^{jj} \le \prod_{i=1}^{n} u^{ii}\,. \end{aligned} \tag{5.6}$$

So we get the inequality involving $\operatorname{tr} u^{ij}$ and $\det u^{ij}$ used above.

As $\det u^{ij}$ is a priori bounded from below by a positive constant in view of

$$\det u^{ij} = (\det u_{ij})^{-1} = \exp\left(-\hat{f} - \dot{u}\right),$$

we may choose μ so large that the first term in (5.4) is greater than $c+1$. For $\delta \le \frac{1}{c\mu} \min\left\{1, \frac{1}{3}\varepsilon\right\}$ we get

$$L\vartheta \ge \frac{1}{3}\varepsilon \operatorname{tr} u^{ij}$$

and furthermore $\vartheta \ge 0$ on $\partial\Omega_\delta$ if we choose δ smaller if necessary.

Let l be an affine linear function such that we have $\pm(\beta^i u_i - \varphi(x)) + l \ge 0$ for $t = 0$ in Ω_δ and $l(x_0) = 0$. For constants A, $B > 0$ consider the function

$$\Theta := A\vartheta + B|x - x_0|^2 \pm (\beta^i u_i - \varphi(x)) + l.$$

We fix $B \gg 1$, get $\Theta \ge 0$ on $\partial\Omega_\delta$, and deduce for $A \gg B$ that $L\Theta \ge 0$ as $\operatorname{tr} u^{ij}$ is bounded from below by a positive constant. The maximum principle yields $\Theta \ge 0$ in Ω_δ. As $\Theta(x_0) = 0$ we have $\Theta_\beta(x_0) \ge 0$ which in turn gives immediately $|u_{\beta\beta}| \le c$. □

5.2. **Remaining C^2-estimates.** We proceed similar as in [7]. Take $T > 0$ slightly smaller than the maximal time interval for which a solution exists. We define for $(x, \tau, t) \in \overline{\Omega} \times S^{n-1} \times [0, T]$ and for positive constants α, γ to be fixed later

$$w(x, \tau, t) := e^{\alpha(\psi - u) + \gamma\cdot|Du|^2} \cdot u_{\tau\tau}.$$

We assume that w attains its maximum over all points of $\partial\Omega$ and all tangential directions τ at a boundary point x_w, in a direction we may take to be e_1 and for $t_0 > 0$. We may choose Euclidean coordinates such that $(u_{ij})(x_w)$ is diagonal for $1 \le i$, $j \le n-1$, where $e_1, \ldots, e_{n-1}$ are tangential directions. We decompose e_1 similar as in (3.1)

$$e_1 = \tau(e_1) + \frac{\langle \nu, e_1\rangle}{\langle \beta, \nu\rangle}\beta,$$

where

$$\tau(e_1) = \tau = e_1 - \langle\nu, e_1\rangle\nu - \frac{\langle\nu, e_1\rangle}{\langle\beta, \nu\rangle}\beta^T, \quad \beta^T = \beta - \langle\beta, \nu\rangle\nu.$$

We differentiate the boundary condition

$$u_\beta = \varphi(x) \quad \text{on } \partial\Omega$$

and multiply the result with the tangential vector τ. Assume that all involved quantities defined on $\partial\Omega$ have been extended smoothly. On $\partial\Omega$ we obtain

$$\frac{2\langle\nu, e_1\rangle}{\langle\nu, \beta\rangle} u_{\beta\tau} = \frac{2\langle\nu, e_1\rangle}{\langle\nu, \beta\rangle}\left(\varphi_j\tau^j - \tau^j\beta^i_j u_i\right) =: \chi(x, Du). \tag{5.7}$$

We remark that χ as defined above is affine linear in Du. From the definitions of τ and χ we get

$$u_{11} = u_{\tau\tau} + \chi + \frac{\langle\nu, e_1\rangle^2}{\langle\beta, \nu\rangle^2} u_{\beta\beta}.$$

Since $\chi(x_w, Du) = 0$, the function

$$\tilde{w} := e^{\alpha(\psi-u)+\gamma|Du|^2}(u_{11} - \chi)$$

satisfies for $x \in \partial\Omega$

$$\begin{aligned}
\tilde{w}(x) &= e^{\alpha(\psi-u)+\gamma|Du|^2}(x)\left\{u_{\tau\tau}(x) + \frac{\langle\nu, e_1\rangle^2}{\langle\beta, \nu\rangle^2} u_{\beta\beta}(x)\right\} \\
&\le \left\{1 - \langle\nu, e_1\rangle^2\left(1 - \frac{\left|\beta^T\right|^2}{\langle\beta, \nu\rangle^2}\right) - \frac{2\langle\nu, e_1\rangle\left\langle\beta^T, e_1\right\rangle}{\langle\beta, \nu\rangle}\right\}\tilde{w}(x_w) \\
&\quad + c\langle\nu, e_1\rangle^2 e^{\alpha(\psi-u)+\gamma|Du|^2}(x) \\
&\le \left\{1 + c\langle\nu, e_1\rangle^2 - \frac{2\langle\nu, e_1\rangle\left\langle\beta^T, e_1\right\rangle}{\langle\beta, \nu\rangle} + c\frac{\langle\nu, e_1\rangle^2}{\max\limits_{\substack{\xi\in T_x\partial\Omega\\|\xi|=1}} u_{\xi\xi}(x)}\right\}\tilde{w}(x_w) \\
&\le \left\{1 + c_1\langle\nu, e_1\rangle^2 - \frac{2\langle\nu, e_1\rangle\left\langle\beta^T, e_1\right\rangle}{\langle\beta, \nu\rangle}\right\}\tilde{w}(x_w).
\end{aligned} \tag{5.8}$$

The proof of (5.8) is different to the proof in [7]. In the respective proof in the cited paper it was not clear to us why the constant could be chosen independent of α and γ. Moreover, for (5.8) we used

Lemma 5.3. *For a solution of our flow equation (1.1)*

$$\max_{\substack{\xi\in T_x\partial\Omega\\|\xi|=1}} u_{\xi\xi}(x)$$

is uniformly bounded from below by a positive constant.

Proof. We have already seen that there is a uniform positive lower bound for $\det D^2u$. In a fixed boundary point we may choose a coordinate system, such that e_n is equal to the inner unit normal of $\partial\Omega$ and $(u_{ij})_{i,j<n}$ is diagonal. Similar as in (5.6) we estimate

$$\det u_{ij} = \prod_{i=1}^{n} u_{ii} - \sum_{i<n} |u_{ni}|^2 \prod_{\substack{j\neq i\\ j<n}} u_{jj} \le \prod_{i=1}^{n} u_{ii} . \tag{5.9}$$

We decompose ν as

$$\nu = \frac{1}{\langle \beta, \nu\rangle} \left(\beta - \beta^T\right)$$

with β^T as above and get immediately in view of the Lemmata 5.1 and 5.2

$$u_{\nu\nu} \le \frac{1}{\langle \beta, \nu\rangle^2} \left(\left|\beta^T\right|^2 \cdot \max_{i<n} u_{ii} + c\right).$$

Finally, we combine the last statement with (5.9). The claim follows. □

We may assume that c_1 in (5.8) is chosen so large and β is so close to ν such that the expression in the last curly brackets in (5.8) is bounded below by $\frac{1}{2}$.

We use χ as above and define

$$W = \frac{e^{\alpha(\psi-u)+\gamma|Du|^2}(u_{11} - \chi)}{1 + c_1\langle \nu, e_1\rangle^2 - \frac{2\langle \nu, e_1\rangle\langle \beta^T, e_1\rangle}{\langle\beta,\nu\rangle}}$$

and consider this quantity in $\overline{\Omega} \times [0, T]$ for some positive time T. We may assume that W attains its maximum at a positive time at $x_W \in \overline{\Omega}$.

We first address to the case $x_W \in \partial\Omega$. Observe that $W(x_W) \le \tilde{w}(x_w) = W(x_w)$, where we evaluate at appropriate times. So we get $W_\beta \le 0$ at x_w, which implies that

$$u_{11\beta} + \alpha\delta_0 u_{11} \le c(1 + (1 + \gamma)u_{11}). \tag{5.10}$$

We differentiate the boundary condition $u_\beta = \varphi(x)$ as in Lemma 5.1, use the a priori estimates obtained so far, and the fact that D^2u restricted to tangential directions is diagonal

$$u_{\beta 11} \ge -c - 2\beta^1_1 u_{11} - 2\beta^n_1 u_{n1}. \tag{5.11}$$

For β C^1-close to ν we see that $\beta^1_1 \approx -\omega_{11} < 0$, $\beta^n_1 \approx 0$, so combining (5.10) and (5.11) yields

$$c(1 + (1 + \gamma)u_{11}) \ge \alpha\delta_0 u_{11} - c.$$

So we see that for $\alpha = \alpha(\gamma)$ sufficiently large we get an upper bound for $u_{11}(x_w)$. This completes the C^2-estimates, when W attains its maximum on $\partial\Omega$.

Now we consider the case when W attains its maximum at $x_W \in \Omega$. We use

$$\Gamma = -\log\left(1 + c_1\langle\nu, e_1\rangle^2 - \frac{2\langle\nu, e_1\rangle\left\langle\beta^T, e_1\right\rangle}{\langle\beta, \nu\rangle}\right)$$

in the following calculations and remark that Γ is well-defined as the argument of the logarithm is bounded below by a positive constant and its $C^2\left(\overline{\Omega}\right)$-norm is uniformly bounded independent of α and γ. We use that

$$\log W = \alpha \cdot (\psi - u) + \gamma \cdot |Du|^2 + \log(u_{11} - \chi) + \Gamma$$

attains its maximum at x_W. Of course we may assume that $(u_{11} - \chi)(x_W) \geq 1$. At the point x_W we get

$$\begin{aligned}
0 = \frac{W_i}{W} &= \alpha(\psi - u)_i + 2\gamma u^k u_{ki} + \frac{u_{11i} - D_i\chi}{u_{11} - \chi} + \Gamma_i, \\
0 &\geq \frac{W_{ij}}{W} - \frac{W_i W_j}{W^2} \\
&= \alpha(\psi - u)_{ij} + 2\gamma u^k_j u_{ki} + 2\gamma u^k u_{kij} \\
&\quad + \frac{u_{11ij} - D_{ij}\chi}{u_{11} - \chi} - \frac{(u_{11i} - D_i\chi)(u_{11j} - D_j\chi)}{(u_{11} - \chi)^2} + \Gamma_{ij}, \\
0 \leq \dot{W} &= \alpha\left(\dot{\psi} - \dot{u}\right) + 2\gamma u^k \dot{u}_k + \frac{\dot{u}_{11} - \frac{d}{dt}\chi}{u_{11} - \chi}
\end{aligned}$$

where we have used that Γ is time-independent. The second line has to be interpreted in the matrix sense. The notations $D.$ and $\frac{d}{dt}$ indicate that the chain rule has not yet been applied. The remaining calculations refer to the point, where W attains its maximum. We get there

$$0 \geq u^{ij}(\log W)_{ij} - \dot{W}.$$

Using estimates for the time derivatives of ψ and u, the strict convexity of ψ (5.2), the fact that $\Gamma \in C^2$ with uniform bounds and the up to twice differentiated flow equation (1.1) yields

$$\begin{aligned}
0 \geq 2\gamma\Delta u &+ \frac{1}{u_{11} - \chi}\left(u^{ir}u^{js}u_{ij1}u_{rs1}\right) - u^{ij}\frac{(u_{11i} - D_i\chi)(u_{11j} - D_j\chi)}{(u_{11} - \chi)^2} \\
&+ \frac{1}{u_{11} - \chi}\left(\hat{f}_{p_i}u_{i11} - c - c \cdot \left|D^2u\right|^2\right) + 2\gamma u^k \hat{f}_{p_i}u_{ik} \\
&+ \frac{1}{u_{11} - \chi}\left(\frac{d}{dt}\chi - u^{ij}D_{ij}\chi\right) - c(\alpha + \gamma) + (\alpha\lambda - c)\operatorname{tr} u^{ij}.
\end{aligned} \tag{5.12}$$

Direct calculations and (1.1) differentiated once yield

$$\frac{d}{dt}\chi - u^{ij}D_{ij}\chi \geq -c\left(1 + \left|D^2u\right| + \operatorname{tr} u^{ij}\right)$$

and $\frac{W_i}{W} = 0$ implies

$$2\gamma u^k \hat{f}_{p_i} u_{ik} + \frac{\hat{f}_{p_i} u_{11i}}{u_{11} - \chi} \ge -c \cdot \alpha - \frac{c \cdot \left(1 + \left|D^2 u\right|\right)}{u_{11} - \chi} - c.$$

These estimates are used to simplify (5.12). For a small constant $\vartheta > 0$ to be fixed later on we may assume that

$$(1 - \vartheta) u_{\eta\eta}(x_W) \equiv (1 - \vartheta) \max_{|\xi|=1} u_{\xi\xi}(x_W) \le (u_{11} - \chi)(x_W),$$

where we have defined a direction η, $|\eta| = 1$, implicitly that corresponds to a maximal eigenvalue. Schwarz's inequality gives

$$u^{ij}(u_{11i} - D_i\chi)(u_{11j} - D_j\chi) \le (1 + \vartheta) u^{ij} u_{11i} u_{11j} + \frac{c}{\vartheta} u^{ij} D_i\chi D_j\chi.$$

From the definition of η we get

$$\begin{aligned} u^{ir} u^{js} u_{ij1} u_{rs1} &\ge \frac{\max\limits_{|\xi|=1} u^{js} u_{\xi j1} u_{\xi s1}}{u_{\eta\eta}} \\ &\ge \frac{1 - \vartheta}{u_{11} - \chi} u^{ij} u_{11i} u_{11j}. \end{aligned}$$

We assume that $\vartheta \le \frac{1}{2}$, use $\frac{W_i}{W} = 0$ and get using the inequalities above

$$\begin{aligned} &u^{ir} u^{js} u_{ij1} u_{rs1} - u^{ij} \frac{1}{u_{11} - \chi} (u_{11i} - D_i\chi)(u_{11j} - D_j\chi) \\ &\ge u^{ir} u^{js} u_{ij1} u_{rs1} - (1 + \vartheta) \frac{1}{u_{11} - \chi} u^{ij} u_{11i} u_{11j} - \frac{c}{\vartheta(1 - \vartheta)} \frac{1}{u_{\eta\eta}} u^{ij} D_i\chi D_j\chi \\ &\ge -\vartheta \frac{2}{u_{11} - \chi} u^{ij} u_{11i} u_{11j} - \frac{2}{\vartheta} \frac{c}{u_{\eta\eta}} \left(1 + \operatorname{tr} u^{ij} + \left|D^2 u\right|\right) \\ &\ge -c\vartheta(u_{11} - \chi) \left(\operatorname{tr} u^{ij} + \alpha^2 \operatorname{tr} u^{ij} + \gamma^2 \left|D^2 u\right|\right) - \frac{1}{\vartheta} \frac{c}{u_{\eta\eta}} \left(1 + \operatorname{tr} u^{ij} + \left|D^2 u\right|\right). \end{aligned}$$

Combining this with (5.12) gives

$$\begin{aligned} 0 \ge &\left(2\gamma - c\,\vartheta\gamma^2 - \frac{c}{\vartheta(\Delta u)^2} - c\right) \Delta u - c(1 + \alpha + \gamma) \\ &+ \left(\alpha\lambda - c - c\,\vartheta\alpha^2 - \frac{c}{\vartheta(\Delta u)^2} - \frac{c}{\Delta u}\right) \operatorname{tr} u^{ij} \end{aligned}$$

Fixing γ, $\alpha = \alpha(\gamma)$ sufficiently large and finally $\vartheta = \vartheta(\gamma, \alpha)$ sufficiently small gives an upper bound for u_{11}. We remark that the constants have been chosen in the same order as above.

It remains to consider the case when

$$(1 - \vartheta) u_{\eta\eta}(x_W) \ge (u_{11} - \chi)(x_W).$$

We may restrict to the non-trivial situation, i. e. we may assume that

$$\left(1 - \frac{\vartheta}{2}\right) u_{\eta\eta}(x_W) \ge u_{11}(x_W).$$

We define

$$\overline{\Omega}\times S^{n-1}\times[0,T]\ni(x,\xi,t)\mapsto\tilde{W}(x,\xi,t)=\frac{e^{\alpha(\psi-u)+\gamma|Du|^2}(u_{\xi\xi}-\chi)}{1-c_1\langle\nu,e_1\rangle^2-\frac{2\langle\nu,e_1\rangle\langle\beta^T,e_1\rangle}{\langle\beta,\nu\rangle}}.$$

Here we use χ as introduced in (5.7). Assume that $\tilde{W}$ attains its maximum at a positive time at $x_{\tilde{W}}$ for a direction $\xi\in S^{n-1}$. We assume first, that $x_{\tilde{W}}\in\partial\Omega$. Using a decomposition of ξ as in (3.1) we obtain for β C^0-close to ν

$$|\tau(\xi)|^2\le 1+c\left\|\beta^T\right\|_{C^0}.$$

As a direct consequence of this decomposition it is easy to see that

$$u_{\xi\xi}\le u_{\tau(\xi)\tau(\xi)}+c.$$

In the next estimate we use the choice of x_w, x_W above (5.10) and remark that the constant c used here may now (in contrast to the inequalities above) also depend on α and γ. We evaluate at a time that corresponds to the choice of $x_\cdot$.

$$\begin{aligned}\tilde{W}(x_{\tilde{W}},\xi)&\le|\tau(\xi)|^2W(x_w)+c\\&\le\left(1+c\left\|\beta^T\right\|_{C^0}\right)W(x_W)+c\\&\le\left(1-\frac{\vartheta}{2}\right)\left(1+c\left\|\beta^T\right\|_{C^0}\right)\tilde{W}(x_W,\eta)+c\\&\le\left(1-\frac{\vartheta}{2}\right)\left(1+c\left\|\beta^T\right\|_{C^0}\right)\tilde{W}(x_{\tilde{W}},\xi)+c.\end{aligned}$$

For β^T sufficiently small, this is only possible if $\tilde{W}(x_{\tilde{W}},\xi)$ is a priori bounded. Similar to the above proof for the interior maximum of W it is possible to obtain a bound for the second derivatives, if $\tilde{W}$ attains its maximum at an interior point $x_{\tilde{W}}\in\Omega$.

6. Longtime existence and convergence

So far we have obtained uniform $\dot{u}$, Du and D^2u-estimates as long as our solution exists. Furthermore, we know that there is a positive constant c such that

$$-c+v^\infty\cdot t\le u\le +c+v^\infty\cdot t,\tag{6.1}$$

where v^∞ is the velocity of a translating solution. This is obtained by enclosing our initial values by translating solutions. These translating solutions will then enclose u as long as our solution u exists due to the maximum principle. We can apply Hölder estimates for the second derivatives due to Evans, Krylov and Safonov as well as Schauder estimates, see [3]. As our problem is invariant in "z-direction", we get longtime existence with uniform bounds for all derivatives of u.

To show convergence to a translating solution we consider $w := u - u^\infty$, similar to [1, 5]. Using the mean value theorem we see that w satisfies a parabolic flow equation of the form

$$\begin{cases} \dot{w} = a^{ij} w_{ij} + b^i w_i & \text{in } \Omega, \\ w_\beta = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus the strong maximum principle implies that w is constant or its oscillation is strictly decreasing in time. In the first case u is already a translating solution. In the second case we wish to exclude that the oscillation is strictly decreasing but does not tend to zero. In the case when the oscillation of $w(\cdot, t)$ tends to $\varepsilon > 0$ as $t \to \infty$, we consider for $t_n \to \infty$

$$u^n(x,t) := u(x, t + t_n) - v^\infty \cdot t_n.$$

Due to the uniform estimates in any C^k-norm for the derivatives of u^n, and the locally (in time) uniform bounds for the C^0-norm, see (6.1), a subsequence of the functions u^n converges locally (in time) uniformly in any C^k-norm to a solution u^* of our flow equation (1.1) that exists for all time. Due to the monotonicity of the oscillation we see that the oscillation of $u^* - u^\infty$ is equal to ε, independent of t. This is a contradiction as the strong maximum principle implies that a positive oscillation is strictly decreasing. As the oscillation of w tends to zero, we see that u tends to a translating solution as $t \to \infty$ in C^0. Adding a constant to the translating solution u^∞ we may assume that $u \to u^\infty$ as $t \to \infty$ in the C^0-norm. Smooth convergence is obtained by using interpolation inequalities of the form

$$\|Dw\|^2_{C^0(\overline{\Omega})} \le c(\Omega) \cdot \|w\|_{C^0(\overline{\Omega})} \cdot \left(\left\| D^2 w \right\|_{C^0(\overline{\Omega})} + \|Dw\|_{C^0(\overline{\Omega})} \right)$$

for $w := u - u^\infty$. This completes the proof of Theorem 1.1.

Appendix A. Prescribing Gauss curvature for entire graphs

Here we present an application of our previous existence result on bounded domains to construct unbounded hypersurfaces with prescribed Gauß curvature. Assuming that the hypersurface is given as an entire graph the problem is then to find a solution to

$$\frac{\det D^2 u}{\left(1 + |Du|^2\right)^{\frac{n+2}{2}}} = g(x). \tag{A.1}$$

Observe that this equation fits in the context of the present paper, cf. (4.1)

$$v^\infty = \log\det D^2 u - \log f(x, Du),$$

just by defining

$$f(x,p) = g(x)/h(p) \quad \text{with} \quad h(p) = h(|p|) = \left(1 + |p|^2\right)^{-\frac{n+2}{2}}$$

and looking for translating solutions with speed $v^\infty = 0$.

Following an argument of Altschuler, Wu [1] we will construct entire rotationally symmetric translating solutions from solutions on growing disk-type domains. Using the graph of the lower half sphere for various radii it is a direct consequence of the strong maximum principle that there cannot exist an entire strictly convex translating solution for any constant value of the Gauß curvature. In the rotationally symmetric case we have the following result.

Theorem A.1. *Let $g(x) = g(|x|)$ be positive, smooth and integrable. There is an entire strictly convex solution u to (A.1) if and only if*

$$v := \log \frac{\int_{\mathbb{R}^n} h(|p|)dp}{\int_{\mathbb{R}^n} g(|x|)dx} \geq 0.$$

The solution constructed here has a uniformly bounded gradient if and only if $v > 0$.

We remark that the rotationally symmetric setting does also allow for a proof by reducing the problem to an ordinary differential equation.

Proof. We will use that for given constants $R > 0$, $\rho > 0$ there is a unique, strictly convex solution (v^∞, u^∞) to the following problem

$$\begin{cases} e^{v^\infty} = \det D^2 u^\infty \frac{h(|Du^\infty|)}{g(|x|)} & \text{in } B_R(0), \\ u^\infty_\nu = -\rho & \text{on } \partial B_R(0). \\ u^\infty(0) = 0 & \end{cases} \qquad (*_{R,\rho})$$

This is a direct application of our elliptic result in Section 4. Furthermore, the uniqueness of the solution also implies its rotational symmetry. The solution could also be found by imposing a second boundary value problem.

From the strict convexity we deduce that Du^∞ is an isomorphism from $B_R(0)$ onto $B_\rho(0)$. Integrating $(*_{R,\rho})$ we obtain

$$e^{v^\infty} \int_{B_R} g(|x|)dx = \int_{B_R} \det D^2 u^\infty h(|Du^\infty|)dx = \int_{B_\rho} h(|p|)dp,$$

which determines the speed

$$v^\infty = v^\infty(R, \rho) := \log \frac{\int_{B_\rho} h(|p|)dp}{\int_{B_R} g(|x|)dx}$$

uniquely as a function of the parameters R, ρ. Note that $v^\infty(R, \rho)$ is strictly decreasing in R and strictly increasing in ρ with $v^\infty(R, \rho) \to -\infty$ for $\rho \to 0$.

1. Nonexistence for $v < 0$: We argue similarly as in the afore mentioned case of constant Gauß curvature. But here we replace lower half spheres by suitably constructed solutions on finite domains with arbitrarily large gradients at the boundary. Since $v < 0$ there exists a unique $\hat{R}$ such that $\int_{B_{\hat{R}}} g(|x|)dx = \int_{\mathbb{R}^n} h(|p|)dp$. Assuming that we have an entire solution u of

(A.1), there is a $\rho > 0$ such that $|Du(x)| < \rho/2$ for all $|x| < \hat{R}$. Now take the unique $R < \hat{R}$ satisfying

$$\int_{B_R} g(|x|)dx = \int_{B_\rho} h(|p|)dp$$

and consider the solution (v^∞, u^∞) of $(*_{R,\rho})$. By definition we know that $v^\infty = v^\infty(R, \rho) = 0$, hence, u^∞ solves equation (A.1) in $B_R(0)$. Furthermore, the Neumann boundary condition and our choice of ρ yield $|Du^\infty| > |Du|$ in a neighborhood of $\partial B_R(0)$. Thus, there is a translate $u^\infty + m$, $m \in \mathbb{R}$, which is strictly greater than u. Now we shift back until the graphs touch first at a point $x \in B_R(0)$. By the strong maximum principle this is impossible as u and u^∞ solve the same elliptic equation in $B_R(0)$.

2. Existence for $v \geq 0$: We construct our solution by choosing a sequence of increasing radii R_k tending to ∞. By the monotonicity properties of the function $v^\infty(R, \rho)$ we can find for each $R > 0$ a unique ρ_R such that $v^\infty(R, \rho_R) = 0$. We remark that ρ_R is an increasing sequence. Again the result from Section 4 gives us a unique, smooth, rotationally symmetric solution u_R to $(*_{R,\rho_R})$, which is defined on $B_R(0)$ and satisfies $v^\infty = 0$. Note that for fixed R the speed v^∞ and the normal derivative at the boundary $-\rho_R$ are uniquely related. Hence the solution u_R must coincide on smaller balls $B_{R'}(0)$, $R' < R$, with the previous solutions $u_{R'}$ to $(*_{R',\rho_{R'}})$. Therefore, as R tends to ∞, the sequence $\{u_R\}$ will converge uniformly on compact sets to a limit u defined on all of $\mathbb{R}^n$. Clearly, u is a rotationally symmetric solution to (A.1). Observe that the sequence ρ_R will diverge in the case $v = 0$, whereas it stays bounded for $v > 0$. This proves the boundedness of $|Du|$ in the latter case. □

Proceeding as in the existence part of the proof we get easily that non-integrable $g(x)$ also allow for solutions provided that $h(p)$ is non-integrable too.

This observation can be extended to the function $h(p)$ arising in the equation of prescribed Gauß curvature in Minkowski space

$$\frac{\det D^2 u}{\left(1 - |Du|^2\right)^{\frac{n+2}{2}}} = g(x). \tag{A.2}$$

Hence, $h(p) = h(|p|) = \left(1 - |p|^2\right)^{-\frac{n+2}{2}}$ and h is not integrable on $B_1(0)$.

Theorem A.2. *For all positive and smooth functions $g(x) = g(|x|)$ there is an entire strictly convex solution u to (A.2). The gradient of a solution satisfies $|Du| < 1$. Moreover, for a solution constructed here, $|Du| \leq 1 - \varepsilon$, $\varepsilon > 0$, if and only if $\int_{\mathbb{R}^n} g < \infty$.*

Proof. We proceed similarly as in the proof of Theorem A.1 and use notations introduced there. Here, the speed function $v^\infty(R, \rho)$ is only defined for

$\rho < 1$. But still $v^\infty(R,\rho)$ is strictly decreasing in R and strictly increasing in ρ and in addition $v^\infty(R,\rho) \to -\infty$ for $\rho \to 0$ and $v^\infty(R,\rho) \to \infty$ for $\rho \to 1$. As in part 2 of the proof of Theorem A.1 we can find for any $R > 0$ a unique ρ_R (with $0 < \rho_R < 1$) satisfying $v^\infty(R,\rho_R) = 0$. Since $\rho_R < 1$ we can choose a smooth function $\tilde{h}(p)$ defined on $\mathbb{R}^n$ such that $h(p) = \tilde{h}(p)$ for all $|p| \le \rho_R$. Again the result from Section 4 gives us a unique, smooth, convex rotationally symmetric solution u_R to $(*_{R,\rho_R})$ with h replaced by $\tilde{h}$, which is defined on $B_R(0)$ and satisfies $v^\infty = 0$. We remark that the convexity of u implies that $|Du| \le \rho_R$ on $B_R(0)$. Thus, u is also a solution for the original function h instead of $\tilde{h}$ in the elliptic equation. Again, for $R > R'$, u_R will coincide with solutions $u_{R'}$ obtained on smaller balls $B_{R'}(0)$. Hence, for R tending to infinity, u_R converges to an entire solution u of (A.2). In the present case the sequence ρ_R stays uniformly bounded away from 1 if $\int_{\mathbb{R}^n} g(|x|)dx < \infty$, whereas ρ_R converges to 1 if g is non-integrable. □

In the general case without rotational symmetry, a theorem corresponding to Theorem 1.1 in Minkowski space can be obtained easily from the techniques of this paper, provided that there holds a uniform a priori bound of the form $|Du| < 1 - \varepsilon$, $\varepsilon > 0$.

Appendix B. Illustrations

The pictures included here show the velocity of a solution to a discrete version to the flow equation

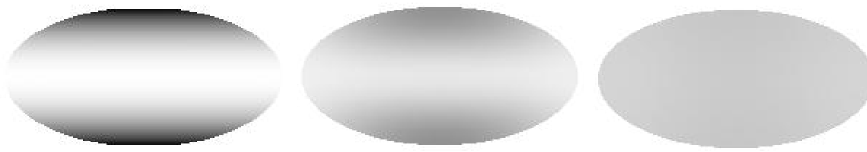
$$\begin{cases} \dot{u} = \log\det D^2 u & \text{in } \Omega \times [0,\infty), \\ \langle Du, \nu\rangle = \langle Du_0, \nu\rangle & \text{on } \Omega \times [0,\infty), \\ u(\cdot,0) = u_0 & \text{on } \Omega, \end{cases}$$

where $\Omega = \left\{(x,y) \in \mathbb{R}^2 : 1.1 \cdot \left(x^2 + (2y)^2\right) < 1\right\}$ and $u_0(x,y) = 1.5x^2 + y^2 - 0.1y^4$. The numerical integration has been carried out on a 200×100 grid corresponding to $[-1,1] \times [-0.5, 0.5] \in \mathbb{R}^2$. Let Ω_0 consist of those points on the grid that belong to Ω and $\partial\Omega_0$ be those points on the grid such that the distance to Ω_0 (with respect to the sup-norm) is exactly the minimal distance of two points on the grid. For simplicity, we do not introduce new notations for the discretized quantities.

In each time step we compute D^2u in Ω_0. Thus, we obtain $\dot{u}$ and set

$$u(x, t + \Delta t) = u(x,t) + \dot{u}(x,t) \cdot \Delta t.$$

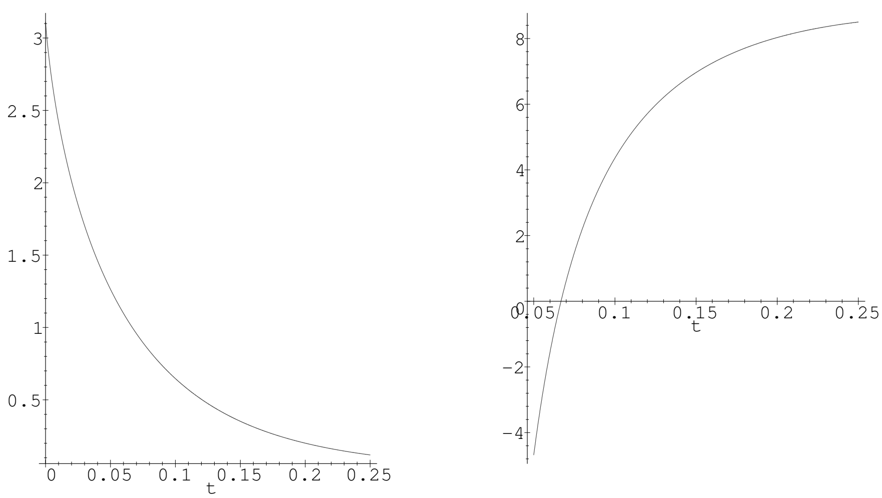
Then we make the corrections due to the boundary condition: For all $x_0 \in \partial\Omega$ let $y_0 := x_0 + \nu(x_0) \cdot \tau_0$, where $\nu(x_0)$ is the normalized negative gradient of $x^2 + 2y^2$ and $\tau_0 = \inf\{\tau : x_0 + \nu(x_0) \cdot \tau \in \text{convex hull}(\Omega_0)\}$. We set $u(x_0) := u_0(x_0) - u_0(y_0) - u(y_0)$. Here $u(y_0)$ is obtained by linear interpolation.



(a) $t = 0.0$ (b) $t = 0.1$ (c) $t = 0.5$

FIGURE 2. Time evolution on an elliptic domain

Figure 2 shows a grayscale plot of the velocity at different times. It can be seen that the velocity tends to a constant, which reflects the property that u tends to a translating solution.



(a) $\delta(t)$ (b) $\frac{-1}{t} \log \delta(t)$

FIGURE 3. Convergence to constant velocity

To illustrate the rate of convergence we show in Figure 3(a) the decay of $\delta(t) = ||v(t) - \bar{v}(t)||^2_{L^2(\Omega)}$, where $\bar{v}(t) = \frac{1}{|\Omega|} \int v(x,t)\, dx$ is the mean velocity. The expected exponential convergence can be seen from Figure 3(b). Here we plot $\frac{-1}{t} \log \delta(t)$, which saturates nicely for larger times.

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