HYPERSURFACES OF PRESCRIBED GAUSS CURVATURE IN EXTERIOR DOMAINS

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ABSTRACT. We prove an existence theorem for convex hypersurfaces of prescribed Gauß curvature in the complement of a compact set in Euclidean space which are close to a cone.

1. INTRODUCTION

Hypersurfaces of prescribed curvature have been subject to intensive studies. To mention a few examples, compact convex hypersurfaces are considered in the closed case in [2], for Neumann boundary conditions in [7], and for Dirichlet boundary conditions in [4, 5, 8, 10], based on the method of [12]. In the non-compact case, complete convex hypersurfaces are found in [1, 9], whereas [6, 13] deal with equations of mean curvature type in exterior domains. In this paper, we consider the Dirichlet problem for convex hypersurfaces of prescribed Gauß curvature in exterior domains. More precisely, let $\emptyset \neq K \subset \mathbb{R}^n$, $n \geq 2$, be a compact set whose boundary ∂K is a smooth submanifold of $\mathbb{R}^n$. Then for $0 < f \in C^\infty\left(\left(\mathbb{R}^n \setminus \mathring{K}\right) \times \mathbb{R}\right)$ and $u_0 \in C^\infty(\partial K)$ we consider strictly convex hypersurfaces of prescribed Gauss curvature represented as graphs over $\mathbb{R}^n \setminus \mathring{K}$, i. e. functions $u \in C^\infty\left(\mathbb{R}^n \setminus \mathring{K}\right)$ such that

$$\left\{\begin{aligned} \mathcal{K}[u] \equiv \frac{\det D^2 u}{\left(1+|Du|^2\right)^{\frac{n+2}{2}}} &= f(x,u) \quad \text{in } \mathbb{R}^n \setminus K, \\ u &= u_0 \qquad\quad \text{on } \partial K\,. \end{aligned}\right. \tag{1.1}$$

The hypersurfaces should be *close to a cone* in the sense that

$$\sup |u - r| < \infty \tag{1.2}$$

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with $u = u(x)$ and $r \equiv |x|$. A function $\underline{u} \in C^\infty\left(\mathbb{R}^n \setminus \overset{\circ}{K}\right)$ is called a *subsolution* if $\underline{u}$ is strictly convex and

$$\begin{cases} \mathcal{K}[\underline{u}] \geq f(x, \underline{u}) & \text{in } \mathbb{R}^n \setminus K, \\ \underline{u} = u_0 & \text{on } \partial K . \end{cases} \tag{1.3}$$

We prove the following main theorem.

Theorem 1.1. *Let $u_0 \in C^\infty(\partial K)$ and $0 < f \in C^\infty\left(\left(\mathbb{R}^n \setminus \overset{\circ}{K}\right) \times \mathbb{R}\right)$ with*

$$f_z \geq 0, \quad \sup\left(f \cdot r^{n+1}\right) + \sup\left(\frac{|Df| + |D^2 f|}{f}\right) < \infty . \tag{1.4}$$

Suppose that $\underline{u} \in C^\infty\left(\mathbb{R}^n \setminus \overset{\circ}{K}\right)$ is a subsolution (1.3) which is close to a cone and satisfies the following decay conditions at infinity,

$$|D(\underline{u} - r)| = \mathcal{O}\left(\frac{1}{r}\right) , \qquad |D^2(\underline{u} - r)| + |D^3 \underline{u}| = \mathcal{O}\left(\frac{1}{r^2}\right) . \tag{1.5}$$

Then there exists a smooth, strictly convex hypersurface of prescribed Gauß curvature (1.1) which is close to a cone (1.2).

Note that $u - r$ has no prescribed values at infinity, because the property that u is close to a cone determines the asymptotics only up to a bounded function. In particular, the function $|u - \underline{u}|$ will in general not become small near infinity. Due to these weak assumptions on the asymptotics, we cannot expect uniqueness of our solutions.

We remark that the upper bound for f in (1.4) is natural because the decay conditions for $\underline{u}$, (1.5), imply that $f(x, \underline{u}) = \mathcal{O}(r^{-n-1})$. Namely, if ν and τ, τ' denote unit vectors in radial and axial directions, respectively, the derivatives of $\underline{u}$ decay at infinity like $|D_{\nu\nu}\underline{u}|, |D_{\nu\tau}\underline{u}| = \mathcal{O}(r^{-2})$, $|D_{\tau\tau'}\underline{u}| = \mathcal{O}(r^{-1})$. Thus $\det D^2\underline{u} = \mathcal{O}(r^{-n-1})$, and the differential inequality (1.3) yields the claimed decay rate for $f(x, \underline{u})$.

The above theorem has the disadvantage that it involves the existence of a strictly convex subsolution. Therefore, we give methods to find subsolutions. In the special case when $K = \overline{B_{\rho_1}} \equiv \overline{B_{\rho_1}(0)}$ is a ball, the boundary values are zero, and f decays even faster than in (1.4), we construct an explicit subsolution to get the following result.

Theorem 1.2. *Let $\rho_1 > 0$ and $a > 2$. If $f \in C^\infty((\mathbb{R}^n \setminus B_{\rho_1}) \times \mathbb{R})$ such that $f_z \geq 0$,*

$$\sup\left(\frac{|Df| + |D^2 f|}{f}\right) < \infty , \qquad 0 < f \leq (a-1)\, 2^{-\frac{3n}{2}-1}\, \rho_1^{a-1}\, r^{1-n-a} ,$$

then there exists a smooth, strictly convex hypersurface of prescribed Gauß curvature,

$$\begin{cases} \mathcal{K}[u] &= f(x,u) & \text{in } \mathbb{R}^n \setminus B_{\rho_1}, \\ u &= 0 & \text{on } \partial B_{\rho_1}\,, \end{cases}$$

which is close to a cone.

A more general method for constructing a subsolution is to combine a subsolution in a bounded set with a subsolution near infinity, in the spirit of viscosity solutions.

Theorem 1.3. *Let* $0 < f \in C^\infty\left(\left(\mathbb{R}^n \setminus \overset{\circ}{K}\right) \times \mathbb{R}\right)$*. Assume that* $\mathbb{R}^n \setminus K = \Omega_1 \cup \Omega_2$*, where* Ω_1 *is a bounded open set,* $\partial K \subset \partial\Omega_1$*, and* Ω_2 *an unbounded open set. Suppose that* u_0 *is a strictly convex subsolution in* Ω_1*,*

$$\mathcal{K}[u_0] \;\geq\; f(x,u_0) \qquad \text{in } \Omega_1,$$

and that $\underline{u}$ *is a strictly convex subsolution in* Ω_2*, both smooth up to the boundary, without prescribed boundary values. Furthermore, we assume that*

$$\begin{aligned} u_0 &< \underline{u} && \text{on } (\mathbb{R}^n \setminus K) \cap \partial\Omega_1, \\ \underline{u} &< u_0 && \text{on } \partial\Omega_2\,, \end{aligned}$$

that $\underline{u}$ *is close to a cone, and (1.4), (1.5) are satisfied. Then there exists a smooth, strictly convex hypersurface of prescribed Gauß curvature (1.1) which is close to a cone.*

This theorem can be applied to the situation when we have a subsolution with correct boundary values u_0 near ∂K and a subsolution $\underline{u}$ near infinity, which can be glued together as made precise above. For $\underline{u}$ one can for example take the subsolution used in the proof of Theorem 1.2, see Lemma 5.1.

The proof of Theorem 1.1 is outlined as follows. It is well known from [4] (see also [10]) that the Dirichlet problem for the prescribed Gauß curvature equation has a strictly convex solution on smooth compact domains for smooth data, provided that for the same boundary values there exists a smooth, strictly convex subsolution $\underline{u}$. Thus we choose R_0 such that $K \subset B_{R_0}$ and consider for any $R > 4R_0$ the Dirichlet problem

$$\begin{cases} \mathcal{K}\left[u^R\right] &= f\left(x,u^R\right) & \text{in } B_R \setminus K, \\ u^R &= \underline{u} & \text{on } \partial B_R \cup \partial K. \end{cases} \tag{1.6}$$

Our main task is to show that the C^2-norms of u^R are uniformly bounded in R. Namely, once this is established, standard Krylov/Shafanov and Schauder regularity theory yields locally uniform bounds for u^R in any C^k-norm [11]. Using a diagonal sequence argument, we get a subsequence u^{R_l}, $R_l \to \infty$, that converges locally smoothly to a strictly convex solution of our original problem. To obtain the required C^2-estimates, we first use barriers to uniformly bound $u^R - |x|$. These C^0-estimates and the convexity of the hypersurfaces allow us to control the asymptotic behavior as $R \to \infty$ of the first derivatives of $u^R(x)$ in the region $R/2 \le |x| \le R$. Using the inner maximum principle for the largest eigenvalue of u_{ij} as well as standard a priori estimates for D^2u^R on ∂K, the C^2-estimates are reduced to controlling D^2u^R on the outer boundary. We bound the tangential second derivatives from above and below by differentiating the Dirichlet boundary conditions and using the C^1-estimates. By adapting the barrier constructions of [4, 10] with constants that have an appropriate scaling in R, we obtain estimates for the mixed second derivatives. In contrast to the compact case [4, 10], the estimates mentioned so far are so strong that when putting them into the equation, we immediately get estimates for the second derivatives in normal direction.

Theorem 1.2 follows immediately from Theorem 1.1 by choosing an explicit subsolution $\underline{u}$, given in Lemma 5.1. To prove Theorem 1.3, we apply Lemma 5.2 to get the solution to the compact Dirichlet problem (1.6) and then proceed exactly as in the proof of Theorem 1.1.

2. Preliminary a priori Estimates

Notation 2.1. In what follows, lower indices denote partial derivatives in $\mathbb{R}^n$. We denote the partial derivative of f with respect to the second argument by f_z. We set $(u^{ij}) = (u_{ij})^{-1}$. All other indices are raised with respect to the Euclidean metric. Unless otherwise stated, we will use the Einstein summation convention. By c we denote a constant independent of R which may change its value from line to line throughout the text.

Without loss of generality we can assume that $0 \in \overset{\circ}{K}$. We choose a constant $L > \max\limits_{\partial K}(u_0 - |x|)$ and set

$$\overline{u} \;=\; |x| + L\,. \tag{2.1}$$

The functions $\overline{u}$ and $\underline{u}$ (as in Theorem 1.1) will be used as upper and lower barriers, respectively. Let u^R be a solution of the Dirichlet problem (1.6).

Lemma 2.2. *As $R \to \infty$, the functions u^R converge locally uniformly to a continuous function u. Moreover, $\underline{u} \le u^R \le \overline{u}$ in $B_R \setminus \overset{\circ}{K}$.*

Proof. From the maximum principle we deduce that

$$\underline{u} \le u^R \le \overline{u} \quad \text{in } B_R \setminus K.$$

Hence for $R_1 < R_2$,

$$u^{R_1} \le u^{R_2} \quad \text{on } \partial B_{R_1} \cup \partial K,$$

and again from the maximum principle,

$$u^{R_1} \le u^{R_2} \quad \text{in } B_{R_1} \setminus K.$$

We conclude that the u^R are monotone in R. Their pointwise limit is convex and thus continuous. So they converge locally uniformly according to Dini's theorem. $\square$

From now on we omit the index R and assume that u is a solution of (1.6) with R fixed sufficiently large.

Lemma 2.3. *$|\nabla u|$ is a priori bounded, uniformly in R.*

Proof. Since u is strictly convex, $|\nabla u|$ attains its maximum at the boundary. Tangential derivatives are uniformly bounded there in view of the Dirichlet boundary conditions. The normal derivatives are estimated as follows. Let $x \in \partial K$ and ν the outer unit normal to ∂K at x. We choose $\lambda_0 > 0$ independent of R such that the line segment $\{x + \lambda\nu,\ 0 \le \lambda \le \lambda_0\}$ is contained in $B_{R_0} \setminus K$. Using the convexity of u as well as the fact that $\underline{u}$ lies below u and $\underline{u}(x) = u(x)$,

$$\nabla_\nu \underline{u}(x) \;\le\; \nabla_\nu u(x) \;\le\; \frac{u(x+\lambda_0\nu) - u(x)}{\lambda_0} \qquad (\text{for } x \in \partial K)$$

($\nabla_\nu u$ denotes the directional derivative). For $x \in \partial B_R$ and $\nu = x/|x|$, we consider similarly the line segment $\{\lambda\nu,\ R_0 \le \lambda \le R\}$ and obtain

$$\frac{u(x) - u(R_0\nu)}{R - R_0} \;\le\; \nabla_\nu u(x) \;\le\; \nabla_\nu \underline{u}(x) \qquad (\text{for } x \in \partial B_R).$$

We finally use the uniform C^0 bounds of Lemma 2.2, in particular that $|u(x) - |x|| \le c$. $\square$

The next lemma controls the asymptotic behavior of ∇u as R gets large.

Lemma 2.4. *For $\frac{R}{2} \le |x| \le R$ let $\nu = x/|x|$ and τ be unit vectors parallel and orthogonal to x, respectively. Then there is a constant c independent of R such that*

$$|\nabla_\nu (u - \underline{u})(x)| \;\le\; \frac{c}{R} \tag{2.2}$$

$$|\nabla_\tau u(x)| \;\le\; \frac{c}{\sqrt{R}}\,. \tag{2.3}$$

Proof. Since u is convex and lies above $\underline{u}$ with $u(R\nu) = \underline{u}(R\nu)$,

$$\frac{u(x) - u(R_0\nu)}{|x| - R_0} \;\le\; \nabla_\nu u(x) \;\le\; \nabla_\nu \underline{u}(R\nu)\,,$$

and thus

$$\frac{u(x) - |x|}{|x| - R_0} - \frac{u(R_0\nu) - R_0}{|x| - R_0} \;\le\; \nabla_\nu u(x) - 1 \;\le\; \nabla_\nu \underline{u}(R\nu) - 1\,.$$

The C^0 estimates of Lemma 2.2 imply that $|u(x) - |x|| \leq c$, and so the left hand side is $\mathcal{O}(R^{-1})$. According to (1.5), $|\nabla_\nu \underline{u}(x) - 1| = \mathcal{O}(R^{-1})$, and thus

$$|\nabla_\nu (u - \underline{u})| \;=\; |\nabla_\nu u(x) - 1| + \mathcal{O}\left(\frac{1}{R}\right) \;\leq\; \frac{c}{R}\,.$$

In order to derive (2.3), we consider u and the barrier functions along the line segment $\{x + \lambda\tau\}$ parametrized by $\lambda \in [-\lambda_0, \lambda_0]$, $\lambda_0 = \sqrt{R^2 - |x|^2}$. The boundary values of u are $u(\pm\lambda_0) = \underline{u}(\pm\lambda_0)$. Thus using that u lies above $\underline{u}$ and is convex, we obtain the estimate

$$\underline{u}'(-\lambda_0) \;\leq\; u'(-\lambda_0) \;\leq\; u'(\lambda = 0) \;\leq\; u'(\lambda_0) \;\leq\; \underline{u}'(\lambda_0)\,,$$

and thus

$$|\nabla_\tau u(x)| \;=\; |u'(\lambda = 0)| \;\leq\; \max\left\{|\underline{u}'(\lambda_0)|,\, |\underline{u}'(-\lambda_0)|\right\}\,.$$

Using (1.5),

$$\underline{u}'(\pm\lambda_0) = \nabla_\tau \underline{u}(x)|_{x=y} = \nabla_\tau(\underline{u}(x) - |x|)|_{x=y} + \nabla_\tau |x||_{x=y} = \mathcal{O}\left(\frac{1}{R}\right) \pm \frac{\lambda_0}{R},$$

where we have set $y = x \pm \lambda_0 \tau$. These estimates imply (2.3) provided that x is sufficiently close to the outer boundary, more precisely if x lies in the strip

$$R^2 - |x|^2 \;=\; \lambda_0^2 \;\leq\; \kappa R\,,$$

where κ is some constant independent of R. Now suppose that x lies outside this strip, $\lambda_0^2 > \kappa R$. We construct straight lines through $(0, u(x)) \in \mathbb{R}^2$ which are tangential to the hyperbola $(\lambda, \overline{u}(\lambda)) = (\lambda, \sqrt{|x|^2 + \lambda^2} + L)$. A short explicit calculation shows that there are exactly two such lines, and that they go through the points $(\pm\lambda_1, \overline{u}(\lambda))$ with

$$\lambda_1^2 \;=\; \frac{|x|^4}{(u(x) - L)^2} - |x|^2\,.$$

Using that $u - |x|$ is bounded uniformly in R, one sees that $\lambda_1^2 = \mathcal{O}(|x|)$, and so we can by increasing κ arrange that

$$\lambda_1^2 \;\leq\; \kappa R\,. \tag{2.4}$$

Thus $\lambda_1^2 < \lambda_0^2$, meaning that the tangential points are inside the ball B_R. Since u is convex and lies below $\overline{u}$, the line segments joining the points $(\lambda = 0, u(\lambda = 0))$ and $(\pm\lambda_1, \overline{u}(\pm\lambda_1))$, respectively, both lie above u. Moreover, these line segments are by construction tangential to the hyperbola $(\lambda, \overline{u}(\lambda))$ at $\lambda = \pm\lambda_1$. Hence

$$\overline{u}'(-\lambda_1) \;\leq\; u'(\lambda = 0) \;\leq\; \overline{u}'(\lambda_1)\,.$$

Using (2.1), we obtain that

$$|\nabla_\tau u(x)| \;\leq\; 2\,\frac{\lambda_1}{R}\,,$$

and (2.4) gives (2.3). ∎

It remains to derive the C^2 a priori estimates. As in [4], one obtains that the second derivatives on ∂K are bounded uniformly in R. Furthermore, it is well known (see e. g. [3]) that considering

$$w = \frac{\beta}{2}|Du|^2 + \log u_{\xi\xi}$$

in an interior maximum taken over $(x,\xi) \in \overline{(B_R \setminus K)} \times S^{n-1}$ yields that

$$\max_{B_R\setminus K} |D^2u| \leq c + \max_{\partial B_R \cup \partial K} |D^2 u| \,.$$

Hence it suffices to bound $|D^2u|$ on the outer boundary ∂B_R. It is here where we shall use the assumption $\sup(|Df| + |D^2 f|)/f < \infty$.

The following lemma gives an estimate for the tangential second derivatives. The mixed second derivatives will be treated in the next section, and the normal second derivatives in Section 4.

Lemma 2.5 (tangential second derivatives at the outer boundary)**.** *Let $x_0 \in \partial B_R$ and τ_1, τ_2 be tangential directions at x_0. Then we have at x_0,*

$$|u_{\tau_1\tau_2} - |x|_{\tau_1\tau_2}| \leq \frac{c}{R^2} \,.$$

Proof. We may assume that $x_0 = R \cdot e_n \equiv R \cdot (0, \ldots, 0, 1)$. Then ∂B_R is represented locally as graph ω, where

$$\begin{aligned} \omega : \hat{B}_R \equiv \{\hat{x} \in \mathbb{R}^{n-1} : |\hat{x}| < R\} &\to \mathbb{R}, \\ \hat{x} &\mapsto \sqrt{R^2 - |\hat{x}|^2}. \end{aligned}$$

According to the Dirichlet boundary conditions,

$$(u - \underline{u})(\hat{x}, \omega(\hat{x})) = 0.$$

We differentiate twice with respect to $\hat{x}^i$, $\hat{x}^j$, $1 \leq i, j \leq n-1$ and obtain that at x_0,

$$(u - \underline{u})_{ij} + (u - \underline{u})_n \, \omega_{ij} = 0 \,.$$

According to the decay conditions at infinity (1.5),

$$\left|\underline{u}_{ij} - |x|_{ij}\right| = \mathcal{O}\left(\frac{1}{R^2}\right),$$

and furthermore

$$\omega_{ij}(x_0) = -\frac{\delta_{ij}}{R} \,.$$

Thus the result follows in view of Lemma 2.4. ∎

3. Mixed C^2-Estimates

Lemma 3.1 (Mixed second derivatives at the outer boundary)**.** *For $x_0 \in \partial B_R$ let τ and ν be unit vectors in tangential and normal directions, respectively. Then*

$$|u_{\tau\nu}|(x_0) \le \frac{c}{\sqrt{R}}\,. \tag{3.1}$$

The proof of this Lemma is split up into several lemmata.

As in the proof of Lemma 2.5 we may assume that $x_0 = R \cdot e_n$ and represent ∂B_R locally as $\operatorname{graph}\omega$ with $\omega(\hat{x}) = \sqrt{R^2 - |\hat{x}|^2}$. We take the logarithm of the differential equation in (1.6),

$$\log\det u_{ij} - \frac{n+2}{2}\log\left(1 + |Du|^2\right) = \log f(x,u),$$

and differentiate with respect to x^k,

$$u^{ij}u_{ijk} - \frac{n+2}{1+|Du|^2}u^i u_{ik} = \frac{f_k + f_z u_k}{f}. \tag{3.2}$$

This gives a motivation for introducing the linear differential operator L by

$$Lw := u^{ij}w_{ij} - \frac{n+2}{1+|Du|^2}u^i w_i.$$

Furthermore, we define for $t < n$ the linear operator

$$T := \frac{\partial}{\partial x^t} + \omega_{tr}(0)\,x^r\frac{\partial}{\partial x^n} \equiv \frac{\partial}{\partial x^t} - \frac{x_t}{R}\frac{\partial}{\partial x^n},$$

where we used the convention $\omega_{tn} = 0$ (thus we sum over $r = 1, \dots, n-1$). In what follows we restrict attention to the domain $\Omega_\delta := B_\delta(x_0) \cap B_R$ with $\delta \le \frac{R}{2}$, notice that $\Omega_\delta \subset B_R \setminus K$.

Lemma 3.2. *The function $u - \underline{u}$ satisfies the following estimates,*

$$\begin{aligned}
|T(u-\underline{u})| &\le \frac{c}{\sqrt{R}} && \text{in } \Omega_\delta,\\
|T(u-\underline{u})| &\le \frac{c}{R^2}\cdot|x-x_0|^2 && \text{on } \partial B_R,\\
|LT(u-\underline{u})| &\le c + \frac{c}{R^2}\operatorname{tr}u^{ij} && \text{in } \Omega_\delta\,,
\end{aligned}$$

where $\operatorname{tr}u^{ij} \equiv u^{ij}\delta_{ij}$.

Proof. Note that $|\omega_i| \le c$, $|\omega_{ij}| \le \frac{c}{R}$ and $|\omega_{ijk}| \le \frac{c}{R^2}$. The first inequality follows directly from the C^1 estimates of Lemma 2.4, whereas for the second inequality we use furthermore that $(u-\underline{u})_t + (u-\underline{u})_n\omega_t = 0$ on ∂B_R, the decay properties of ω_{ijk}, and the fact that $u - \underline{u}$ vanishes on ∂B_R. To prove the last inequality, we apply (3.2), the relation $u^{ij}u_{jk} = \delta^i_k$, and the C^1-estimates to obtain that

$$|LT(u-\underline{u})| \le c\cdot\frac{|Df|}{f} + \frac{c}{R} + c\cdot|D^2\underline{u}| + c\cdot \operatorname{tr} u^{ij}\cdot\left(|D^3\underline{u}| + \frac{1}{R}|D^2\underline{u}|\right).$$

Now we use the conditions (1.4) and (1.5). □

The function ϑ introduced in the next Lemma will be the main part of a barrier function which we shall construct in what follows.

Lemma 3.3. *There exists a positive constant ε independent of R such that*

$$\vartheta := (u-\underline{u}) + \frac{1}{\sqrt{R}}\,d - \frac{1}{2R^{\frac{5}{4}}}\,d^2$$

fulfills the estimates

$$\begin{cases} L\vartheta & \le & -\varepsilon\, R^{\frac{3}{4n}} - \varepsilon\, R^{-\frac{5}{4}}\operatorname{tr} u^{ij} & \text{in } \Omega_\delta, \\ \vartheta & \ge & 0 & \text{on } \partial\Omega_\delta\,, \end{cases}$$

provided that $\delta = R^{\frac{3}{4}}$ and R is sufficiently large. Here $d = R - |x|$ is the distance from ∂B_R.

Proof. It is obvious that $\vartheta \ge 0$ on $\partial\Omega_\delta$. We fix $x_0 \in \Omega_\delta$ and set $\nu = x_0/|x_0|$. Let τ, τ' belong to an orthonormal basis for the orthogonal complement of ν which we choose such that the submatrix $u^{\tau\tau'}$ is diagonal. Assume that ν and τ, τ' correspond to the indices n and $1, \ldots, n-1$, respectively. We use the Einstein summation convention for τ, τ'. The matrix u^{ij} is positive, and thus testing with the vectors $\nu \pm \tau$ gives

$$|u^{\nu\tau}| \;\le\; \frac{1}{2}\left(u^{\nu\nu} + u^{\tau\tau}\right). \tag{3.3}$$

We introduce the abbreviation

$$\operatorname{tr} u^{\tau\tau'} \;=\; u^{ij}\left(\delta_{ij} - \frac{x_i}{|x|}\frac{x_j}{|x|}\right).$$

Direct computations using (1.5) and (3.3) give

$$\begin{aligned}
Lu &= u^{ij}u_{ij} - (n+2)\,\frac{|Du|^2}{1+|Du|^2} \;\le\; c, \\
L\underline{u} &\ge -c \,+\, u^{\tau\tau'}\,\underline{u}_{\tau\tau'} \,+\, 2\,u^{\tau\nu}\,\underline{u}_{\tau\nu} \,+\, u^{\nu\nu}\,\underline{u}_{\nu\nu} \\
&\ge -c + \left(\frac{1}{|x|} - \frac{c}{|x|^2}\right)\operatorname{tr} u^{\tau\tau'} - \frac{c}{|x|^2}\,u^{\nu\nu}, \\
L\vartheta &\le c\cdot\left(1+\frac{1}{\sqrt{R}}\right) - \left(\frac{1}{|x|} - \frac{c}{|x|^2} + \frac{1}{|x|}\left(\frac{1}{\sqrt{R}} - \frac{d}{R^{\frac{5}{4}}}\right)\right)\operatorname{tr} u^{\tau\tau'} \\
&\quad - \left(\frac{1}{R^{\frac{5}{4}}} - \frac{c}{|x|^2}\right)u^{\nu\nu},
\end{aligned}$$

and thus for R sufficiently large,

$$L\vartheta \;\le\; c - \frac{1}{2R}\mathrm{tr}\, u^{\tau\tau'} - \frac{1}{2R^{\frac{5}{4}}}\, u^{\nu\nu}.$$

Expanding the determinant and using that $u^{\tau\tau'}$ is diagonal gives

$$\begin{aligned}
\det u^{ij} \;&=\; \det\begin{pmatrix} u^{11} & 0 & \cdots & 0 & u^{1n} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & u^{n-1\,n-1} & u^{n-1\,n} \\ u^{1n} & \cdots & \cdots & u^{n-1\,n} & u^{nn} \end{pmatrix} \\
&=\; \prod_i u^{ii} - \sum_\tau |u^{n\tau}|^2 \prod_{\tau'\neq\tau} u^{\tau'\tau'} \;\le\; \prod_i u^{ii}\,.
\end{aligned}$$

Hence the inequality for geometric and arithmetic means as well as (1.6) and (1.4) show that for large values of R,

$$\begin{aligned}
L\vartheta \;&\le\; c - \frac{1}{c}\cdot(\det u_{ij})^{-\frac{1}{n}}\cdot R^{-\frac{n-1}{n}-\frac{5}{4}\frac{1}{n}} - \frac{1}{4R^{\frac{5}{4}}}\,\mathrm{tr}\, u^{ij} \\
&\le\; c - \frac{1}{c}\,|x|^{\frac{n+1}{n}}\, R^{-\frac{1}{n}\left((n-1)+\frac{5}{4}\right)} - \frac{1}{4R^{\frac{5}{4}}}\,\mathrm{tr}\, u^{ij} \\
&\le\; -\frac{1}{c}\cdot R^{\frac{3}{4n}} - \frac{1}{4R^{\frac{5}{4}}}\,\mathrm{tr}\, u^{ij}.
\end{aligned}$$

□

Lemma 3.4. *There exists a positive constant A independent of R such that*

$$\Theta \;:=\; \vartheta + A\cdot\frac{1}{R^2}\cdot|x-x_0|^2 \pm T(u-\underline{u})$$

satisfies the inequalities

$$\begin{cases} L\Theta \;\le\; 0 & \text{in } \Omega_\delta, \\ \;\;\Theta \;\ge\; 0 & \text{on } \partial\Omega_\delta, \end{cases} \tag{3.4}$$

where $\delta = R^{\frac{3}{4}}$ and ϑ is as in Lemma 3.3.

Proof. According to Lemma 3.3, the condition $\Theta \ge 0$ on $\partial\Omega_\delta$ follows if

$$A\cdot\frac{1}{R^2}\cdot|x-x_0|^2 \pm T(u-\underline{u}) \ge 0 \quad \text{on } \partial\Omega_\delta.$$

In view of Lemma 3.2, this can be arranged by choosing A sufficiently large. The property $L\Theta \le 0$ now follows from the inequality

$$-\varepsilon R^{\frac{3}{4n}} - \varepsilon R^{-\frac{5}{4}}\,\mathrm{tr}\, u^{ij} + c\cdot\frac{A}{R} + c + c\cdot\frac{1+A}{R^2}\,\mathrm{tr}\, u^{ij} \le 0,$$

which holds for R sufficiently large. □

Proof of Lemma 3.1. The maximum principle applied to (3.4) yields that $\Theta \geq 0$ in Ω_δ. Since $\Theta(x_0) = 0$, it follows that

$$\Theta_\nu(x_0) \geq 0$$

with $\nu = -\frac{x_0}{|x_0|}$. Thus we obtain

$$\vartheta_\nu(x_0) \geq |(T(u - \underline{u}))_\nu|(x_0) ,$$

and this finally gives (3.1). $\square$

4. Remaining A Priori Estimates

The decay rate in Lemma 3.1 has been chosen such that the following lemma follows immediately.

Lemma 4.1 (Double normal C^2-estimates at the outer boundary)**.** *Under the assumptions of Lemma 3.1 and of one of our Theorems 1.1-1.3,*

$$|u_{\nu\nu}|(x_0) \leq c.$$

Proof. We choose for fixed $x_0 \in \partial B_R$ an orthonormal basis as in Lemma 3.3, now such that the submatrix $u_{\tau\tau'}$ is diagonal. We expand the determinant,

$$\begin{aligned} f(x,u) \cdot \left(1 + |Du|^2\right)^{\frac{n+2}{2}} &= \det u_{ij} \;=\; u_{nn} \cdot \prod_{i<n} u_{ii} - \sum_{k<n} u_{kn}^2 \cdot \prod_{k \neq i < n} u_{ii} \\ &= u_{nn} \cdot \prod_{i<n} u_{ii} - \prod_{i<n} u_{ii} \cdot \sum_{k<n} u_{kn}^2 \, \frac{1}{u_{kk}} \, . \end{aligned}$$

Now we substitute in the estimates of Lemma 2.5 and Lemma 3.1,

$$\begin{aligned} u_{nn} &\leq \frac{c}{\prod\limits_{i<n} u_{ii}} \cdot f(x,u) + \sum_{k<n} \frac{u_{kn}^2}{u_{kk}} \\ &\leq c \cdot R^{n-1} \cdot R^{-n-1} + \sum_{k<n} \frac{\left(\frac{c}{\sqrt{R}}\right)^2}{\frac{c}{R}} \;\leq\; c \, . \end{aligned}$$

$\square$

5. Barrier Constructions

The following lemma gives a simple example of a barrier construction.

Lemma 5.1. *Let* $K = \overline{B_{\rho_1}}(0)$ *with* $\rho_1 > 0$ *and* $u_0 \equiv 0$. *Suppose that* $f \in C^\infty\left(\mathbb{R}^n \setminus \overset{\circ}{K}\right)$ *satisfies*

$$0 \;<\; f \;\leq\; (a-1)\, 2^{-\frac{3n}{2}-1}\, \rho_1^{a-1}\, r^{1-n-a}$$

with $a > 2$. Then there is a strictly convex subsolution $\underline{u}$ which is close to a cone such that (1.5) is satisfied.

Proof. We introduce functions φ and ψ by

$$\begin{aligned} \varphi(r) &= \frac{a-1}{2}\,\rho_1^{a-1}\,r^{-a} \\ \psi(r) &= -\int_{\rho_1}^{r}\left(\int_{\rho}^{\infty}\varphi(\tau)\,d\tau\right)d\rho \end{aligned}$$

and define $\underline{u}$ by

$$\underline{u}\;:\;\mathbb{R}^n\setminus B_{\rho_1}\to\mathbb{R}\;:\;x\mapsto|x|-\rho_1+\psi(|x|)\,.$$

This $\underline{u}$ is zero on ∂B_{ρ_1}, is close to a cone, and satisfies the regularity conditions (1.5). Furthermore,

$$0\;\le\;-\psi'(r)\;=\;\int_r^{\infty}\varphi(\tau)\,d\tau\;\le\;\int_{\rho_1}^{\infty}\varphi(\tau)\,d\tau\;=\;\frac{1}{2}\,.$$

We compute the Gauß curvature of graph $\underline{u}$,

$$\begin{aligned} \underline{u}_i &= \frac{x_i}{|x|}+\psi'\frac{x_i}{|x|}, \quad |D\underline{u}|=1+\psi', \\ \underline{u}_{ij} &= \frac{1}{|x|}(1+\psi')\left(\delta_{ij}-\frac{x_i}{|x|}\frac{x_j}{|x|}\right)+\varphi(|x|)\frac{x_i}{|x|}\frac{x_j}{|x|}\,. \end{aligned}$$

We conclude that in $\mathbb{R}^n\setminus B_{\rho_1}$

$$\begin{aligned} \mathcal{K}[\underline{u}] &= \varphi\cdot r^{1-n}\cdot(1+\psi')^{n-1}\cdot\left(1+(1+\psi')^2\right)^{-\frac{n+2}{2}} \\ &\ge (a-1)\cdot\rho_1^{a-1}\cdot r^{1-n-a}\cdot 2^{-n}\cdot 2^{-\frac{n+2}{2}}\,. \end{aligned}$$

□

The above construction could be extended to more general domains and non-zero boundary values, provided that the boundary values are sufficiently close to a cone, i. e. $|u_0-|x||_{C^2}<\varepsilon$, and f is sufficiently small. For non-zero boundary values, it seems preferable to use the following construction of a subsolution, based on the idea of viscosity subsolutions.

Lemma 5.2. *For a smooth domain $\Omega\subset\mathbb{R}^n$ and a smooth positive function $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ with $f_z\ge0$, we assume that there exist two smooth, strictly convex subsolutions, i. e.*

$$u_i:\overline{\Omega_i}\to\mathbb{R},\quad \mathcal{K}[u_i]\ge f(x,u_i)\ \textit{in}\ \Omega_i,\quad i=1,2,$$

such that $\Omega_1\cup\Omega_2=\Omega$ (Ω_i = open), $u_i<u_{3-i}$ on $\partial\Omega_i$, $i=1,2$, and $\{u_1=u_2\}\Subset\Omega_1\cap\Omega_2$ (in particular $\{u_1=u_2\}\cap\partial\Omega=\emptyset$). Then there exists a smooth function $u:\overline{\Omega}\to\mathbb{R}$ such that

$$\begin{cases} \mathcal{K}[u] = f(x,u) & \textit{in }\Omega, \\ u = \max\{u_1,u_2\} & \textit{on }\partial\Omega, \\ u \ge \max\{u_1,u_2\} & \textit{in }\Omega. \end{cases}$$

Proof. We choose a smooth, strictly convex function $w : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} w & \geq & \max\{u_1, u_2\} & \text{in } \Omega, \\ w & = & \max\{u_1, u_2\} & \text{on } \partial\Omega \end{cases}$$

and define $f_w(x) := \mathcal{K}[w](x)$. Then w is a solution of the Dirichlet problem

$$\begin{cases} \mathcal{K}[w] & = & f_w(x) & \text{in } \Omega, \\ w & = & \max\{u_1, u_2\} & \text{on } \partial\Omega. \end{cases}$$

We may assume that $f_w < f(\cdot, \max\{u_1, u_2\})$ in Ω because otherwise we solve a Dirichlet problem (with small f) by using [4] with w as the subsolution, and take the solution to this Dirichlet problem instead of w. Finally, we apply the continuity method and solve the Dirichlet problems

$$\begin{cases} \mathcal{K}\left[u^t\right] & = & (1-t) f_w(x) + t f(x, u^t) & \text{in } \Omega, \\ u^t & = & \max\{u_1, u_2\} & \text{on } \partial\Omega, \\ u^t & \geq & \max\{u_1, u_2\} & \text{in } \Omega, \end{cases}$$

for smooth strictly convex functions $u^t : \overline{\Omega} \to \mathbb{R}$, $0 \leq t \leq 1$. The solvability of this Dirichlet problem is a consequence of the a priori estimates of [4] and the fact that

$$u^t > \max\{u_1, u_2\} \quad \text{in } \Omega \text{ for } 0 \leq t < 1.$$

The last inequality follows from the maximum principle because $(1-t) f_w(x) + t f(x, \max\{u_1, u_2\}) < f(x, \max\{u_1, u_2\})$ for $0 \leq t < 1$. For $t = 1$ we obtain the function u we are looking for. □

Appendix A. Notes

We remark that the C^2-estimates at the outer boundary ∂B_R cannot be obtained from the estimates on compact domains by a simple scaling argument. The reason is that if one considers the function $\frac{1}{R}\, u(Rx)$ near $B_1 \setminus \left(\frac{1}{R} \cdot K\right)$, the corresponding rescaled function f tends to zero near ∂B_1 as $R \to \infty$.

The purpose of this paper was to find a method for proving the existence of strictly convex hypersurfaces of prescribed Gauß curvature in exterior domains. But for technical simplicity, we did not consider the most general situation to which our methods apply.

First of all, our regularity assumptions could be weakened; indeed, it is sufficient to assume the same regularity as in the compact case, i. e. similar to [10]

$$\partial K \in C^4\,, \qquad \underline{u} \in C^4\Big(\mathbb{R}^n \setminus \overset{\circ}{K}\Big)\,, \qquad f \in C^2\Big(\Big(\mathbb{R}^n \setminus \overset{\circ}{K}\Big) \times \mathbb{R}\Big)\,.$$

Furthermore, it is possible to treat equations of the form

$$\det u_{ij} = f(x, u, Du)$$

with $f > 0$. We do not consider this case here because the explicit construction of a subsolution becomes more complicated for this more general ansatz.

Moreover, it is possible to remove the technical condition $f_z \geq 0$ by using a mapping degree argument and assuming that $f_z \geq 0$ only outside the set $\{(x, z) \in (\mathbb{R}^n \setminus K) \times \mathbb{R} : \underline{u}(x) \leq z \leq \overline{u}(x)\}$. But the uniqueness as described in Lemma 2.2 is lost, i. e. the limit of u^{R_k} as $R_k \to \infty$ may depend on the choice of the subsequence. We also remark that the decay condition

$$\sup \left(\frac{|Df| + |D^2 f|}{f} \right) < \infty$$

is clearly only needed between the two barriers.

Finally, one could consider hypersurfaces of prescribed Gauß curvature which, instead of being close to a cone, have a different asymptotic behavior near infinity. One example would be to take a smooth convex domain $\Omega \subset \mathbb{R}^{n-1}$, $0 \in \Omega$, and to consider hypersurfaces which are close to the cone $\mathbb{R}_+ \cdot (\partial\Omega \times \{1\})$. It even seems possible to adapt our methods to hypersurfaces of prescribed Gauß curvature which are close to a more general convex hypersurface, provided that its Gauß curvature decays sufficiently fast at infinity, and that one can find barriers that give good C^1-estimates.

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