SCHOUTEN TENSOR EQUATIONS IN CONFORMAL GEOMETRY WITH PRESCRIBED BOUNDARY METRIC

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Abstract. On a manifold with boundary, we deform the metric conformally. This induces a deformation of the Schouten tensor. We fix the metric at the boundary and realize a prescribed value for the product of the eigenvalues of the Schouten tensor in the interior, provided that there exists a subsolution.

1. Introduction

Let (M^n, g_{ij}) be an n-dimensional Riemannian manifold, $n \geq 3$. The Schouten tensor (S_{ij}) of (M^n, g_{ij}) is defined as

$$S_{ij} = \tfrac{1}{n-2}\left(R_{ij} - \tfrac{1}{2(n-1)} R g_{ij}\right),$$

where (R_{ij}) and R denote the Ricci and scalar curvature of (M^n, g_{ij}), respectively. Consider the manifold $\left(\tilde{M}^n, \tilde{g}_{ij}\right) = \left(M^n, e^{-2u} g_{ij}\right)$, where we have used $u \in C^2(M^n)$ to deform the metric conformally. The Schouten tensors S_{ij} of g_{ij} and $\tilde{S}_{ij}$ of $\tilde{g}_{ij}$ are related by

$$\tilde{S}_{ij} = u_{ij} + u_i u_j - \tfrac{1}{2}|\nabla u|^2 g_{ij} + S_{ij},$$

where indices of u denote covariant derivatives with respect to the background metric g_{ij}, moreover $|\nabla u|^2 = g^{ij} u_i u_j$ and $\left(g^{ij}\right) = (g_{ij})^{-1}$. Eigenvalues of the Schouten tensor are computed with respect to the background metric g_{ij}, so the product of the eigenvalues of the Schouten tensor $(\tilde{S}_{ij})$ equals a given function $s : M^n \to \mathbb{R}$, if

$$\frac{\det\left(u_{ij} + u_i u_j - \frac{1}{2}|\nabla u|^2 g_{ij} + S_{ij}\right)}{e^{-2nu}\det(g_{ij})} = s(x). \tag{1.1}$$

We say that u is an admissible solution for (1.1), if the tensor in the determinant in the numerator is positive definite. At admissible solutions, (1.1) becomes an elliptic equation. As we are only interested in admissible solutions, we will always assume that s is positive.

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Let now M^n be compact with boundary and $\underline{u} : M^n \to \mathbb{R}$ be a smooth (up to the boundary) admissible subsolution to (1.1)

$$\frac{\det\left(\underline{u}_{ij} + \underline{u}_i\underline{u}_j - \frac{1}{2}|\nabla\underline{u}|^2 g_{ij} + S_{ij}\right)}{e^{-2n\underline{u}}\det\left(g_{ij}\right)} \ge s(x). \tag{1.2}$$

Assume that there exists a supersolution $\overline{u}$ to (1.1) fulfilling some technical conditions specified in Definition 2.1. Assume furthermore that M^n admits a strictly convex function χ. Without loss of generality, we have $\chi_{ij} \ge g_{ij}$ for the second covariant derivatives of χ in the matrix sense.

The conditions of the preceding paragraph are automatically fulfilled if M^n is a compact subset of flat $\mathbb{R}^n$ and $\underline{u}$ fulfills (1.2) and in addition $\det(\underline{u}_{ij}) \ge s(x)e^{-2n\underline{u}}\det(g_{ij})$ with $\underline{u}_{ij} > 0$ in the matrix sense. Then Lemma 2.2 implies the existence of a supersolution and we may take $\chi = |x|^2$.

We impose the boundary condition that the metric $\tilde{g}_{ij}$ at the boundary is prescribed,

$$\tilde{g}_{ij} = e^{-2\underline{u}} g_{ij} \quad \text{on } \partial M^n.$$

Assume that all data are smooth up to the boundary. We prove the following

Theorem 1.1. *Let M^n, g_{ij}, $\underline{u}$, $\overline{u}$, χ, and s be as above. Then there exists a metric $\tilde{g}_{ij}$, conformally equivalent to g_{ij} with $\tilde{g}_{ij} = e^{-2\underline{u}}g_{ij}$ on ∂M^n such that the product of the eigenvalues of the Schouten tensor induced by $\tilde{g}_{ij}$ equals s.*

This follows readily from

Theorem 1.2. *Under the assumptions stated above, there exists an admissible function $u \in C^0(M^n) \cap C^\infty(M^n \setminus \partial M^n)$ solving* (1.1) *such that $u = \underline{u}$ on ∂M^n.*

Recently, in a series of papers, Jeff Viaclovsky studied conformal deformations of metrics on closed manifolds and elementary symmetric functions S_k, $1 \le k \le n$, of the eigenvalues of the associated Schouten tensor, see e. g. [15] for existence results. Pengfei Guan, Jeff Viaclovsky, and Guofang Wang provide an estimate that can be used to show compactness of manifolds with lower bounds on elementary symmetric functions of the eigenvalues of the Schouten tensor [8]. A similar equation arises in geometric optics [10, 16]. Xu-Jia Wang proved the existence of solutions to Dirichlet boundary value problems for an equation similar to (1.1), provided that the domains are small. In [12] we provide a transformation that shows the similarity between reflector and Schouten tensor equations. Pengfei Guan and Xu-Jia Wang obtained local C^2-estimates [10]. This was extended by Pengfei Guan and Guofang Wang to local C^1- and C^2-estimates in the case of elementary symmetric functions S_k of the Schouten tensor of a conformally deformed metric [9]. Boundary value problems for Monge-Ampère equations have been studied by Luis Caffarelli, Louis Nirenberg, and Joel Spruck in [1] an many other people later on. For us, those articles using subsolutions as used by Bo Guan and Joel Spruck will be especially useful [6, 7, 11, 13].

It follows directly from the proof that we can also solve Equation (1.1) on non-compact complete manifolds provided that there exist appropriate sub- and supersolutions with locally bounded difference in C^0. Then we can solve (1.1) with an artificially introduced Dirichlet boundary condition on a sequence of growing domains exhausting the non-compact manifold. A subsequence of these solutions converges then to a solution of (1.1) on the manifold. This works as the local C^2-estimates in [9] depend only on a local bound for $|u|$. Note that either $s(x)$ has to decay at infinity or the manifold with metric $e^{-2u}g_{ij}$ is non-complete. Otherwise, [8] implies a positive lower bound on the Ricci tensor, i. e. $\tilde{R}_{ij} \geq \frac{1}{c}\tilde{g}_{ij}$ for some positive constant c. This yields compactness of the manifold [5].

It is a further issue to solve similar problems for other elementary symmetric functions of the Schouten tensor. As the induced mean curvature of ∂M^n is related to the Neumann boundary condition, this is another natural boundary condition.

To show existence for a boundary value problem for fully nonlinear equations like Equation (1.1), one usually proves C^2-estimates up to the boundary. Then standard results imply C^k-bounds for $k \in \mathbb{N}$ and existence results. In our situation, however, we don't expect that C^2-estimates up to the boundary can be proved. This is due to the gradient terms appearing in the determinant in (1.1). It is possible to overcome these difficulties by considering only small domains [16]. Our method is different. We regularize the equation and prove full regularity up to the boundary for the regularized equation. Then we use the fact, that interior C^k-estimates [9] can be obtained independent of the regularization. Moreover, we can prove uniform C^1-estimates. Thus we can pass to a limit and get a solution in $C^0(M^n) \cap C^\infty(M^n \setminus \partial M^n)$.

To be more precise, we rewrite (1.1) in the form

$$\log\det\left(u_{ij} + u_i u_j - \tfrac{1}{2}|\nabla u|^2 g_{ij} + S_{ij}\right) = f(x, u), \tag{1.3}$$

where $f \in C^\infty(M^n \times \mathbb{R})$. Our method can actually be applied to any equation of that form provided that we have sub- and supersolutions. Thus we consider in the following equations of the form (1.3). Equation (1.3) makes sense in any dimension provided that we replace S_{ij} by a smooth tensor. In this case Theorem 1.2 is valid in any dimension. Note that even without the factor $\frac{1}{n-2}$ in the definition of the Schouten tensor, our equation is not elliptic for $n = 2$ for any function u as the trace $g^{ij}(R_{ij} - \frac{1}{2}Rg_{ij})$ equals zero, so there has to be a non-positive eigenvalue of that tensor. Let $\psi : M^n \to [0, 1]$ be smooth, $\psi = 0$ in a neighborhood of the boundary. Then our strategy is as follows. We consider a sequence ψ_k of those functions that fulfill $\psi_k(x)=1$ for $\operatorname{dist}(x,\partial M^n)>\frac{2}{k}$, $k\in\mathbb{N}$, and boundary value problems

$$\begin{cases} \log\det\left(u_{ij}+\psi u_iu_j-\frac12\psi|\nabla u|^2g_{ij}+T_{ij}\right) &= f(x,u) \quad \text{in } M^n,\\ u &= \underline{u} \quad \text{on } \partial M^n. \end{cases} \tag{1.4}$$

We dropped the index k to keep the notation simple. The tensor T_{ij} coincides with S_{ij} on $\left\{x\in M^n:\operatorname{dist}(x,\partial M^n)>\frac{2}{k}\right\}$ and interpolates smoothly to S_{ij} plus a sufficiently large constant multiple of the background metric g_{ij} near the boundary.

Our sub- and supersolutions act as barriers and imply uniform C^0-estimates. We prove uniform C^1-estimates based on the admissibility of solutions. Admissibility means here that $u_{ij}+\psi u_iu_j-\frac12\psi|\nabla u|^2+S_{ij}$ is positive definite for those solutions. As mentioned above, we can't prove uniform C^2-estimates for u, but we get C^2-estimates that depend on ψ. These estimates guarantee, that we can apply standard methods (Evans-Krylov-Safanov theory, Schauder estimates for higher derivatives, and mapping degree theory for existence, see e. g. [4, 6, 14]) to prove existence of a smooth admissible solution to (1.4). Then we use [9] to get uniform interior a priori estimates on compact subdomains of M^n as $\psi=1$ in a neighborhood of these subdomains for all but a finite number of regularizations. These a priori estimates suffice to pass to a subsequence and to obtain an admissible solution to (1.3) in $M^n\setminus\partial M^n$. As $u^k=u=\underline{u}$ for all solutions u^k of the regularized equation and those solutions have uniformly bounded gradients, the boundary condition is preserved when we pass to the limit and we obtain Theorem 1.2 provided that we can prove $\left\|u^k\right\|_{C^1(M^n)}\le c$ uniformly and $\left\|u^k\right\|_{C^2(M^n)}\le c(\psi)$. These estimates are proved in Lemmata 4.1 and 5.4, the crux of this paper.

The rest of the article is organized as follows. We introduce supersolutions and some notation in Section 2. We mention C^0-estimates in Section 3. In Section 4, we prove uniform C^1-estimates. Then the C^2-estimates proved in Section 5 complete the a priori estimates and the proof of Theorem 1.2.

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2. Supersolutions and Notation

Before we define a supersolution, we explain more explicitly, how we regularize the equation. For fixed $k\in\mathbb{N}$ we take ψ_k such that

$$\psi_k(x)=\begin{cases}0 & \operatorname{dist}(x,\partial M^n)<\frac1k,\\ 1 & \operatorname{dist}(x,\partial M^n)>\frac2k\end{cases}$$

and ψ_k is smooth with values in $[0,1]$. Again, we drop the index k to keep the notation simple. We fix $\lambda \ge 0$ sufficiently large so that

$$\log\det\left(\underline{u}_{ij} + \psi\underline{u}_i\underline{u}_j - \tfrac{1}{2}\psi|\nabla\underline{u}|^2 g_{ij} + S_{ij} + \lambda(1-\psi)g_{ij}\right) \ge f(x,\underline{u}) \tag{2.1}$$

for any $\psi = \psi_k$, independent of k. As $\log\det(\cdot)$ is a concave function on positive definite matrices, (2.1) follows for k sufficiently large, if

$$\log\det\left(\underline{u}_{ij} + \underline{u}_i\underline{u}_j - \tfrac{1}{2}|\nabla u|^2 g_{ij} + S_{ij}\right) \ge f(x,\underline{u}) \quad \text{on } M^n$$

and

$$\log\det\left(\underline{u}_{ij} + S_{ij} + \lambda g_{ij}\right) \ge f(x,\underline{u}) \quad \text{near } \partial M^n,$$

provided that the arguments of the determinants are positive definite.

We define

Definition 2.1 (supersolution)**.** A smooth function $\overline{u} : M^n \to \mathbb{R}$ is called a supersolution, if $\overline{u} \ge \underline{u}$ and for any ψ as considered above,

$$\log\det\left(\overline{u}_{ij} + \psi\overline{u}_i\overline{u}_j - \tfrac{1}{2}\psi|\nabla\overline{u}|^2 g_{ij} + S_{ij} + \lambda(1-\psi)g_{ij}\right) \le f(x,\underline{u})$$

holds for those points in M^n for which the tensor in the determinant is positive definite.

Lemma 2.2. *If M^n is a compact subdomain of flat $\mathbb{R}^n$, the subsolution $\underline{u}$ fulfills* (1.2) *and in addition*

$$\det(\underline{u}_{ij}) \ge s(x)e^{-2n\underline{u}}\det(g_{ij})$$

holds, where $\underline{u}_{ij} > 0$ in the matrix sense, then there exists a supersolution.

Proof. In flat $\mathbb{R}^n$, we have $S_{ij} = 0$. The inequality

$$\frac{\det\left(\underline{u}_{ij} + \psi\underline{u}_i\underline{u}_j - \frac{1}{2}\psi|\nabla\underline{u}|^2 g_{ij}\right)}{e^{-2n\underline{u}}\det\left(g_{ij}\right)} \ge s(x) \tag{2.2}$$

is fulfilled if ψ equals 0 or 1 by assumption. As above, (2.2) follows for any $\psi \in [0,1]$. Thus (2.1) is fulfilled for $\lambda = 0$.

Let $\overline{u} = \sup\limits_{M^n}\underline{u} + 1 + \varepsilon|x|^2$ for $\varepsilon > 0$. It can be verified directly that $\overline{u}$ is a supersolution for $\varepsilon > 0$ fixed sufficiently small. □

Notation 2.3. We set

$$\begin{aligned} w_{ij} =& u_{ij} + \psi u_i u_j - \tfrac{1}{2}\psi|\nabla u|^2 g_{ij} + S_{ij} + \lambda(1-\psi)g_{ij} \\ =& u_{ij} + \psi u_i u_j - \tfrac{1}{2}\psi|\nabla u|^2 g_{ij} + T_{ij} \end{aligned}$$

and use $\left(w^{ij}\right)$ to denote the inverse of (w_{ij}). The Einstein summation convention is used. We lift and lower indices using the background metric. Vectors of length one are called directions. Indices, sometimes preceded by a semi-colon, denote covariant derivatives. We use indices preceded by a colon for partial derivatives. Christoffel symbols of the background metric are denoted by Γ^k_{ij}, so $u_{ij} = u_{;ij} = u_{,ij} - \Gamma^k_{ij} u_k$. Using the Riemannian curvature tensor (R_{ijkl}), we can interchange covariant differentiation

$$\begin{aligned} u_{ijk} =& u_{kij} + u_a g^{ab} R_{bijk}, \\ u_{iklj} =& u_{ikjl} + u_{ka} g^{ab} R_{bilj} + u_{ia} g^{ab} R_{bklj}. \end{aligned} \tag{2.3}$$

We write $f_z = \frac{\partial f}{\partial u}$ and $\operatorname{tr} w^{ij} = w^{ij} g_{ij}$. The letter c denotes estimated positive constants and may change its value from line to line. It is used so that increasing c keeps the estimates valid. We use (c_j), (c^k), ... to denote estimates tensors.

3. Uniform C^0-Estimates

The techniques of this section are quite standard, but they simplify the C^0-estimates used before for Schouten tensor equations, see [15, Prop. 3]. Here, we interpolate between the expressions for the Schouten tensors rather than between the functions inducing the conformal deformations.

For the existence proof of the regularized problem, we apply a mapping degree argument. In view of our sub- and supersolutions, we only have to ensure that we can apply the maximum principle or the Hopf boundary point lemma at a point, where a solution touches a barrier for the first time during the deformation associated with the mapping degree argument to prove C^0-estimates. Note that u can touch $\overline{u}$ only in those points from below where $\overline{u}$ is admissible. Compare this to [3]. Without loss of generality, we may assume that u touches $\underline{u}$ from above. Here, touching means $u = \underline{u}$ and $\nabla u = \nabla \underline{u}$ at a point, so our considerations include the case of touching at the boundary. It suffices to prove an inequality of the form

$$0 \le a^{ij} (\underline{u} - u)_{ij} + b^i (u - \underline{u})_i + d(\underline{u} - u) \tag{3.1}$$

with positive definite a^{ij}.

Define

$$S^\psi_{ij}[v] = v_{ij} + \psi v_i v_j - \tfrac{1}{2} \psi |\nabla v|^2 g_{ij} + T_{ij}.$$

We apply the mean value theorem and get for a symmetric positive definite tensor a^{ij} and a function d

$$\begin{aligned} 0 \le& \log\det S^\psi_{ij}[\underline{u}] - \log\det S^\psi_{ij}[u] - f(x, \underline{u}) + f(x, u) \\ =& \int\limits_0^1 \frac{d}{dt} \log\det \left\{ t S^\psi_{ij}[\underline{u}] + (1-t) S^\psi_{ij}[u] \right\} dt - \int\limits_0^1 \frac{d}{dt} f(x, t\underline{u} + (1-t)u) dt \\ =& a^{ij} \left(\left(\underline{u}_{ij} + \psi \underline{u}_i \underline{u}_j - \tfrac{1}{2} \psi |\nabla \underline{u}|^2 g_{ij} \right) - \left(u_{ij} + \psi u_i u_j - \tfrac{1}{2} \psi |\nabla u|^2 g_{ij} \right) \right) \\ &+ d \cdot (\underline{u} - u). \end{aligned}$$

The first integral is well-defined as the set of positive definite tensors is convex. We have $|\nabla \underline{u}|^2 - |\nabla u|^2 = \langle \nabla(\underline{u} - u), \nabla(\underline{u} + u)\rangle$ and

$$\begin{aligned} a^{ij}(\underline{u}_i\underline{u}_j - u_iu_j) =& a^{ij}\int_0^1 \frac{d}{dt}\big((t\underline{u}_i + (1-t)u_i)(t\underline{u}_j + (1-t)u_j)\big)\,dt \\ =& 2a^{ij}\int_0^1 (t\underline{u}_j + (1-t)u_j)dt \cdot (\underline{u} - u)_i, \end{aligned}$$

so we obtain an inequality of the form (3.1). Thus, we may assume in the following that we have $\underline{u} \le u \le \overline{u}$.

4. Uniform C^1-Estimates

Lemma 4.1. *An admissible solution of* (1.4) *has uniformly bounded gradient.*

Proof. We apply a method similar to [13, Lemma 4.2]. Let

$$W = \tfrac{1}{2}\log|\nabla u|^2 + \mu u$$

for $\mu \gg 1$ to be fixed. Assume that W attains its maximum over M^n at an interior point x_0. This implies at x_0

$$0 = W_i = \frac{u^j u_{ji}}{|\nabla u|^2} + \mu u_i$$

for all i. Multiplying with u^i and using admissibility gives

$$\begin{aligned} 0 =& u^iu^ju_{ij} + \mu|\nabla u|^4 \\ \ge& -\psi|\nabla u|^4 + \tfrac{1}{2}\psi|\nabla u|^4 - c|\nabla u|^2 - \lambda|\nabla u|^2 + \mu|\nabla u|^4. \end{aligned}$$

The estimate follows for sufficiently large μ as λ, see (2.1), does not depend on ψ. If W attains its maximum at a boundary point x_0, we introduce normal coordinates such that W_n corresponds to a derivative in the direction of the inner unit normal. We obtain in this case $W_i = 0$ for $i < n$ and $W_n \le 0$ at x_0. As the boundary values of u and $\underline{u}$ coincide and $u \ge \underline{u}$, we may assume that $u_n \ge 0$. Otherwise, $0 \ge u_n \ge \underline{u}_n$ and $u_i = \underline{u}_i$, so a bound for $|\nabla u|$ follows immediately. Thus we obtain $0 \ge u^iW_i$ and the rest of the proof is identical to the case where W attains its maximum in the interior. □

Note that in order to obtain uniform C^1-estimates, we used admissibility, but did not differentiate (1.3).

5. C^2-Estimates

5.1. **C^2-Estimates at the Boundary.** Boundary estimates for an equation of the form $\det(u_{ij} + S_{ij}) = f(x)$ have been considered in [1]. It is straight forward to handle the additional term that is independent of u in the determinant and to use subsolutions like in [6, 7, 11, 13]. We want to point out, that we were only able to obtain estimates for the second derivatives of u at the boundary by introducing ψ and thus removing gradient terms of u in the determinant near the boundary. The C^2-estimates at the boundary are very similar to [13]. We do not repeat the proofs for the double tangential and double normal estimates, but repeat that for the mixed tangential normal derivatives as we can slightly streamline this part. Our method does not imply uniform a priori estimates at the boundary as we look only at small neighborhoods of the boundary depending on the regularization or, more precisely, on the set, where $\psi = 0$.

Lemma 5.1 (Double Tangential Estimates)**.** *An admissible solution of* (1.4) *has uniformly bounded partial second tangential derivatives, i. e. for tangential directions τ_1 and τ_2, $u_{,ij}\tau_1^i\tau_2^j$ is uniformly bounded.*

Proof. This is identical to [13, Section 5.1], but can also be found at various other places. It follows directly by differentiating the boundary condition twice tangentially. □

Lemma 5.2 (Mixed Estimates)**.** *An admissible solution of* (1.4) *has uniformly bounded partial second mixed tangential normal derivatives, i. e. for a tangential direction τ and for the inner unit normal ν, $u_{,ij}\tau^i\nu^j$ is uniformly bounded.*

Proof. The proof is similar to [13, Section 5.2]. The main differences are as follows. The modified definition of the linear operator T in (5.4) clarifies the relation between T and the boundary condition. The term T_{ij} does (in general) not vanish in a fixed boundary point for appropriately chosen coordinates. In [13], we could choose such coordinates. Similarly, we choose coordinates such that the Christoffel symbols become small near a fixed boundary point. Here, we can add and subtract the term T_{ij} in (5.5) as it is independent of u. Finally, we explain here more explicitly how to apply the inequality for geometric and arithmetic means in (5.7).

Fix normal coordinates around a point $x_0 \in \partial M^n$, so $g_{ij}(x_0)$ equals the Kronecker delta and the Christoffel symbols fulfill $\left|\Gamma^k_{ij}\right| \le c\,\mathrm{dist}(\cdot, x_0) = c|x - x_0|$, where the distance is measured in the flat metric using our chart, but is equivalent to the distance with respect to the background metric. Abbreviate the first $n-1$ coordinates by $\hat{x}$ and assume that M^n is locally given by $\{x^n \ge \omega(\hat{x})\}$ for a smooth function ω. We may assume that $(0, \omega(0))$

corresponds to the fixed boundary point x_0 and $\nabla\omega(0) = 0$. We restrict our attention to a neighborhood of x_0, $\Omega_\delta = \Omega_\delta(x_0) = M^n \cap B_\delta(x_0)$ for $\delta > 0$ to be fixed sufficiently small, where $\psi = 0$. Thus the equation takes the form

$$\log\det(u_{ij} + T_{ij}) = \log\det\left(u_{,ij} - \Gamma^k_{ij}u_k + T_{ij}\right) = f(x,u). \tag{5.1}$$

Assume furthermore that $\delta > 0$ is chosen so small that the distance function to $\partial\Omega$ is smooth in Ω_δ.

We differentiate the boundary condition tangentially

$$0 = (u - \underline{u})_{,t}(\hat{x}, \omega(\hat{x})) + (u - \underline{u})_{,n}(\hat{x}, \omega(\hat{x}))\omega_{,t}(\hat{x}), \quad t < n. \tag{5.2}$$

Differentiating (5.1) yields

$$w^{ij}\left(u_{,ijk} - \Gamma^l_{ij}u_{,lk}\right) = f_k + f_z u_k + w^{ij}\left(\Gamma^l_{ij,k}u_l - T_{ij,k}\right). \tag{5.3}$$

This motivates the definition of the differential operators T and L. Here $t < n$ is fixed and ω is evaluated at the projection of x to the first $n-1$ components

$$\begin{aligned} Tv :=& v_t + v_n\omega_t, \quad t < n, \\ Lv :=& w^{ij}v_{,ij} - w^{ij}\Gamma^l_{ij}v_l. \end{aligned} \tag{5.4}$$

On ∂M^n, we have $T(u - \underline{u}) = 0$, so we obtain

$$|T(u - \underline{u})| \le c(\delta)\cdot|x - x_0|^2 \quad \text{on } \partial\Omega_\delta.$$

Derivatives of $\underline{u}$ are a priorily bounded, thus

$$|LT(u - \underline{u})| \le c\cdot\left(1 + \operatorname{tr} w^{ij}\right) \quad \textit{in } \Omega_\delta.$$

Set $d := \operatorname{dist}(\cdot, \partial M^n)$, measured in the Euclidean metric of the fixed coordinates. We define for $1 \gg \alpha > 0$ and $\mu \gg 1$ to be chosen

$$\vartheta := (u - \underline{u}) + \alpha d - \mu d^2.$$

The function ϑ will be the main part of our barrier. As $\underline{u}$ is admissible, there exists $\varepsilon > 0$ such that

$$\underline{u}_{,ij} - \Gamma^l_{ij}\underline{u}_l + T_{ij} \ge 3\varepsilon g_{ij}.$$

We apply the definition of L

$$\begin{aligned} L\vartheta =& w^{ij}\left(u_{,ij} - \Gamma^l_{ij}u_l + T_{ij}\right) - w^{ij}\left(\underline{u}_{,ij} - \Gamma^l_{ij}\underline{u}_l + T_{ij}\right) \\ &+ \alpha w^{ij}d_{,ij} - \alpha w^{ij}\Gamma^l_{ij}d_l \\ &- 2\mu d w^{ij}d_{,ij} - 2\mu w^{ij}d_i d_j + 2\mu d w^{ij}\Gamma^l_{ij}d_l \end{aligned} \tag{5.5}$$

We have $w^{ij}\left(u_{,ij} - \Gamma^l_{ij}u_l + T_{ij}\right) = w^{ij}w_{ij} = n$. Due to the admissibility of $\underline{u}$, we get $-w^{ij}\left(\underline{u}_{,ij} - \Gamma^l_{ij}\underline{u}_l + T_{ij}\right) \le -3\varepsilon \operatorname{tr} w^{ij}$. We fix $\alpha > 0$ sufficiently small and obtain

$$\alpha w^{ij}d_{,ij} - \alpha w^{ij}\Gamma^l_{ij}d_l \le \varepsilon \operatorname{tr} w^{ij}.$$

Obviously, we have

$$-2\mu d w^{ij} d_{,ij} + 2\mu d w^{ij} \Gamma^l_{ij} d_l \le c(\mu\delta) \operatorname{tr} w^{ij}.$$

To exploit the term $-2\mu w^{ij} d_i d_j$, we use that $|d_i - \delta^n_i| \le c \cdot |x - x_0| \le c \cdot \delta$, so

$$-2\mu w^{ij} d_i d_j \le -\mu w^{nn} + c(\mu\delta) \max_{k,\,l} \left| w^{kl} \right|.$$

As w^{ij} is positive definite, we obtain by testing $\begin{pmatrix} w^{kk} & w^{kl} \\ w^{kl} & w_{ll} \end{pmatrix}$ with the vectors $(1, 1)$ and $(1, -1)$ that $\left| w^{kl} \right| \le \operatorname{tr} w^{ij}$. Thus (5.5) implies

$$L\vartheta \le -2\varepsilon \operatorname{tr} w^{ij} - \mu w^{nn} + c + c(\mu\delta) \operatorname{tr} w^{ij} \tag{5.6}$$

We may assume that $\left(w^{ij}\right)_{i,\,j<n}$ is diagonal. Then

$$\begin{aligned} e^{-f} = \det\left(w^{ij}\right) &= \det \begin{pmatrix} w^{11} & 0 & \cdots & 0 & w^{1n} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & w^{n-1\,n-1} & w^{n-1\,n} \\ w^{1n} & \cdots & \cdots & w^{n-1\,n} & w^{nn} \end{pmatrix} \\ &= \prod_{i=1}^n w^{ii} - \sum_{i<n} \left|w^{ni}\right|^2 \prod_{\substack{j \ne i \\ j<n}} w^{jj} \le \prod_{i=1}^n w^{ii}. \end{aligned} \tag{5.7}$$

implies that $\operatorname{tr} w^{ij}$ tends to infinity if w^{nn} tends to zero. So we can fix $\mu \gg 1$ such that the absolute constant in (5.6) can be absorbed. Note also that the geometric arithmetic means inequality implies

$$\tfrac{1}{n} \operatorname{tr} w^{ij} = \tfrac{1}{n} \sum_{i=1}^n w^{ii} \ge \left(\prod_{i=1}^n w^{ii} \right)^{1/n},$$

so (5.7) yields a positive lower bound for $\operatorname{tr} w^{ij}$. Finally, we fix $\delta = \delta(\mu)$ sufficiently small and use (5.6) to deduce that

$$L\vartheta \le -\varepsilon \operatorname{tr} w^{ij}. \tag{5.8}$$

We may assume that δ is fixed so small that $\vartheta \ge 0$ in Ω_δ.

Define for A, $B \gg 1$ the function

$$\Theta^\pm := A\vartheta + B|x - x_0|^2 \pm T(u - \underline{u}).$$

Our estimates imply that $\Theta^\pm \ge 0$ on $\partial\Omega_\delta$ for $B \gg 1$ fixed sufficiently large and $L\Theta^\pm \le 0$ in Ω_δ, when $A \gg 1$, depending also on B, is fixed sufficiently large. Thus the maximum principle implies that $\Theta^\pm \ge 0$ in Ω_δ. As $\Theta^\pm(x_0) = 0$, we deduce that $\Theta^\pm_{,n} \ge 0$, so we obtain a bound for $(Tu)_{,n}$ and the lemma follows. □

Lemma 5.3 (Double Normal Estimates)**.** *An admissible solution of* (1.4) *has uniformly bounded partial second normal derivatives, i. e. for the inner unit normal* ν, $u_{,ij}\nu^i\nu^j$ *is uniformly bounded.*

Proof. The proof is identical to [13, Section 5.3]. Note however, that the notation there is slightly different. There $-u_{,ij}+a_{ij}$ is positive definite instead of $u_{,ij}-\Gamma^k_{ij}u_k+T_{ij}$ here. □

5.2. **Interior C^2-Estimates.**

Lemma 5.4 (Interior Estimates)**.** *An admissible solution of* (1.4) *has uniformly bounded second derivatives.*

Proof. Note the admissibility implies that w_{ij} is positive definite. This implies a lower bound on the eigenvalues of u_{ij}.

For $\lambda \gg 1$ to be chosen sufficiently large, we maximize the functional

$$W = \log\left(w_{ij}\eta^i\eta^j\right) + \lambda\chi$$

over M^n and all $\left(\eta^i\right)$ with $g_{ij}\eta^i\eta^j = 1$. In view of the boundary estimates obtained above, we may assume that W attains its maximum at an interior point x_0 of M^n. As in [2] we may choose normal coordinates around x_0 and an appropriate extension of $\left(\eta^i\right)$ corresponding to the maximum value of W. In this way, we can pretend that w_{11} is a scalar function that equals $w_{ij}\eta^i\eta^j$ at x_0 and we obtain

$$0 = W_i = \frac{1}{w_{11}}w_{11;i} + \lambda\chi_i \tag{5.9}$$

and

$$0 \ge W_{ij} = \frac{1}{w_{11}}w_{11;ij} - \frac{1}{w_{11}^2}w_{11;i}w_{11;j} + \lambda\chi_{ij} \tag{5.10}$$

in the matrix sense, $1 \le i,\, j \le n$. Here and below, all quantities are evaluated at x_0. We may assume that w_{ij} is diagonal and $w_{11} \ge 1$. Differentiating (1.4) yields

$$w^{ij}w_{ij;k} = f_k + f_z u_k, \tag{5.11}$$

$$w^{ij}w_{ij;11} - w^{ik}w^{jl}w_{ij;1}w_{kl;1} = f_{11} + 2f_{1z}u_1 + f_{zz}u_1u_1 + f_z u_{11}. \tag{5.12}$$

Combining the convexity assumption on χ, (5.10) and (5.12) gives

$$\begin{aligned}
0 \geq & \frac{1}{w_{11}} w^{ij} w_{11;ij} - \frac{1}{w_{11}^2} w^{ij} w_{11;i} w_{11;j} + \lambda \operatorname{tr} w^{ij} \\
= & \frac{1}{w_{11}} w^{ij} (w_{11;ij} - w_{ij;11}) \\
& + \frac{1}{w_{11}} w^{ik} w^{jl} w_{ij;1} w_{kl;1} - \frac{1}{w_{11}^2} w^{ij} w_{11;i} w_{11;j} \\
& + \frac{1}{w_{11}} (f_{11} + 2 f_{1z} u_1 + f_{zz} u_1 u_1 + f_z u_{11}) + \lambda \operatorname{tr} w^{ij}.
\end{aligned} \tag{5.13}$$

It will be convenient to decompose w_{ij} as follows

$$\begin{aligned}
w_{ij} = & u_{ij} + r_{ij}, \\
r_{ij} = & \psi u_i u_j - \tfrac{1}{2} \psi |\nabla u|^2 g_{ij} + T_{ij}.
\end{aligned} \tag{5.14}$$

The quantity r_{ij} is a priorily bounded, so the right-hand side of (5.12) is bounded from below by $-c(1 + w_{11})$.

Let us first consider some terms involving at most third derivatives of u

$$\begin{aligned}
w^{ik} w^{jl} w_{ij;1} w_{kl;1} - \frac{1}{w_{11}} w^{ij} w_{11;i} w_{11;j} \geq & \frac{1}{w_{11}} w^{ij} (w_{i1;1} w_{j1;1} - w_{11;i} w_{11;j}) \\
= & \frac{1}{w_{11}} w^{ij} ((u_{i11} + r_{i1;1})(u_{j11} + r_{j1;1}) - (u_{11i} + r_{11;i})(u_{11j} + r_{11;j})) \\
\geq & \frac{1}{w_{11}} w^{ij} (u_{i11} u_{j11} - u_{11i} u_{11j} + 2 u_{i11} r_{j1;1} - 2 u_{11i} r_{11;j} - r_{11;i} r_{11;j}).
\end{aligned} \tag{5.15}$$

We will bound each term on the right-hand side individually. The term $r_{11;i}$ is of the form $c_i + c^k u_{ki}$. We rewrite $u_{ki} = w_{ki} - r_{ki}$, use $w^{ij} w_{jk} = \delta^i_k$ and obtain

$$\left| \frac{1}{w_{11}} w^{ij} r_{11;i} r_{11;j} \right| \leq \frac{1}{w_{11}} c \left(1 + w_{11} + \operatorname{tr} w^{ij}\right).$$

Note that w_{ij} is diagonal, so the maximality of W implies $|w_{ij}| \leq w_{11}$ for any i, j. We use (5.14), (5.9) and rewrite $r_{11;i}$ as above

$$\begin{aligned}
-2 \frac{1}{w_{11}} w^{ij} u_{11i} r_{11;j} = & -2 \frac{1}{w_{11}} w^{ij} (w_{11i} - r_{11;i}) r_{11;j} \\
= & 2 \lambda w^{ij} \chi_i r_{11;j} + 2 \frac{1}{w_{11}} w^{ij} r_{11;i} r_{11;j} \\
\geq & -c\lambda \left(1 + \operatorname{tr} w^{ij}\right) - \frac{1}{w_{11}} c \left(1 + w_{11} + \operatorname{tr} w^{ij}\right).
\end{aligned}$$

To estimate the next term, we use (2.3), (5.14), (5.9) and the fact that the second derivatives of u in $r_{j1;1}$ appear with a factor ψ

$$\begin{aligned}\frac{2}{w_{11}}w^{ij}u_{i11}r_{j1;1} =&\frac{2}{w_{11}}w^{ij}\left(w_{11;i}-r_{11;i}+u_ag^{ab}R_{b1i1}\right)r_{j1;1}\\ =&-2\lambda w^{ij}\chi_i\left(c_j+\psi c_j^kw_{k1}+c^kw_{kj}\right)\\ &-\frac{c}{w_{11}}w^{ij}\left(c_i+c^kw_{ki}\right)\left(c_j+\psi c_j^kw_{k1}+c^kw_{kj}\right)\\ \geq&-c\lambda\left(1+\operatorname{tr}w^{ij}+\psi w_{11}\operatorname{tr}w^{ij}\right)-c\left(1+\operatorname{tr}w^{ij}\right).\end{aligned}$$

We interchange third covariant derivatives and get

$$\begin{aligned}&\frac{1}{w_{11}}w^{ij}(u_{i11}u_{j11}-u_{11i}u_{11j})\\ &=\frac{1}{w_{11}}w^{ij}\left(u_{i11}u_{j11}-\left(u_{i11}+u_ag^{ab}R_{b11i}\right)\left(u_{j11}+u_cg^{cd}R_{d11j}\right)\right)\\ &\geq-2\frac{1}{w_{11}}w^{ij}u_{i11}u_ag^{ab}R_{b11j}-c\frac{1}{w_{11}}\operatorname{tr}w^{ij}\\ &=2\lambda w^{ij}\chi_iu_ag^{ab}R_{b11j}+2\frac{1}{w_{11}}w^{ij}r_{i1;1}u_ag^{ab}R_{b11j}-c\frac{1}{w_{11}}\operatorname{tr}w^{ij}\\ &\geq-c(1+\lambda)\left(1+\operatorname{tr}w^{ij}\right).\end{aligned}$$

Recall that $\operatorname{tr}w^{ij}$ is bounded below by a positive constant. We employ (5.15) and get the estimate

$$\frac{1}{w_{11}}w^{ik}w^{jl}w_{ij;1}w_{kl;1}-\frac{1}{w_{11}^2}w^{ij}w_{11;i}w_{11;j}\geq-c\left(1+\lambda\psi+\frac{\lambda}{w_{11}}\right)\operatorname{tr}w^{ij}. \tag{5.16}$$

Next, we consider the terms in (5.13) involving fourth derivatives. Equation (2.3) implies

$$\begin{aligned}u_{11ij}=&u_{ij11}+u_{a1}g^{ab}R_{bi1j}+u_ag^{ab}R_{bi1j;1}+u_{1a}g^{ab}R_{bij1}+u_{ia}g^{ab}R_{b1j1}\\ &+u_{aj}g^{ab}R_{b11i}+u_ag^{ab}R_{b11i;j}\\ \geq&u_{ij11}-c_{ij}(1+w_{11}).\end{aligned}$$

We use (5.14)

$$w^{ij}(w_{11;ij}-w_{ij;11})=w^{ij}(u_{11ij}-u_{ij11})+w^{ij}(r_{11;ij}-r_{ij;11})$$

$$
\begin{aligned}
&\geq w^{ij}(r_{11;ij}-r_{ij;11})-cw_{11}\operatorname{tr}w^{ij}\\
&= w^{ij}\left(\psi_{ij}u_1^2+4\psi_i u_1u_{1j}+2\psi u_{1j}u_{1i}+2\psi u_1u_{1ij}\right)\\
&\quad+w^{ij}\left(-\psi_{11}u_iu_j-4\psi_1u_{i1}u_j-2\psi u_{1i}u_{1j}-2\psi u_iu_{j11}\right)\\
&\quad+w^{ij}\left(-\tfrac12\psi_{ij}|\nabla u|^2g_{11}-2\psi_iu^ku_{kj}g_{11}-\psi u^k_ju_{ki}g_{11}-\psi u^ku_{kij}g_{11}\right)\\
&\quad+w^{ij}\left(\tfrac12\psi_{11}|\nabla u|^2g_{ij}+2\psi_1u^ku_{k1}g_{ij}+\psi u^k_1u_{k1}g_{ij}+\psi u^ku_{k11}g_{ij}\right)\\
&\quad+w^{ij}\left(T_{11;ij}-T_{ij;11}\right)-cw_{11}\operatorname{tr}w^{ij}.
\end{aligned}
$$

Some terms cancel. We use (5.14) and the fact that w^{ij} is the inverse of w_{ij}. Then we interchange covariant third derivatives (2.3) and employ once again (5.14)

$$
\begin{aligned}
&w^{ij}(w_{11;ij}-w_{ij;11})\geq\\
&\quad\geq w^{ij}\left(2\psi u_1u_{1ij}-2\psi u_iu_{j11}-\psi u^ku_{kij}g_{11}+\psi u^ku_{k11}g_{ij}\right)\\
&\quad\quad+w^{ij}\left(-\psi u^k_ju_{ki}g_{11}+\psi u^k_1u_{k1}g_{ij}\right)-cw_{11}\operatorname{tr}w^{ij}\\
&\quad=2\psi u_1w^{ij}u_{ij1}+2\psi u_1w^{ij}u_ag^{ab}R_{bi1j}\\
&\quad\quad-\psi g_{11}u^kw^{ij}u_{ijk}-\psi g_{11}u^kw^{ij}u_ag^{ab}R_{bikj}\\
&\quad\quad-2\psi u_iw^{ij}u_{11j}-2\psi u_iw^{ij}u_ag^{ab}R_{b1j1}\\
&\quad\quad+\psi u^ku_{11k}\operatorname{tr}w^{ij}+\psi u^ku_ag^{ab}R_{b1k1}\operatorname{tr}w^{ij}\\
&\quad\quad-\psi g_{11}w^{ij}(w_{ik}-r_{ik})(w_{jl}-r_{jl})g^{kl}\\
&\quad\quad+\psi(w_{1k}-r_{1k})(w_{1l}-r_{1l})g^{kl}\operatorname{tr}w^{ij}-cw_{11}\operatorname{tr}w^{ij}.
\end{aligned}
$$

We replace third derivatives of u by derivatives of w_{ij}. Equations (5.11) and (5.9) allow to replace these terms by terms involving at most second derivatives of u

$$
\begin{aligned}
&w^{ij}(w_{11;ij}-w_{ij;11})\geq\\
&\quad\geq 2\psi u_1w^{ij}w_{ij;1}-2\psi u_1w^{ij}r_{ij;1}-\psi g_{11}u^kw^{ij}w_{ij;k}+\psi g_{11}u^kw^{ij}r_{ij;k}\\
&\quad\quad-2\psi u_iw^{ij}w_{11;j}+2\psi u_iw^{ij}r_{11;j}+\psi u^kw_{11;k}\operatorname{tr}w^{ij}-\psi u^kr_{11;k}\operatorname{tr}w^{ij}\\
&\quad\quad+\psi w_{11}^2\operatorname{tr}w^{ij}-cw_{11}\operatorname{tr}w^{ij}\\
&\quad\geq 2\lambda\psi w_{11}w^{ij}u_i\chi_j-\lambda\psi w_{11}u^k\chi_k\operatorname{tr}w^{ij}+\psi w_{11}^2\operatorname{tr}w^{ij}-cw_{11}\operatorname{tr}w^{ij}\\
&\quad\geq -c\lambda\psi w_{11}\operatorname{tr}w^{ij}+\psi w_{11}^2\operatorname{tr}w^{ij}-cw_{11}\operatorname{tr}w^{ij}.
\end{aligned}
$$

This gives

$$
\frac{1}{w_{11}}w^{ij}(w_{11;ij}-w_{ij;11})\geq -c\lambda\psi\operatorname{tr}w^{ij}+\psi w_{11}\operatorname{tr}w^{ij}-c\operatorname{tr}w^{ij}. \tag{5.17}
$$

We estimate the respective terms in (5.13) using (5.16) and (5.17) and obtain

$$0 \geq \left\{ \psi(w_{11} - c\lambda) + \left(\lambda - c - \frac{c\lambda}{w_{11}} \right) \right\} \operatorname{tr} w^{ij}. \tag{5.18}$$

Assume that all c's in (5.18) are equal. Now we fix λ equal to $c+1$. Then (5.18) implies that w_{11} is bounded above. □

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