ASYMPTOTICS FOR GEOMETRIC EVOLUTION EQUATIONS

OLIVER C. SCHNÜRER

Abstract. In these lectures, we study hypersurfaces that solve geometric evolution equations. The normal velocity of the evolving hypersurfaces depends on their Gauß or mean curvature. We will discuss the asymptotic behaviour of those solutions for large times. Depending on the initial conditions there are many known results that describe different ways solutions might evolve. We will focus on two aspects related to the work of the lecturer:

- Hypersurfaces that become round as they shrink to points.
- The behaviour of complete non-compact solutions.

Contents

1. Overview and Plan for the Winter School 1
2. Differential Geometry of Submanifolds 2
3. Evolving Submanifolds 5
4. Evolution Equations for Submanifolds 10
5. Convex Hypersurfaces 15
6. Entire solutions to Gauß curvature flows 22
7. Mean Curvature Flow of Complete Graphs 27
Appendix A. Parabolic Maximum Principles 31
Appendix B. Some Linear Algebra 32
References 33

1. Overview and Plan for the Winter School

We consider flow equations that deform hypersurfaces according to their curvature.

If $X_0 : M^n \to \mathbb{R}^{n+1}$ is an embedding of an n-dimensional manifold, we can define principal curvatures $(\lambda_i)_{1\le i\le n}$ and a normal vector ν. We deform the embedding vector according to

$$\begin{cases} \frac{d}{dt}X = -F\nu, \\ X(\cdot,0) = X_0, \end{cases}$$

where F is a symmetric function of the principal curvatures, e. g. the mean curvature $H = \lambda_1 + \cdots + \lambda_n$. In this way, we obtain a family $X(\cdot,t)$ of embeddings and study their behavior near singularities and for large times. We consider hypersurfaces that contract to a point in finite time and, after appropriate rescaling, to a round sphere. Graphical solutions are shown to exist for all times or to disappear to infinity.

Date: October 18, 2017.

2000 *Mathematics Subject Classification.* 53C44.

Lecture notes for the “Winter School on Geometric Evolution Equations and Related Topics”, Regensburg, 08.-10.10.2012, organized by H. Abels, H. Garcke and L. Müller.

Nowadays classical results in this was obtained by G. Huisken [17] for mean curvature flow.

Remark 1.1.
(i) We will use geometric flow equations as a tool to deform a manifold.
(ii) The flow equations considered are parabolic equations like the heat equation.
(iii) In order to control the behavior of the flow, we will look for properties of the manifold that are preserved under the flow.
(iv) For precise control on the behavior of the evolving manifold, we will look for quantities that are monotone and have geometric significance, i. e. their boundedness implies geometric properties of the evolving manifold.

Plan for the Winter School. These notes contain a lot of background material that will be covered as necessary during this winter school. Each day, we will give an overview of some geometric problem. We will also study the details for the main estimate involved.

- Convex surfaces contracting to a round point and an estimate for Gauß curvature flow, Theorem 5.6, measuring the deviation from being umbilic.
- Gauß curvature flows of entire graphs and the model case for a local C^2-bound, Theorem 6.8.
- Mean curvature flow of complete graphs and local C^1-bounds, Theorem 7.6.

2. Differential Geometry of Submanifolds

We will only consider hypersurfaces in Euclidean space.

We use $X = X(x, t) = (X^\alpha)_{1\le\alpha\le n+1}$ to denote the time-dependent embedding vector of a manifold M^n into $\mathbb{R}^{n+1}$ and $\frac{d}{dt}X = \dot{X}$ for its total time derivative. Set $M_t := X(M, t) \subset \mathbb{R}^{n+1}$. We will often identify an embedded manifold with its image. We will assume that X is smooth. Assume furthermore that M^n is smooth, orientable, connected, complete and $\partial M^n = \emptyset$. We choose $\nu = \nu(x) = (\nu^\alpha)_{1\le\alpha\le n+1}$ to be the outer (or downward pointing) unit normal vector to M_t at $x \in M_t$. The embedding $X(\cdot, t)$ induces at each point on M_t a metric $(g_{ij})_{1\le i,\,j\le n}$ and a second fundamental form $(h_{ij})_{1\le i,\,j\le n}$. Let (g^{ij}) denote the inverse of (g_{ij}). These tensors are symmetric. The principal curvatures $(\lambda_i)_{1\le i\le n}$ are the eigenvalues of the second fundamental form with respect to that metric. That is, at $p \in M$, for each principal curvature λ_i, there exists $0 \ne \xi \in T_pM \cong \mathbb{R}^n$ such that

$$\lambda_i \sum_{l=1}^n g_{kl}\xi^l = \sum_{l=1}^n h_{kl}\xi^l \text{ or, equivalently, } \lambda_i\xi^l = \sum_{k,r=1}^n g^{lk}h_{kr}\xi^r.$$

As usual, eigenvalues are listed according to their multiplicity. A hypersurface is called strictly convex, if all principal curvatures are strictly positive. The inverse of the second fundamental form is denoted by $\left(\tilde{h}^{ij}\right)_{1\le i,\,j\le n}$.

Latin indices range from 1 to n and refer to geometric quantities on the hypersurface, Greek indices range from 1 to $n+1$ and refer to components in the ambient space $\mathbb{R}^{n+1}$. In $\mathbb{R}^{n+1}$, we will always choose Euclidean coordinates. We use the Einstein summation convention for repeated upper and lower indices. Latin indices are raised and lowered with respect to the induced metric or its inverse $\left(g^{ij}\right)$, for Greek indices we use the flat metric $(\overline{g}_{\alpha\beta})_{1\le\alpha,\beta\le n+1} = (\delta_{\alpha\beta})_{1\le\alpha,\beta\le n+1}$ of $\mathbb{R}^{n+1}$. So the defining equation for the principal curvatures becomes $\lambda_i g_{kl}\xi^l = h_{kl}\xi^l$.

Denoting by $\langle\cdot,\,\cdot\rangle$ the Euclidean scalar product in $\mathbb{R}^{n+1}$, we have

$$g_{ij} = \langle X_{,\,i},\, X_{,\,j}\rangle = X^\alpha_{,\,i}\delta_{\alpha\beta}X^\beta_{,\,j},$$

where we used indices, preceded by commas, to denote partial derivatives. We write indices, preceded by semi-colons, e. g. $h_{ij;\,k}$ or $v_{;k}$, to indicate covariant differentiation with respect to the induced metric. Later, we will also drop the commas and semi-colons, if the meaning is clear from the context. We set $X^\alpha_{;i} \equiv X^\alpha_{,i}$ and

$$X^\alpha_{;\,ij} = X^\alpha_{,\,ij} - \Gamma^k_{ij} X^\alpha_{,\,k}, \tag{2.1}$$

where

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl} (g_{il,\,j} + g_{jl,\,i} - g_{ij,\,l})$$

are the Christoffel symbols of the metric (g_{ij}). So $X^\alpha_{;ij}$ becomes a tensor.

The Gauß formula relates covariant derivatives of the position vector to the second fundamental form and the normal vector

$$X^\alpha_{;\,ij} = -h_{ij}\nu^\alpha. \tag{2.2}$$

The Weingarten equation allows to compute derivatives of the normal vector

$$\nu^\alpha_{;\,i} = h^k_i X^\alpha_{;\,k}. \tag{2.3}$$

We can use the Gauß formula (2.2) or the Weingarten equation (2.3) to compute the second fundamental form.

Symmetric functions of the principal curvatures are well-defined, we will use the mean curvature $H = \lambda_1 + \ldots + \lambda_n$, the square of the norm of the second fundamental form $|A|^2 = \lambda_1^2 + \ldots + \lambda_n^2$, $\operatorname{tr} A^k = \lambda_1^k + \ldots + \lambda_n^k$, and the Gauß curvature $K = \lambda_1 \cdot \ldots \cdot \lambda_n$. It is often convenient to choose coordinate systems such that, at a fixed point, the metric tensor equals the Kronecker delta, $g_{ij} = \delta_{ij}$, and (h_{ij}) is diagonal, $(h_{ij}) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, e. g.

$$\sum \lambda_k h^2_{ij;k} = \sum_{i,\,j,\,k=1}^{n} \lambda_k h^2_{ij;k} = h^{kl} h^i_{j;\,k} h^j_{i;\,l} = h_{rs} h_{ij;\,k} h_{ab;\,l} g^{ia} g^{jb} g^{rk} g^{sl}.$$

Whenever we use this notation, we will also assume that we have fixed such a coordinate system.

A normal velocity F can be considered as a function of $(\lambda_1, \ldots, \lambda_n)$ or $(h_{ij},\, g_{ij})$. If $F(\lambda_i)$ is symmetric and smooth, then $F(h_{ij}, g_{ij})$ is also smooth [13, Theorem 2.1.20]. We set $F^{ij} = \frac{\partial F}{\partial h_{ij}}$, $F^{ij,\,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}$. Note that in coordinate systems with diagonal h_{ij} and $g_{ij} = \delta_{ij}$ as mentioned above, F^{ij} is diagonal. For $F = |A|^2$, we have $F^{ij} = 2h^{ij} = 2\lambda_i g^{ij}$, and for $F = K^\alpha$, $\alpha > 0$, we have $F^{ij} = \alpha K^\alpha \tilde{h}^{ij} = \alpha K^\alpha \lambda_i^{-1} g^{ij}$.

The Gauß equation expresses the Riemannian curvature tensor of the hypersurface in terms of the second fundamental form

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}. \tag{2.4}$$

As we use only Euclidean coordinate systems in $\mathbb{R}^{n+1}$, $h_{ij;\,k}$ is symmetric according to the Codazzi equations.

The Ricci identity allows to interchange covariant derivatives. We will use it for the second fundamental form

$$h_{ik;\,lj} = h_{ik;\,jl} + h^a_k R_{ailj} + h^a_i R_{aklj}. \tag{2.5}$$

For tensors A and B, $A_{ij} \ge B_{ij}$ means that $(A_{ij} - B_{ij})$ is positive definite.

Finally, we use c to denote universal, estimated constants.

Graphical Submanifolds.

Lemma 2.1. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be smooth. Then* graph u *is a submanifold in $\mathbb{R}^{n+1}$. The metric g_{ij}, the lower unit normal vector ν, the second fundamental form h_{ij},* *the mean curvature H, and the Gauß curvature K are given by*

$$
\begin{aligned}
g_{ij} =& \delta_{ij} + u_i u_j, \\
g^{ij} =& \delta^{ij} - \frac{u^i u^j}{1+|Du|^2}, \\
\nu =& \frac{((u_i), -1)}{\sqrt{1+|Du|^2}} \equiv \frac{((u_i), -1)}{v}, \\
h_{ij} =& \frac{u_{ij}}{\sqrt{1+|Du|^2}} \equiv \frac{u_{ij}}{v}, \\
H =& \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right),
\end{aligned}
$$

and

$$
K = \frac{\det D^2 u}{\left(1+|Du|^2\right)^{\frac{n+2}{2}}},
$$

where $u_i \equiv \frac{\partial u}{\partial x^i}$ and $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$. Note that in Euclidean space, we don't need to distinguish between Du and ∇u.

Proof.

(i) We use the embedding vector $X(x) := (x, u(x))$, $X : \mathbb{R}^n \to \mathbb{R}^{n+1}$. The induced metric is the pull-back of the metric in Euclidean $\mathbb{R}^{n+1}$, $g := X^* g_{\mathbb{R}^{n+1}_{\text{Eucl.}}}$. We have $X_{,i} = (e_i, u_i)$. Hence

$$
g_{ij} = X^\alpha_{,i} \delta_{\alpha\beta} X^\beta_{,j} = \langle X_{,i}, X_{,j}\rangle = \langle (e_i, u_i), (e_j, u_j)\rangle = \delta_{ij} + u_i u_j.
$$

(ii) It is easy to check, that g^{ij} is the inverse of g_{ij}. Note that $u^i := \delta^{ij} u_j$, i.e., we lift the index with respect to the flat metric. It is convenient to choose a coordinate system such that $u_i = 0$ for $i < n$.

(iii) The vectors $X_{,i} = (e_i, u_i)$ are tangent to $\operatorname{graph} u$. The vector $((-u_i), 1) \equiv (-Du, 1)$ is orthogonal to these vectors, hence, up to normalization, a unit normal vector.

(iv) We combine (2.1), (2.2) and compute the scalar product with ν to get

$$
\begin{aligned}
h_{ij} =& -\langle X_{;ij}, \nu\rangle = -\langle X_{,ij} - \Gamma^k_{ij} X_{,k}, \nu\rangle = -\langle X_{,ij}, \nu\rangle \\
=& -\left\langle (0, u_{ij}), \frac{((u_i), -1)}{v}\right\rangle = \frac{u_{ij}}{v}.
\end{aligned}
$$

(v) We obtain

$$
\begin{aligned}
H =& \sum_{i=1}^n \lambda_i = g^{ij} h_{ij} = \left(\delta^{ij} - \frac{u^i u^j}{1+|Du|^2}\right) \frac{u_{ij}}{\sqrt{1+|Du|^2}} \\
=& \frac{\delta^{ij} u_{ij}}{\sqrt{1+|Du|^2}} - \frac{u^i u^j u_{ij}}{\left(1+|Du|^2\right)^{3/2}} \\
=& \frac{\Delta u}{\sqrt{1+|Du|^2}} - \frac{u^i u^j u_{ij}}{\left(1+|Du|^2\right)^{3/2}}
\end{aligned}
$$

and, on the other hand,

$$\begin{aligned}\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) &= \sum_{i=1}^n \frac{\partial}{\partial x^i}\frac{u_i}{\sqrt{1+|Du|^2}}\\ &= \sum_{i=1}^n \frac{u_{ii}}{\sqrt{1+|Du|^2}} - \sum_{i,j=1}^n \frac{u_i u_j u_{ji}}{(1+|Du|^2)^{3/2}}\\ &= H.\end{aligned}$$

(vi) From the defining equation for the principal curvatures, we obtain

$$\begin{aligned}K = \prod_{i=1}^n \lambda_i &= \det\left(g^{ij}h_{jk}\right) = \det g^{ij}\cdot\det h_{ij} = \frac{\det h_{ij}}{\det g_{ij}}\\ &= \frac{v^{-n}\det u_{ij}}{v^2} = \frac{\det D^2u}{(1+|Du|^2)^{\frac{n+2}{2}}}.\end{aligned}$$

□

Exercise 2.2 (Spheres)**.** The lower part of a sphere of radius R is locally given as graph u with $u: B_R(0)\to\mathbb{R}$ defined by $u(x) := -\sqrt{R^2-|x|^2}$. Compute explicitly for that example all the quantities mentioned in Lemma 2.1 and the principal curvatures.

Exercise 2.3. Give a geometric definition of the (principal) curvature of a curve in $\mathbb{R}^2$ in terms of a circle approximating that curve in an optimal way.

Use the min-max characterization of eigenvalues to extend that geometric definition to n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$.

Exercise 2.4 (Rotationally symmetric graphs)**.**
Assume that the function $u:\mathbb{R}^n\to\mathbb{R}$ is smooth and $u(x)=u(y)$, if $|x|=|y|$. Then $u(x)=f(|x|)$ for some $f:\mathbb{R}_+\to\mathbb{R}$. Compute once again all the geometric quantities mentioned in Lemma 2.1.

3. Evolving Submanifolds

General Definition. We will only consider the evolution of manifolds of dimension n embedded into $\mathbb{R}^{n+1}$, i. e. the evolution of hypersurfaces in Euclidean space. (Mean curvature flow is also considered for manifolds of arbitrary codimension. Another generalization is to study flow equations of hypersurfaces immersed into a (Riemannian or Lorentzian) manifold.)

Definition 3.1. Let M^n denote an orientable manifold of dimension n. Let $X(\cdot,t): M^n\to\mathbb{R}^{n+1}$, $0\le t\le T\le\infty$, be a smooth family of smooth embeddings. Let ν denote one choice of the normal vector field along $X(M^n,t)$. Then $M_t := X(M^n,t)$ is said to move with normal velocity F, if

$$\frac{d}{dt}X = -F\nu \quad \text{in } M^n\times[0,T).$$

In codimension 1, we often don't need to assume that M^n is orientable.

Remark 3.2. Let $X: M^n\to N^{n+1}$ be a C^2-immersion and $H_1(N;\mathbb{Z}/2\mathbb{Z})=0$. Assume that X is proper, $X^{-1}(\partial N)=\partial M$, and X is transverse to ∂N. Then $N\setminus f(M)$ is not connected [10]. Hence, if M^n is closed and embedded in $\mathbb{R}^{n+1}$, M^n is orientable.

In the following we will often identify an embedded submanifold and its image under the embedding.

Evolution of Graphs.

Lemma 3.3. *Let $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a smooth function such that* $\operatorname{graph} u$ *evolves according to $\frac{d}{dt}X = -F\nu$. Then*

$$\dot{u} = \sqrt{1 + |Du|^2} \cdot F.$$

Proof. Beware of assuming that considering the $(n+1)$-st component in the evolution equation $\frac{d}{dt}X = -F\nu$ were equal to $\dot{u}$ as a hypersurface evolving according to $\frac{d}{dt}X = -F\nu$ does not only move in vertical direction but also in horizontal direction.

Let p denote a point on the abstract manifold embedded via X into $\mathbb{R}^{n+1}$. As our embeddings are graphical, we see that

$$X(p, t) = (x(p, t), u(x(p, t), t)).$$

We consider the scalar product of both sides of the evolution equation with ν and obtain

$$F = \langle F\nu, \nu\rangle = \left\langle -\frac{d}{dt}X, \nu \right\rangle = -\left\langle \left(\left(\dot{x}^k \right), u_i \dot{x}^i + \dot{u} \right), \frac{((u_i), -1)}{\sqrt{1 + |Du|^2}} \right\rangle = \frac{\dot{u}}{\sqrt{1 + |Du|^2}}.$$

□

Corollary 3.4. *Let $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a smooth function such that* $\operatorname{graph} u$ *solves mean curvature flow $\frac{d}{dt}X = -H\nu$. Then*

$$\dot{u} = \sqrt{1 + |Du|^2} \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right).$$

Exercise 3.5 (Rotationally symmetric translating solutions)**.** Let $u := \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be rotationally symmetric. Assume that $\operatorname{graph} u$ is a translating solution to mean curvature flow $\frac{d}{dt}X = -H\nu$, i. e. a solution such that $\dot{u}$ is constant.

Why does it suffice to consider the case $\dot{u} = 1$?

Similar to Exercise 2.4, derive an ordinary differential equation for translating rotationally symmetric solutions to mean curvature flow.

Remark 3.6. Consider a physical system consisting of a domain $\Omega \subset \mathbb{R}^3$. Assume that the energy of the system is proportional to the surface area of $\partial\Omega$. Then the L^2-gradient flow for the area is mean curvature flow. We check that in a model case for graphical solutions in Lemma 3.7.

Lemma 3.7. *Let $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be smooth. Assume that $u(x, 0) \equiv 0$ for $|x| \geq R$. Then the surface area is maximally reduced among all normal velocities F with given L^2-norm, if the normal velocity of* $\operatorname{graph} u$ *is given by H, i. e. if $\dot{u} = \sqrt{1 + |Du|^2}\, H$.*

Proof. The area of $\operatorname{graph} u(\cdot, t)|_{B_R}$ is given by

$$A(t) = \int\limits_{B_R} \sqrt{1 + |Du|^2}\, dx.$$

Define the induced area element $d\mu$ by $d\mu := \sqrt{1+|Du|^2}\,dx$. We obtain using integration by parts

$$\begin{aligned}\frac{d}{dt}A(t)\bigg|_{t=0} &= \int\limits_{B_R}\frac{d}{dt}\sqrt{1+|Du|^2}\,dx\,\bigg|_{t=0} = \int\limits_{B_R(0)}\frac{1}{\sqrt{1+|Du|^2}}\langle Du, D\dot u\rangle\bigg|_{t=0}\\ &= -\int\limits_{B_R}\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\frac{\dot u}{v}\cdot v\,dx\,\bigg|_{t=0} = -\int\limits_{B_R}H\,F\,d\mu\,\bigg|_{t=0}\\ &\geq -\left(\int\limits_{B_R}H^2\,d\mu\right)^{1/2}\left(\int\limits_{B_R}F^2\,d\mu\right)^{1/2}\bigg|_{t=0}.\end{aligned}$$

Here, we have used Hölder's inequality $\|ab\|_{L^1} \leq \|a\|_{L^2}\cdot\|b\|_{L^2}$. There, we get equality precisely if a and b differ only by a multiplicative constant. Hence the surface area is reduced most efficiently among all normal velocities F with $\|F\|_{L^2} = \|H\|_{L^2}$, if we choose $F = H$. In this sense, mean curvature flow is the L^2-gradient flow for the area integral. □

Examples.

Lemma 3.8. *Consider mean curvature flow, i. e. the evolution equation* $\frac{d}{dt}X = -H\nu$, *with* $M_0 = \partial B_R(0)$. *Then a smooth solution exists for* $0 \leq t < T := \frac{1}{2n}R^2$ *and is given by* $M_t = \partial B_{r(t)}(0)$ *with* $r(t) = \sqrt{2n(T-t)} = \sqrt{R^2 - 2nt}$.

Proof. The mean curvature of a sphere of radius $r(t)$ is given by $H = \frac{n}{r(t)}$. Hence we obtain a solution to mean curvature flow, if $r(t)$ fulfills

$$\dot r(t) = \frac{-n}{r(t)}.$$

A solution to this ordinary differential equation is given by $r(t) = \sqrt{2n(T-t)}$.

(The theory of partial differential equations implies that this solution is actually unique and hence no solutions exist that are not spherical.) □

Exercise 3.9. Find a solution to mean curvature flow with $M_0 = \partial B_R(0)\times\mathbb{R}^k \subset \mathbb{R}^l\times\mathbb{R}^k$. This includes in particular cylinders. Note that for $k > 1$, it is not obvious, whether these solutions are unique.

Exercise 3.10. Find solutions for $\frac{d}{dt}X = -|A|^2\nu$, $\frac{d}{dt}X = -K\nu$, $\frac{d}{dt}X = \frac{1}{H}\nu$, and $\frac{d}{dt}X = \frac{1}{K}\nu$ if $M_0 = \partial B_R(0) \subset \mathbb{R}^{n+1}$, especially for $n = 2$.

Remark 3.11 (Level-set flow)**.** Let M_t be a family of smooth embedded hypersurfaces in $\mathbb{R}^{n+1}$ that move according to $\frac{d}{dt}X = -F\nu$ with $F > 0$. Impose the global assumption that each point $x \in \mathbb{R}^{n+1}$ belongs to at most one hypersurface M_t. Then we can (at least locally) define a function $u : \mathbb{R}^{n+1} \to \mathbb{R}$ by setting $u(x) = t$, if $x \in M_t$. That is $u(x)$ is the time, at which the hypersurface passes the point x. We obtain the equation $F \cdot |Du| = 1$.

If $F < 0$, we get $F\cdot|Du| = -1$.

This formulation is used to describe weak solutions, where singularities in the classical formulation occur. See for example [18], where the inverse mean curvature flow $F = -\frac{1}{H}$ is considered to prove the Riemannian Penrose inequality. Note that $H = \operatorname{div}\left(\frac{Du}{|Du|}\right)$ as the outer unit normal vector to a closed expanding hypersurface $M_t = \{u = t\}$ is given by $\frac{Du}{|Du|}$. According to (2.3), the divergence of the unit normal yields the mean curvature as the derivative of the unit normal in the direction of the unit normal vanishes. Hence the evolution equation $\frac{d}{dt}X = \frac{1}{H}\nu$ can be rewritten as

$$\operatorname{div}\left(\frac{Du}{|Du|}\right) = |Du|.$$

Mean curvature flow can be rewritten as $|Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) = -1$.

Exercise 3.12. Verify the formula for the mean curvature in the level-set formulation. Compute level-set solutions to the flow equations $\frac{d}{dt}X = -H\nu$ and $\frac{d}{dt}X = \frac{1}{H}\nu$, where u depends only on $|x|$, i. e. the hypersurfaces M_t are spheres centered at the origin. Compare the result to your earlier computations.

We will use the level-set formulation to study a less trivial solution to mean curvature flow which can be written down in closed form.

Exercise 3.13 (Paper-clip solution)**.** Let $v \neq 0$. Consider the set

$$M_t := \left\{(x, y) \in \mathbb{R}^2 : e^{v^2 t} \cosh(vy) = \cos(vx)\right\}.$$

Show that M_t solves mean curvature flow. Describe the shape of M_t for $t \to -\infty$ and for $t \uparrow 0$ (after appropriate rescaling).

Compare this to Theorem 5.1.

Note that you may also rewrite solutions equivalently (on an appropriate domain) as

$$y_\pm := \frac{1}{v} \log\left(\cos(vx) \pm \sqrt{\cos^2(vx) - e^{2v^2 t}}\right) - vt.$$

Hint: You should obtain $t_x = u_x = -\frac{\sin(vx)}{v\cos(vx)}$ and $u_y = -\frac{\sinh(vy)}{v\cosh(vy)}$.

Short-Time Existence and Avoidance Principle. In the case of closed initial hypersurfaces, short-time existence is guaranteed by the following

Theorem 3.14 (Short-time existence)**.** *Let $X_0 : M^n \to \mathbb{R}^{n+1}$ be an embedding describing a smooth closed hypersurface. Let $F = F(\lambda_i)$ be smooth, symmetric, and $\frac{\partial F}{\partial \lambda_i} > 0$ everywhere on $X(M^n)$ for all i. Then the initial value problem*

$$\begin{cases} \frac{d}{dt}X = -F\nu, \\ X(\cdot, 0) = X_0 \end{cases}$$

has a smooth solution on some (short) time interval $[0, T)$, $T > 0$.

Idea of Proof. Represent solutions locally as graphs in a tubular neighborhood of $X_0(M^n)$. Then $\frac{\partial F}{\partial \lambda_i} > 0$ ensures that the evolution equation for the height function in this coordinate system is strictly parabolic. Linear theory and the implicit function theorem guarantee that there exists a solution on a short time interval.

For details see [19, Theorem 3.1]. □

Exercise 3.15. (i) Check, for which initial data the conditions in Theorem 3.14 are fulfilled if $F = H$, K, $|A|^2$, $-1/H$, $-1/K$.

(ii) Find examples of closed hypersurfaces such that
   a) $H > 0$,
   b) $K > 0$,
   c) H is not positive everywhere,
   d) $H > 0$, but K changes sign.

(iii) Show that on every smooth closed hypersurface $M^n \subset \mathbb{R}^{n+1}$, there is a point, where M^n is strictly convex, i. e. $\lambda_i > 0$ is fulfilled for every i.

On the other hand, starting with a closed hypersurface gives rise to solutions that exist at most on a finite time interval. This is a consequence of the following

Theorem 3.16 (Avoidance principle). *Let $F = F(\lambda_i)$ be smooth and symmetric. Let M_t^1 and $M_t^2 \subset \mathbb{R}^{n+1}$ be two embedded closed hypersurfaces and smooth solutions to a strictly parabolic flow equation $\frac{d}{dt}X = -F\nu$, i. e. $\frac{\partial F}{\partial \lambda_i} > 0$ during the flow. Assume that F, considered as a function of (D^2u, Du) for graphs, is elliptic on a set, which is convex and independent of Du. If M_0^1 and M_0^2 are disjoint, M_t^1 and M_t^2 can only touch if the respective normal vectors fulfill $\nu^1 = -\nu^2$ there. Hence, if M_0^1 is contained in a bounded component of $\mathbb{R}^{n+1} \setminus M_0^2$, then M_t^1 is contained in a bounded component of $\mathbb{R}^{n+1} \setminus M_t^2$ unless the hypersurfaces touch each other in a point with opposite normals.*

The technical condition on the convexity of the domain, where F, considered as a function of (D^2u, Du), is convex, is technical and always fulfilled for the evolution equations considered here (besides for the inverse mean curvature flow). It can be relaxed, but makes the proof less transparent. Only for $F = -\frac{1}{H}$, a separate argument is needed. (It suffices to choose coordinates such that $|Du| \ll 1$. Then interpolation does not destroy positivity of the denominator. The technical details are left as an exercise.)

The normal velocity F is a symmetric function of the principal curvatures. Thus it is well-defined, as the principal curvatures are defined only up to permutations.

We have considered F as a function of the principal curvatures. Writing an evolving hypersurface locally as graph u, we also wish to express F in terms of $\left(D^2u, Du\right)$. We continue to call this function F. In the cases considered here, it is clear from the explicit expressions, that F is also a smooth function of $\left(D^2u, Du\right)$. In general, this is a theorem [13, Theorem 2.1.20].

A similar statement is true for the condition $\frac{\partial F}{\partial \lambda_i} > 0$ and the ellipticity of $F\left(D^2, Du\right)$, i.e. $0 < \frac{F(r,p)}{r_{ij}} < \infty$ in the sense of matrices. Once again, it can be checked by direct computations for the normal velocities considered here, that these two statements are equivalent.

Proof of Theorem 3.16. Otherwise there would be some $t_0 > 0$ such that $M_{t_0}^2$ touches $M_{t_0}^1$ at some point $p \in \mathbb{R}^{n+1}$ with normal vectors $\nu^1 = \nu^2$ at p. Writing M_t^i locally as graph u^i over the common tangent hyperplane $T_pM_{t_0}^i \subset \mathbb{R}^{n+1}$, we see that the functions u^i fulfill $\dot{u}^i = F\left(D^2u^i, Du^i\right)$ for some strictly elliptic differential operator F corresponding to the normal velocity F. We may assume that $u^1 > u^2$ for $t < t_0$. The evolution equation for the difference $w := u_1 - u_2$ fulfills $w > 0$ for $t < t_0$ locally in space-time and $w(0, t_0)$, if we have $p = (0, 0)$ in our coordinate system. The evolution equation for w can be computed as follows

$$
\begin{aligned}
\dot{w} &= \dot{u}^1 - \dot{u}^2 = F\left(D^2u^1, Du^1\right) - F\left(D^2u^2, Du^2\right)\\
&= \int\limits_0^1 \frac{d}{d\tau} F\left(\tau D^2u^1 + (1-\tau)D^2u^2, \tau Du^1 + (1-\tau)Du^2\right) d\tau\\
&= \int\limits_0^1 \frac{\partial F}{\partial r_{ij}}(\ldots)\, d\tau \cdot \left(u^1 - u^2\right)_{ij} + \int\limits_0^1 \frac{\partial F}{\partial p_i}(\ldots)\, d\tau \cdot \left(u^1 - u^2\right)_i\\
&\equiv a^{ij}w_{ij} + b^iw_i.
\end{aligned}
$$

Hence we can apply the parabolic Harnack inequality or the strong parabolic maximum principle and see that it is impossible that $w(x,t) > 0$ for small $|x|$ and $t < t_0$, but $w(0, t_0) = 0$. Hence M_t^1 can't touch M_t^2 in a point, where $\nu^1 = \nu^2$. The theorem follows. ∎

Exercise 3.17. Show that the normal velocities as considered in Exercise 3.15 can be represented (in an appropriate domain) as smooth functions of $\left(D^2u, Du\right)$ for hypersurfaces that are locally represented as graph u.

Denote by Γ_F the set of $(\lambda_i) \subset \mathbb{R}^n$ such that $\frac{\partial F}{\partial \lambda_i} > 0$. Show that this set is a convex cone. Prove that F as a function $\left(D^2u, Du\right)$ is strictly elliptic precisely if the principal curvatures corresponding to (D^2u, Du) lie in Γ_F.

Corollary 3.18. *Let M_0 be a smooth closed embedded hypersurface in $\mathbb{R}^{n+1}$. Then a smooth solution M_t to $\frac{d}{dt}X = -H\nu$ can only exist on some finite time interval $[0, T)$, $T < \infty$.*

Proof. Choose a large sphere that encloses M_0. According to Lemma 3.8, that sphere shrinks to a point in finite time. Thus the solution M_t can exist smoothly at most up to that time. □

Exercise 3.19. Deduce similar corollaries for the normal velocities in Exercise 3.15. You may use Exercise 3.10.

Consider T maximal such that a smooth solution M_t as in Corollary 3.18 exists on $[0, T)$. Then the embedding vector X is uniformly bounded according to Theorem 3.16. Then some spatial derivative of the embedding $X(\cdot, t)$ has to become unbounded as $t \uparrow T$. For otherwise we could apply Arzelà-Ascoli and obtain a smooth limiting hypersurface M_T such that M_t converges smoothly to M_T as $t \uparrow T$. This, however, is impossibly, as Theorem 3.14 would allow to restart the flow from M_T. In this way, we could extend the flow smoothly all the way up to $T + \varepsilon$ for some $\varepsilon > 0$, contradicting the maximality of T.

It can often be shown that extending a solution beyond T is possible provided that $\|X(\cdot, t)\|_{C^2}$ is uniformly bounded. For mean curvature flow, this follows from explicit estimates. For other normal velocities, additional assumptions (the principal curvatures stay in a region, where F has nice properties) and Krylov-Safonov-estimates can imply such a result.

4. Evolution Equations for Submanifolds

In this chapter, we will compute evolution equations of geometric quantities, see e. g. [17, 19, 23].

For a family M_t of hypersurfaces solving the evolution equation

$$\frac{d}{dt}X = -F\nu \tag{4.1}$$

with $F = F(\lambda_i)$, where F is a smooth symmetric function, we have the following evolution equations.

Lemma 4.1. *The metric g_{ij} evolves according to*

$$\frac{d}{dt}g_{ij} = -2Fh_{ij}. \tag{4.2}$$

Proof. By definition, $g_{ij} = \langle X_{,i}, X_{,j}\rangle = X^\alpha_{,i}\delta_{\alpha\beta}X^\beta_{,j}$. We differentiate with respect to time. Derivatives of $\delta_{\alpha\beta}$ vanish. The term $X^\alpha_{,i}$ involves only partial derivatives. We obtain

$$\frac{d}{dt}g_{ij} = \left(\dot{X}^\alpha\right)_{,i}\delta_{\alpha\beta}X^\beta_{,j} + X^\alpha_{,i}\delta_{\alpha\beta}\left(\dot{X}^\beta\right)_{,j}$$

(we may exchange partial spatial and time derivatives)

$$= (-F\nu^\alpha)_{,i}\delta_{\alpha\beta}X^\beta_{,j} + X^\alpha_{,i}\delta_{\alpha\beta}(-F\nu\beta)_{,j}$$

(in view of the evolution equation $\frac{d}{dt}X = -F\nu$)

$$= -F\nu^\alpha_{;i}\delta_{\alpha\beta}X^\beta_{,j} - X^\alpha_{,i}\delta_{\alpha\beta}F\nu_{;j}$$

(terms involving derivatives of F vanish as ν and $X^\alpha_{,i}$ are orthogonal to each other; as the background metric $\overline{g}_{\alpha\beta} = \delta_{\alpha\beta}$ is flat, covariant and partial derivatives of ν coincide)

$$= -Fh^k_i X^\alpha_{,k}\delta_{\alpha\beta}X^\beta_{,j} - FX^\alpha_{,i}\delta_{\alpha\beta}h^k_j X^\beta_{,k}$$

(in view of the Weingarten equation (2.3))

$$= -Fh^k_i g_{kj} - Fg_{ik}h^k_j$$

(by the definition of the metric)

$$= -2Fh_{ij}$$

(by the definition of $h^i_j := h_{jk}g^{ki}$).

The lemma follows. $\square$

Corollary 4.2. *The evolution equation of the volume element $d\mu := \sqrt{\det g_{ij}}\,dx$ is given by*

$$\frac{d}{dt}d\mu = -FH\,d\mu. \tag{4.3}$$

Proof. Exercise. Recall the formulae for differentiating the determinant and the inverse of a matrix. $\square$

Lemma 4.3. *The unit normal ν evolves according to*

$$\frac{d}{dt}\nu^\alpha = g^{ij}F_{;\,i}X^\alpha_{;\,j}. \tag{4.4}$$

Proof. By definition, the unit normal vector ν has length one, $\langle\nu,\nu\rangle = 1 = \nu^\alpha\delta_{\alpha\beta}\nu^\beta$. Differentiating yields

$$0 = \dot\nu^\alpha\delta_{\alpha\beta}\nu^\beta.$$

Hence it suffices to show that the claimed equation is true if we take on both sides the scalar product with an arbitrary tangent vector. The vectors $X_{,i}$ (which we will also denote henceforth by X_i as there is no danger of confusion; we will also adopt this convention if partial and covariant derivatives of some quantity coincide) form a basis of the tangent plane at a fixed point. We differentiate the relation

$$0 = \langle\nu, X_i\rangle = \nu^\alpha\delta_{\alpha\beta}X^\beta_i$$

and obtain

$$\begin{aligned} 0 &= \frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_i + \nu^\alpha\delta_{\alpha\beta}\frac{d}{dt}X^\beta_i \\ &= \frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_i + \nu^\alpha\delta_{\alpha\beta}\left(\frac{d}{dt}X^\beta\right)_i \\ &= \frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_i - \nu^\alpha\delta_{\alpha\beta}\left(F\nu^\beta\right)_i. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_i &= \nu^\alpha\delta_{\alpha\beta}\nu^\beta F_i + F\nu^\alpha\delta_{\alpha\beta}\nu^\beta_i \\ &= F_i + F\tfrac{1}{2}\langle\nu,\nu\rangle_i = F_i \end{aligned}$$

and the lemma follows as taking the scalar product of the claimed evolution equation with X_k, i.e. multiplying it with $\delta_{\alpha\beta}X^\beta_k$, yields

$$\frac{d}{dt}\nu^\alpha\delta_{\alpha\beta}X^\beta_k = g^{ij}F_iX^\alpha_j\delta_{\alpha\beta}X^\beta_k = g^{ij}F_ig_{jk} = \delta^i_kF_i = F_k. \qquad \square$$

Lemma 4.4. *The second fundamental form h_{ij} evolves according to*

$$\frac{d}{dt}h_{ij} = F_{;\,ij} - Fh^k_ih_{kj}. \tag{4.5}$$

Proof. The Gauß formula (2.2) implies that $h_{ij} = -X^\alpha_{;ij}\nu_\alpha$. Differentiating yields

$$\begin{aligned}
\frac{d}{dt}h_{ij} &= -\frac{d}{dt}\langle X_{;ij},\nu\rangle\\
&= -\left\langle \frac{d}{dt}X_{;ij},\nu\right\rangle - \left\langle -h_{ij}\nu,\frac{d}{dt}\nu\right\rangle\\
&= -\left\langle \frac{d}{dt}X_{;ij},\nu\right\rangle + h_{ij}\left\langle \nu,\frac{d}{dt}\nu\right\rangle\\
&= -\left\langle \frac{d}{dt}X_{;ij},\nu\right\rangle\\
&= -\frac{d}{dt}\left(X^\alpha_{,ij} - \Gamma^k_{ij}X^\alpha_k\right)\nu_\alpha\\
&= -\left(\frac{d}{dt}X^\alpha\right)_{,ij}\nu_\alpha + \Gamma^k_{ij}\left(\frac{d}{dt}X^\alpha\right)_{,k}\nu_\alpha
\end{aligned}$$

(where no time derivatives of Γ^k_{ij} show up as $X^\alpha_i\nu_\alpha = 0$)

$$= (F\nu^\alpha)_{,ij}\nu_\alpha - \Gamma^k_{ij}(F\nu^\alpha)_{,k}\nu_\alpha$$

(in view of the evolution equation)

$$\begin{aligned}
&= F_{,ij}\nu^\alpha\nu_\alpha + F_{,i}\nu^\alpha_{,j}\nu_\alpha + F_{,j}\nu^\alpha_{,i}\nu_\alpha + F\nu^\alpha_{,ij}\nu_\alpha - \Gamma^k_{ij}F_{,k}\nu^\alpha\nu_\alpha - \Gamma^k_{ij}F\nu^\alpha_{,k}\nu_\alpha\\
&= F_{;ij} + F\nu^\alpha_{,ij}\nu_\alpha
\end{aligned}$$

as $F_{;ij} = F_{,ij} - \Gamma^k_{ij}F_{,k}$ and $\nu^\alpha_{,j}\nu_\alpha = \frac{1}{2}(\nu^\alpha\nu_\alpha)_j = 0$. It remains to show that $\nu^\alpha_{,ij}\nu_\alpha = -h^k_ih_{kj}$. We obtain

$$\nu^\alpha_{,ij}\nu_\alpha = \nu^\alpha_{;i,j}\nu_\alpha$$

(as $\nu^\alpha_i = \nu^\alpha_{;i}$)

$$= \nu^a_{;ij}\nu_\alpha$$

($\nu^\alpha_{;ij} = (\nu^\alpha_{;i})_{,j} - \Gamma^k_{ij}\nu^\alpha_k$ and $0 = \nu^\alpha_k\nu_\alpha$)

$$= \left(h^k_iX^\alpha_k\right)_{;j}\nu_\alpha$$

(according to the Weingarten equation (2.3))

$$= h^k_i(-h_{kj}\nu^\alpha)\nu_\alpha$$

(due to the Gauß equation (2.2) and the orthogonality $X^\alpha_k\nu_\alpha=0$)

$$=-h^k_ih_{kj}$$

as claimed. The Lemma follows. □

Lemma 4.5. *The normal velocity F evolves according to*

$$\frac{d}{dt}F-F^{ij}F_{;ij}=FF^{ij}h^k_ih_{kj}. \tag{4.6}$$

Proof. We have, see [25, Lemma 5.4], the proof of [13, Theorem 2.1.20], or check this explicitly for the normal velocity considered,

$$\frac{\partial F}{\partial g_{kl}}=-F^{il}h^k_i$$

and compute the evolution equation of the normal velocity F

$$\begin{aligned}\frac{d}{dt}F-F^{ij}F_{;ij}=&-F^{il}h^k_i\frac{d}{dt}g_{kl}+F^{ij}\frac{d}{dt}h_{ij}-F^{ij}F_{;ij}\\=&FF^{ij}h^k_ih_{kj},\end{aligned}$$

where we used (4.2) and (4.5). □

We will need more explicit evolution equations for geometric quantities $\boxplus$ involving $\frac{d}{dt}\boxplus-F^{ij}\boxplus_{;ij}$.

Lemma 4.6. *The second fundamental form h_{ij} evolves according to*

$$\begin{aligned}\frac{d}{dt}h_{ij}-F^{kl}h_{ij;\,kl}=&F^{kl}h^a_kh_{al}\cdot h_{ij}-F^{kl}h_{kl}\cdot h^a_ih_{aj}\\&-Fh^k_ih_{kj}+F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j}.\end{aligned} \tag{4.7}$$

Proof. Direct calculations yield

$$\begin{aligned}\frac{d}{dt}h_{ij}-F^{ij}h_{ij;\,kl}=&F_{;ij}-Fh^k_ih_{kj}-F^{ij}h_{ij;kl} && \text{by (4.5)}\\=&F^{kl}h_{kl;\,ij}+F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j}\\&-Fh^k_ih_{kj}-F^{ij}h_{ij;\,kl}\\=&F^{kl}h_{ik;\,lj}+F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j}\\&-Fh^k_ih_{kj}-F^{ij}h_{ik;\,jl} && \text{by Codazzi}\\=&F^{kl}\left(h^a_kR_{ailj}+h^a_iR_{aklj}\right)-Fh^k_ih_{kj}\\&+F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j} && \text{by (2.5)}\\=&F^{kl}h^a_kh_{al}h_{ij}-F^{kl}h^a_kh_{aj}h_{il}\\&+F^{kl}h^a_ih_{al}h_{kj}-F^{kl}h^a_ih_{aj}h_{kl}\\&-Fh^k_ih_{kj}+F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j} && \text{by (2.4)}\\=&F^{kl}h^a_kh_{al}h_{ij}-F^{kl}h^a_ih_{aj}h_{kl}\\&-Fh^k_ih_{kj}+F^{kl,\,rs}h_{kl;\,i}h_{rs;\,j}.\end{aligned}$$

□

Remark 4.7. A direct consequence of (4.1) and (2.2) is

$$\frac{d}{dt}X^\alpha-F^{ij}X^\alpha_{;\,ij}=\left(F^{ij}h_{ij}-F\right)\nu^\alpha. \tag{4.8}$$

Hence

(4.9)

$$\frac{d}{dt}|X|^2 - F^{ij}\left(|X|^2\right)_{;ij} = 2\left(F^{ij}h_{ij} - F\right)\langle X,\,\nu\rangle - 2F^{ij}g_{ij},$$

(4.10)

Proof. We have

$$\begin{aligned}\frac{d}{dt}|X|^2 - F^{ij}\left(|X|^2\right)_{;ij} &= 2\left\langle X,\frac{d}{dt}X\right\rangle - 2F^{ij}\langle X_i, X_j\rangle - 2F^{ij}\langle X, X_{;ij}\rangle\\ &= 2\langle X, -F\nu\rangle - 2F^{ij}g_{ij} - 2F^{ij}\langle X, -h_{ij}\nu\rangle. \qquad \square\end{aligned}$$

Lemma 4.8. *The evolution equation for the unit normal ν is*

$$\frac{d}{dt}\nu^\alpha - F^{ij}\nu^\alpha_{;ij} = F^{ij}h^k_i h_{kj}\cdot\nu^\alpha. \tag{4.11}$$

Proof. We compute

$$\begin{aligned}\frac{d}{dt}\nu^\alpha - F^{ij}\nu^\alpha_{;ij} &= g^{ij}F_{;\,i}X^\alpha_{;\,j} - F^{ij}\left(h^k_i X^\alpha_{;\,k}\right)_{;\,j} && \text{by (4.4) and (2.3)}\\ &= g^{ij}F^{kl}h_{kl;\,i}X^\alpha_{;\,j} - F^{ij}h^k_{i;\,j}X^\alpha_{;\,k} - F^{ij}h^k_i X^\alpha_{;\,kj}\\ &= F^{ij}h^k_i h_{kj}\nu^\alpha && \text{by (2.2).} \qquad \square\end{aligned}$$

Lemma 4.9. *The evolution equation for the scalar product $\langle X,\nu\rangle$ is*

$$\frac{d}{dt}\langle X,\nu\rangle - F^{ij}\langle X,\nu\rangle_{;ij} = -F^{ij}h_{ij} - F + F^{ij}h^k_i h_{kj}\langle X,\,\nu\rangle. \tag{4.12}$$

Proof. We obtain

$$\begin{aligned}\frac{d}{dt}\langle X,\,\nu\rangle - F^{ij}\langle X,\,\nu\rangle_{;ij} &= X^\alpha\delta_{\alpha\beta}\left(\frac{d}{dt}\nu^\beta - F^{ij}\nu^\alpha_{;ij}\right)\\ &\quad + \left(\frac{d}{dt}X^\alpha - F^{ij}X^\alpha_{;\,ij}\right)\delta_{\alpha\beta}\nu^\beta\\ &\quad - 2F^{ij}X^\alpha_{;\,i}\delta_{\alpha\beta}\nu^\beta_{;\,j}\\ &= F^{ij}h^k_i h_{kj}\langle X,\,\nu\rangle + \left(F^{ij}h_{ij} - F\right)\langle\nu,\,\nu\rangle\\ &\quad - 2F^{ij}X^\alpha_{;\,i}\delta_{\alpha\beta}h^k_j X^\beta_{;\,k}\end{aligned}$$

by (2.3), (4.8), and (4.11)

$$= F^{ij}h^k_i h_{kj}\langle X,\,\nu\rangle - F^{ij}h_{ij} - F. \qquad \square$$

Lemma 4.10. *Let $\eta_\alpha = (-e_{n+1})_\alpha = (0,\ldots,0,-1)$. Then $\tilde v := \langle\eta,\nu\rangle \equiv \eta_\alpha\nu^\alpha$ fulfills*

$$\frac{d}{dt}\tilde v - F^{ij}\tilde v_{;ij} = F^{ij}h^k_i h_{kj}\tilde v \tag{4.13}$$

and $v := \tilde v^{-1}$ fulfills

$$\frac{d}{dt}v - F^{ij}v_{;ij} = -vF^{ij}h^k_i h_{kj} - 2\frac{1}{v}F^{ij}v_i v_j. \tag{4.14}$$

Proof. The evolution equation for $\tilde{v}$ is a direct consequence of (4.11). For the proof of the evolution equation of v observe that

$$v_i = -\tilde{v}^{-2}\tilde{v}_i = -v^2\tilde{v}_i$$

and

$$v_{;ij} = -\tilde{v}^{-2}\tilde{v}_{;ij} + 2\tilde{v}^{-3}\tilde{v}_i\tilde{v}_j = -v^2\tilde{v}_{;ij} + 2v^{-1}v_iv_j. \qquad \square$$

5. Convex Hypersurfaces

G. Huisken obtained the following theorem [17] for $n \geq 2$. The corresponding result for curves by M. Gage, R. Hamilton, and M. Grayson is even better, see [12, 15]. It is only required that $M \subset \mathbb{R}^2$ is a closed embedded curve.

Theorem 5.1. *Let $M \subset \mathbb{R}^{n+1}$ be a smooth closed convex hypersurface. Then there exists a smooth family M_t of hypersurfaces solving*

$$\begin{cases} \frac{d}{dt}X = -H\nu & \text{for } 0 \leq t < T, \\ M_0 = M \end{cases}$$

for some $T > 0$.

As $t \nearrow T$,

- *$M_t \to Q$ in Hausdorff distance for some $Q \in \mathbb{R}^{n+1}$ (convergence to a point),*
- *$(M_t - Q)\cdot(2n(T-t))^{-1/2} \to \mathbb{S}^n$ smoothly (convergence to a "round point").*

The key step in the proof of Theorem 5.1 (in the case $n \geq 2$) is the following

Theorem 5.2. *Let $M_t \subset \mathbb{R}^{n+1}$ be a family of convex closed hypersurfaces flowing according to mean curvature flow. Then there exists some $\delta > 0$ such that*

$$\max_{M_t} \frac{n|A|^2 - H^2}{H^{2-\delta}}$$

is bounded above.

The proof involves complicated integral estimates.

Exercise 5.3. Prove Theorem 5.2 for $\delta = 0$.

Hint: Use Kato's inequality.

Theorem 5.4 (Kato's inequality)**.** *We have*

$$|\nabla|A||^2 \leq |\nabla A|^2.$$

Proof. We prove this inequality if $|A| \neq 0$. In the exercise above, we only need that case. As $\nabla|A|^2 = 2|A|\nabla|A|$, the claim is equivalent to $\frac{1}{4}\left|\nabla|A|^2\right|^2 \leq |A|^2\cdot|\nabla A|^2$. We choose a coordinate system such that $g_{ij} = \delta_{ij}$ and h_{ij} is diagonal with eigenvalues λ_i. We obtain there

$$\begin{aligned} \frac{1}{4}\left|\nabla|A|^2\right|^2 &= \frac{1}{4}\sum_k \left(\nabla_k|A|^2\right)^2 = \sum_{i,j,k}\lambda_i h_{ii;k}\lambda_j h_{jj;k} \\ &\leq \sum_{i,j,k}\left(\frac{1}{2}\lambda_i^2 h_{jj;k}^2 + \frac{1}{2}\lambda_j^2 h_{ii;k}^2\right) = \sum_{i,j,k}\lambda_i^2 h_{jj;k}^2 \leq \sum_{i,j,k,l} h_{ij;k}^2\lambda_l^2 \\ &= |A|^2\cdot|\nabla A|^2. \end{aligned} \qquad \square$$

Remark 5.5. For simplicity, we will illustrate the significance of the quantity considered in Theorem 5.2 only in the case $n = 2$. These considerations extend to higher dimensions.

As

$$\begin{aligned}2|A|^2 - H^2 &= 2(\lambda_1^2+\lambda_2^2) - (\lambda_1+\lambda_2)^2\\ &= 2\lambda_1^2 + 2\lambda_2^2 - \lambda_1^2 - 2\lambda_1\lambda_2 - \lambda_2^2\\ &= \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2\\ &= (\lambda_1-\lambda_2)^2,\end{aligned}$$

it measures the difference from being umbilic ($\lambda_1 = \lambda_2$) and vanishes precisely if M_t is a sphere. Recall from differential geometry that, according to Codazzi, $\lambda_1 = \lambda_2$ everywhere implies that M_t is locally part of a sphere or hyperplane.

Assume that $\min\limits_{M_t} H \to \infty$ as $t \nearrow T$. Assume also that $\lambda_1 \le \lambda_2$ and that the surfaces stay strictly convex, i. e. $\min\limits_{M_t} \lambda_1 > 0$. Then Theorem 5.2 implies for any ε there exists t_ε, such that for $t_\varepsilon \le t < T$

$$\varepsilon \ge \frac{n|A|^2 - H^2}{H^2} = \frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)^2} \ge \frac{(\lambda_1-\lambda_2)^2}{4\lambda_2^2} = \frac{1}{4}\left(\frac{\lambda_1}{\lambda_2} - 1\right)^2.$$

Hence $\frac{\lambda_1}{\lambda_2} \approx 1$ and thus this implies that M_t is, in terms of the principal curvatures λ_i, close to a sphere.

There are many results showing that convex hypersurfaces converge to round points under certain flow equations, see e. g. [1, 3, 4, 11, 12, 14, 20, 23, 24, 31].

Let us consider normal velocities of homogeneity bigger than one. In this case, the calculations, that lead to a theorem corresponding to Theorem 5.2 for mean curvature flow, are much simpler and rely only on the maximum principle.

Theorem 5.6. *[[3, Proposition 3]] Let M_t be a smooth family of closed strictly convex solutions to Gauß curvature flow $\frac{d}{dt}X = -K\nu$. Then*

$$t \mapsto \max_{M_t} (\lambda_1 - \lambda_2)^2$$

is non-increasing.

Proof. Recall that $H^2 - 4K = (\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2 = (\lambda_1-\lambda_2)^2 =: w$. For Gauß curvature flow, we have, according to Appendix B,

$$\begin{aligned}F^{ij} &= K^{ij} = \frac{\partial}{\partial h_{ij}} \frac{\det h_{kl}}{\det g_{kl}} = \frac{\det h_{kl}}{\det g_{kl}} \tilde{h}^{ij} = K\tilde{h}^{ij},\\ F^{ij,kl} &= K\tilde{h}^{ij}\tilde{h}^{kl} - K\tilde{h}^{ik}h^{lj},\end{aligned}$$

where $\tilde{h}^{ij}$ is the inverse of h_{ij}. Recall the evolution equations (4.2), (4.6), and (4.7) which become for Gauß curvature flow

$$\begin{aligned}\frac{d}{dt} g_{ij} &= -2Kh_{ij},\\ \frac{d}{dt}K - K\tilde{h}^{kl}K_{kl} &= KK\tilde{h}^{ij}h_i^k h_{kj} = K^2H,\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}h_{ij} - K\tilde{h}^{kl}h_{ij;kl} &= K\tilde{h}^{kl}h_k^a h_{al}h_{ij} - K\tilde{h}^{kl}h_{kl}h_i^a h_{aj} - Kh_i^k h_{kj}\\ &\quad + K\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right) h_{kl;i}h_{rs;j}\\ &= KHh_{ij} - (n+1)Kh_i^a h_{aj} + K\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right) h_{kl;i}h_{rs;j},\end{aligned}$$

where $n = 2$. We have

$$\begin{aligned}\frac{d}{dt}H - K\tilde{h}^{ij}H_{;ij} =& - h_{ij}g^{ik}g^{jl}\frac{d}{dt}g_{kl} + g^{ij}\left(\frac{d}{dt}h_{ij} - K\tilde{h}^{kl}h_{ij;kl}\right)\\ =& 2K|A|^2 + KH^2 - 3K|A|^2 + Kg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j}\\ =& K\left(H^2 - |A|^2\right) + Kg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j}\\ =& 2K^2 + Kg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j},\end{aligned}$$

hence

$$\begin{aligned}\frac{d}{dt}w - K\tilde{h}^{ij}w_{;ij} =& 2H\left(\frac{d}{dt}H - K\tilde{h}^{ij}H_{;ij}\right) - 2K\tilde{h}^{ij}H_iH_j\\ &- 4\left(\frac{d}{dt}K - K\tilde{h}^{ij}K_{;ij}\right)\\ =& 2H\left(2K^2 + Kg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j}\right)\\ &- 2K\tilde{h}^{ij}H_iH_j - 4K^2H\\ =& 2HKg^{ij}\left(\tilde{h}^{kl}\tilde{h}^{rs} - \tilde{h}^{kr}\tilde{h}^{sl}\right)h_{kl;i}h_{rs;j} - 2K\tilde{h}^{ij}H_iH_j.\end{aligned}$$

In a coordinate system, such that $g_{ij} = \delta_{ij}$ and $h_{ij} = \operatorname{diag}(\lambda_1, \lambda_2)$, we obtain

$$\begin{aligned}\frac{d}{dt}w - K\tilde{h}^{ij}w_{;ij} =& 2KH\sum_{i,j,k=1}^{2}\frac{1}{\lambda_i\lambda_j}h_{ii;k}h_{jj;k} - 2KH\sum_{i,j,k=1}^{2}\frac{1}{\lambda_i\lambda_j}h^2_{ij;k}\\ &- 2K\sum_{i,j,k=1}^{2}\frac{1}{\lambda_k}h_{ii;k}h_{jj;k}\\ =& 2KH\sum_{\substack{i,j,k=1\\ i\neq j}}^{2}\frac{1}{\lambda_i\lambda_j}h_{ii;k}h_{jj;k} - 2KH\sum_{\substack{i,j,k=1\\ i\neq j}}^{2}\frac{1}{\lambda_i\lambda_j}h^2_{ij;k} - 2K\sum_{i,j,k=1}^{2}\frac{1}{\lambda_k}h_{ii;k}h_{jj;k}\\ =& \frac{4KH}{\lambda_1\lambda_2}\left(h_{11;1}h_{22;1} - h^2_{12;1} + h_{11;2}h_{22;2} - h^2_{12;2}\right)\\ &- \frac{2K}{\lambda_1}(h_{11;1} + h_{22;1})^2 - \frac{2K}{\lambda_2}(h_{11;2} + h_{22;2})^2.\end{aligned}$$

From now on, we consider a positive spatial maximum of $H^2 - 4K$. There, we get $2Hg^{ij}h_{ij;k} - 4K\tilde{h}^{ij}h_{ij;k} = 0$ for $k = 1, 2$. In a coordinate system as above, this (divided by 2) becomes

$$\begin{aligned}0 =& Hh_{11;k} + Hh_{22;k} - 2\frac{K}{\lambda_1}h_{11;k} - 2\frac{K}{\lambda_2}h_{22;k}\\ =& (\lambda_1 + \lambda_2 - 2\lambda_2)h_{11;k} + (\lambda_1 + \lambda_2 - 2\lambda_1)h_{22;k}\\ =& (\lambda_1 - \lambda_2)(h_{11;k} - h_{22;k}).\end{aligned}$$

This enables us to replace $h_{11;2}$ in the evolution equation in a positive critical point by $h_{22;2}$: $h_{11;2} = h_{22;2}$ and $h_{22;1} = h_{11;1}$. Using also the Codazzi equations, we can rewrite the evolution equation in a positive critical point as

$$\begin{aligned}\frac{d}{dt}w - K\tilde{h}^{ij}w_{;ij} =& 4(\lambda_1+\lambda_2)\left(h_{11;1}^2 - h_{22;2}^2 + h_{22;2}^2 - h_{11;1}^2\right)\\ &- \frac{2K}{\lambda_1}(h_{11;1}+h_{22;1})^2 - \frac{2K}{\lambda_2}(h_{11;2}+h_{22;2})^2\\ \le& 0.\end{aligned}$$

Hence, by the parabolic maximum principle, Theorem A.1, the claim follows. $\square$

A consequence of Theorem 5.6 is the following result, see [3, Theorem 1].

Theorem 5.7. *Let $M \subset \mathbb{R}^3$ be a smooth closed strictly convex surface. Then there exists a smooth family of closed strictly convex hypersurfaces solving Gauß curvature flow $\frac{d}{dt}X = -K\nu$ for $0 \le t < T$. As $t \nearrow T$, M_t converges to a round point.*

Sketch of proof. The main steps are

(i) The convergence to a point is due to K. Tso [30]. There, the problem is rewritten in terms of the support function and considered in all dimensions. It is shown that a positive lower bound on the Gauß curvature is preserved during the evolution. This ensures that the surfaces stay convex. The evolution equation of $\frac{K}{\langle X,\nu\rangle - \frac{1}{2}R}$ is used to bound the principal curvatures as long as the surface encloses $B_R(0)$. Thus a positive lower bound on the principal curvatures follows. Parabolic Krylov-Safonov estimates imply bounds on higher derivatives.
(ii) Theorem 5.6,
(iii) Show that M_t is between spheres of radius $r_+(t)$ and $r_-(t)$ and center $q(t)$ with $\frac{r_+(t)}{r_-(t)} \to 1$ as $t \nearrow T$.
(iv) Show that the quotient $\frac{K(p,t)}{K_{r(t)}}$ converges to 1 as $t \nearrow T$. Here $r(t) = (3(T-t))^{1/3}$ is the radius of a sphere flowing according to Gauß curvature flow that becomes singular at $t = T$ and $K_{r(t)} = (3(T-t))^{-2/3}$ its Gauß curvature. This involves a Harnack inequality for the normal velocity.
(v) Show that $\frac{\lambda_i}{(3(T-t))^{-1/3}} \to 1$ as $t \nearrow T$.
(vi) Obtain uniform a priori estimates for a rescaled version of the flow and hence smooth convergence to a round sphere. $\square$

We see directly from the parabolic maximum principle for tensors that a positive lower bound on the principal curvatures is preserved for surfaces moving with normal velocity $|A|^2$.

Lemma 5.8. *For a smooth closed strictly convex surface M in $\mathbb{R}^3$, flowing according to $\frac{d}{dt}X = -|A|^2\nu$, the minimum of the principal curvatures is non-decreasing.*

Proof. We have $F = |A|^2 = h_{ij}g^{jk}h_{kl}g^{li}$, $F^{ij} = 2g^{ia}h_{ab}g^{bj}$, and $F^{ij,kl} = 2g^{ik}g^{jl}$. Consider $M_{ij} = h_{ij} - \varepsilon g_{ij}$ with $\varepsilon > 0$ so small that M_{ij} is positive semi-definite for some time t_0. We wish to show that M_{ij} is positive semi-definite for $t > t_0$. Using (4.7), we obtain

$$\frac{d}{dt}h_{ij} - F^{kl}h_{ij;\,kl} = 2\operatorname{tr}A^3 h_{ij} - 3|A|^2 h_i^k h_{kj} + 2g^{kr}g^{ls}h_{kl;\,i}h_{rs;\,j}.$$

In the evolution equation for M_{ij}, we drop the positive definite terms involving derivatives of the second fundamental form

$$\frac{d}{dt}M_{ij} - F^{kl}M_{ij;\,kl} \ge 2\operatorname{tr}A^3 h_{ij} - 3|A|^2 h_i^k h_{kj} + 2\varepsilon|A|^2 h_{ij}.$$

Let ξ be a zero eigenvalue of M_{ij} with $|\xi| = 1$, $M_{ij}\xi^j = h_{ij}\xi^j - \varepsilon g_{ij}\xi^j = 0$. So we obtain in a point with $M_{ij} \ge 0$

$$\begin{aligned}\left(2\operatorname{tr} A^3 h_{ij} - 3|A|^2 h_i^k h_{kj} + 2\varepsilon|A|^2 h_{ij}\right)\xi^i\xi^j =&2\varepsilon \operatorname{tr} A^3 - 3\varepsilon^2|A|^2 + 2\varepsilon^2|A|^2\\ =&2\varepsilon \operatorname{tr} A^3 - \varepsilon^2|A|^2\\ \ge&2\varepsilon^2|A|^2 - \varepsilon^2|A|^2 > 0\end{aligned}$$

and the maximum principle for tensors, Theorem A.2, which extends to the case $\frac{d}{dt}M_{ij} \ge \ldots$, gives the result. □

Exercise 5.9. Show that under mean curvature flow of closed hypersurfaces, the following inequalities are preserved during the flow.

(i) $0 \le H$, $0 < H$,
(ii) $h_{ij} \ge 0$,
(iii) $\varepsilon H g_{ij} \le h_{ij} \le \beta H g_{ij}$ for $0 < \varepsilon \le \frac{1}{n} < \beta < 1$.

Such estimates exist also for other normal velocities.

Theorem 5.10 ([23]). *Let M_t be a family of closed strictly convex hypersurfaces evolving according to $\frac{d}{dt}X = -|A|^2\nu$. Then*

$$t \mapsto \max_{M_t} \frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$$

is non-increasing.

Exercise 5.11.

(i) Prove Theorem 5.10.

Hint: In a positive critical point of $w := \frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$, for $F = |A|^2$, the evolution equation of w is given by

$$\begin{aligned}\frac{d}{dt}w - F^{ij}w_{;ij} =& -4(\lambda_1-\lambda_2)^2\lambda_1\lambda_2\\ &-2\frac{5\lambda_1^8 - 4\lambda_1^7\lambda_2 + 46\lambda_1^6\lambda_2^2 + 48\lambda_1^5\lambda_2^3 + 72\lambda_1^4\lambda_2^4}{\left(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2\right)^2\lambda_1^4}h_{11;1}^2\\ &-2\frac{44\lambda_1^3\lambda_2^5 + 34\lambda_1^2\lambda_2^6 + 8\lambda_1\lambda_2^7 + 3\lambda_2^8}{\left(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2\right)^2\lambda_1^4}h_{11;1}^2\\ &-2\frac{5\lambda_2^8 - 4\lambda_2^7\lambda_1 + 46\lambda_2^6\lambda_1^2 + 48\lambda_2^5\lambda_1^3 + 72\lambda_2^4\lambda_1^4}{\left(\lambda_2^2+\lambda_2\lambda_1+\lambda_1^2\right)^2\lambda_2^4}h_{22;2}^2\\ &-2\frac{44\lambda_2^3\lambda_1^5 + 34\lambda_2^2\lambda_1^6 + 8\lambda_2\lambda_1^7 + 3\lambda_1^8}{\left(\lambda_2^2+\lambda_2\lambda_1+\lambda_1^2\right)^2\lambda_2^4}h_{22;2}^2.\end{aligned}$$

(This is a longer calculation.)

(ii) Show that the only closed strictly convex surfaces contracting self-similarly (by homotheties) under $\frac{d}{dt}X = -|A|^2\nu$, are round spheres. A surface M_t is said to evolve by homotheties, if for every t_1, t_2, there exists $\lambda \in \mathbb{R}$ such that $M_{t_1} = \lambda M_{t_2}$.

(iii) Show that for closed strictly convex initial data M, there exists some $c > 0$ such that $\frac{1}{c} \le \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} \le c$ for surfaces evolving according to $\frac{d}{dt}X = -|A|^2\nu$ for all $0 \le t < T$, where T is, as usual, the maximal existence time.

Similar results also exist for expanding surfaces

Theorem 5.12 ([24]). *Let M_t be a family of closed strictly convex hypersurfaces evolving according to $\frac{d}{dt}X = \frac{1}{K}\nu$. Then*

$$t \mapsto \max_{M_t} \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1^2 \lambda_2^2}$$

is non-increasing.

Exercise 5.13. Prove Theorem 5.12 and deduce consequences similar to those in Exercise 5.11.

Hint: In a critical point of $w := \frac{(\lambda_1-\lambda_2)^2}{\lambda_1^2\lambda_2^2}$, the evolution equation of w reads

$$\frac{d}{dt}w - F^{ij}w_{;ij} = -2\frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1^3\lambda_2^3} - \frac{8}{\lambda_1^6\lambda_2}h_{11;\,1}^2 - \frac{8}{\lambda_1\lambda_2^6}h_{22;\,2}^2.$$

Isoperimetric Inequalities. In a situation, where we know, that mean curvature flow exists until the enclosed volume shrinks to zero, it can be used to prove isoperimetric inequalities. Flow equations and isoperimetric inequalities are studied by G. Huisken, F. Schulze [26], and P. Topping [29]. We want to describe such an approach in a model situation.

Consider a family M_t^2 of smooth closed surfaces that moves by mean curvature flow until the volume of the enclosed area Ω_t shrinks to zero. In this case, the isoperimetric inequality reads

$$\frac{1}{3\sigma}\left(\mathcal{H}^2(M_t)\right)^{3/2} - \mathcal{H}^3(\Omega_t) \equiv \frac{1}{3\sigma}|M_t|^{3/2} - |\Omega_t| \geq 0$$

for $\sigma = \sqrt{4\pi}$. (Recall that $|\partial B_1^3| = 4\pi$ and $|B_1^3| = \frac{4\pi}{3}$.) We want to prove this inequality using mean curvature flow for surfaces M_t. Note first that by Hölder's inequality

$$\int\limits_{M_t} H \leq \left(\int\limits_{M_t} 1\right)^{1/2} \left(\int\limits_{M_t} H^2\right)^{1/2}.$$

Secondly, by Gauß-Bonnet,

$$\int\limits_{M_t} H^2 = \int\limits_{M_t} (\lambda_1-\lambda_2)^2 + \int\limits_{M_t} 4K \geq 4\int\limits_{M_t} K = 4\mathcal{H}^2(\partial B_1(0)) \equiv 4|\partial B_1(0)| = 16\pi.$$

If M_t is not a topological sphere, the integration has to be restricted to a set $\tilde{M}_t \subset M_t$ such that $\nu\colon \tilde{M}_t \to \mathbb{S}^2$ is bijective. Hence, under the evolution by mean curvature flow, we get according to (4.3) the following estimate for the isoperimetric difference (we may assume that $\int_{M_t} H > 0$ for otherwise the inequality derived in the following follows already from the first line)

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{3\sigma}|M_t|^{3/2} - |\Omega_t|\right) &= \frac{1}{2\sigma}|M_t|^{1/2}\int\limits_{M_t} -H^2\,d\mu - \int\limits_{M_t} -H\,d\mu \\
&\leq -\frac{1}{2\sigma}\left(\int\limits_{M_t} H^2\right)^{1/2}\int\limits_{M_t} H + \int\limits_{M_t} H \\
&= \left(1 - \frac{1}{2\sigma}\left(\int\limits_{M_t} H^2\right)^{1/2}\right)\int\limits_{M_t} H \\
&\leq \left(1 - \frac{1}{2\sigma}\sqrt{16\pi}\right)\int H
\end{aligned}$$

$$= 0.$$

Setting $f(t) := \frac{1}{3\sigma}|M_t|^{3/2} - |\Omega_t|$, we have $f(T) \ge 0$ (considered as a limit as $t \nearrow T$) as we have assumed that $|\Omega_T| = 0$ (in the sense of a limit). Integrating backwards in time yields $f(t) \ge 0$ for $t < T$, which is the isoperimetric inequality claimed above.

Calculations on a Computer Algebra System. For checking the monotonicity of

$$t \mapsto \max_{M_t} \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2},$$

see Theorem 5.10, the calculations become quite long. In the following we describe how the calculations leading to this theorem can be done by a computer provided that you trust these machines.

(i) Rewrite $w = \frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$ in terms of H and K, H and K in terms of g_{ij} and h_{ij} and finally g_{ij} and h_{ij} as a function of Du and D^2u, provided that the surface is locally described as graph u.

(ii) Proceed similarly with the normal velocity $|A|^2 = F\left(Du, D^2u\right)$. Then u fulfills the partial differential equation

$$\dot{u} = \sqrt{1 + |Du|^2} \cdot F(Du, D^2u) \equiv vF.$$

(iii) Differentiating this equation

$$\dot{u}_k = vF_{r_{ij}} u_{ijk} + vF_{p_i} u_{ik} + \frac{u^i}{v} u_{ik}$$

and dropping lower order terms suggests to consider the linearized operator

$$LW := \dot{W} - vF_{r_{ij}} W_{ij},$$

where v and F are evaluated at $\left(Du, D^2u\right)$.

(iv) We would like to show that w is non-increasing. This follows from the maximum principle if we can show that $\frac{d}{dt}w - F^{ij}w_{;ij} \equiv \frac{d}{dt}w - \frac{\partial F}{\partial h_{ij}} w_{;ij} \le 0$ in a positive maximum of w. By the chain rule, we get

$$\frac{\partial F}{\partial r_{ij}} = \frac{\partial F}{\partial h_{kl}} \cdot \frac{\partial h_{kl}}{\partial r_{ij}} = \frac{\partial F}{\partial h_{ij}} \cdot \frac{1}{v}.$$

(v) The considerations in the last paragraph do not depend on the coordinate system. We choose a coordinate system such that a positive maximum is attained at the origin and $Du(0) = 0$. We may assume in addition that $D^2u(0)$ is diagonal. At the origin, both factors that distinguish covariant and partial derivatives in $w_{;ij} = w_{,ij} - \Gamma^k_{ij} w_{,k}$ vanish. Hence it suffices to show that $Lw|_{x=0} \le 0$. This can be carried out with the help of a computer.

The algorithm in words:

(1) Write $w = w\left(Du, D^2u\right)$ and $F = F\left(Du, D^2u\right)$.
(2) Compute the following derivatives in terms of derivatives of u: $F_{r_{ij}}$, $\dot{w}$, w_i, w_{ij}.
(3) Combine those derivatives and get $Lw =: N_1$ in terms of derivatives of u.
(4) Use the relations obtained from differentiating $\dot{u} = vF$, $\dot{u}_k = (vF)_k$ and $\dot{u}_{kl} = (vF)_{kl}$ to remove any time derivative from N_1: Call the result N_2.
(5) As w is positive and maximal at the point we want to consider, we can solve $w_k = 0$ for u_{11k} and u_{22k}. We use this to replace the terms u_{112} and u_{221} in N_2 and get N_3.
(6) Assume that $Du(0) = 0$ and $D^2u(0) = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ in N_3 to get N_4.

(7) N_4 consists of three terms:
$$N_4 = A + Bu_{111}^2 + Cu_{222}^2,$$
no terms involving $u_{111}u_{222}$ show up. Observe that A, B and C do only depend on a and b and that B and C are equal up to interchanging a and b.

(8) It is easy to see that $A \le 0$ and $B \le 0$ for $a, b \ge 0$ in the situation of Theorem 5.10.

If it is not obvious, whether these inequalities hold, Sturm's algorithm [28] can be used to check the underlying polynomials for positivity.

(9) Applying the steps above for different choices of w can be used to find monotone quantities, see [23, 24].

Two warnings:

- Do not use the simplifications valid at a single point, especially $Du = 0$, before differentiating.
- The computer might identify u_{12} and u_{21}. Take this into account when computing $F_{r_{12}}$.

Exercise 5.14. Prove Theorem 5.10 based on computer algebra calculations.

6. ENTIRE SOLUTIONS TO GAUSS CURVATURE FLOWS

Mean Curvature Flow. For mean curvature flow of entire graphs, K. Ecker and G. Huisken proved the following existence theorem [9, Theorem 5.1]

Theorem 6.1. *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous. Then there exists a function $u \in C^\infty(\mathbb{R}^n \times (0,\infty)) \cap C^0(\mathbb{R}^n \times [0,\infty))$ solving*
$$\begin{cases} \dot u = \sqrt{1+|Du|^2}\,\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) & \text{in } \mathbb{R}^n \times (0,\infty),\\ u(\cdot,t) \to u_0 & \text{as } t \searrow 0 \text{ in } C^0_{loc}(\mathbb{R}^n). \end{cases}$$

The key ingredient in the existence proof is the following localized gradient estimate.

Theorem 6.2. *Let $u : B_R(0) \times [0,T] \to \mathbb{R}$ be a smooth solution to graphical mean curvature flow. Then*
$$\sqrt{1+|Du|^2(0,t)} \le c(n) \sup_{B_R(0)} \sqrt{1+|Du|^2(\cdot,0)} \cdot \exp\left(c(n)\,R^{-2}\left(\operatorname*{osc}_{B_R(0)\times[0,T]} u\right)^2\right).$$

Theorem 6.1 has been extended to continuous initial data by J. Clutterbuck [5] and T. Colding and W. Minicozzi [7].

If u is initially close to a cone in an appropriate sense, graphical mean curvature flow converges, as $t \to \infty$, after appropriate rescaling, to a self-similarly expanding solution "coming out of a cone", see the papers by K. Ecker and G. Huisken [9] and N. Stavrou [27].

Stability of translating solutions to graphical mean curvature flow without rescaling is considered in [6].

Gauß Curvature Flows. The results of this section are joint work with J. Urbas and not yet published elsewhere.

Let $\alpha > 0$. An entire graphical solution moving with normal velocity K^α fulfills
$$\frac{d}{dt}X = -K^\alpha \nu \tag{6.1}$$

or, equivalently, with an initial condition

$$\text{(6.2)} \qquad \begin{cases} \dot{u} = \sqrt{1+|Du|^2}\left(\dfrac{\det D^2u}{\left(1+|Du|^2\right)^{\frac{n+2}{2}}}\right)^{\alpha} & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^n. \end{cases}$$

It is a parabolic equation if u is strictly convex.

In order to formulate our existence theorem, we need the following definition.

Definition 6.3 (ν-condition)**.** Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then u (or $\operatorname{graph} u$) fulfills the ν-condition if for every $\varepsilon > 0$ there exists $r > 0$ such that for points $p, q \in \operatorname{graph} u$ with $|p-q| \le 1$ and $|p|, |q| \ge r$

$$|\nu(p) - \nu(q)| \le \varepsilon$$

holds.

For general functions u, this condition is restrictive. This is different for convex functions.

Example 6.4.

(i) $u(x) = |x|^k$ fulfills the condition for every $k > 1$ and for every $k > 0$ outside the origin.

(ii) We say that $k\colon \mathbb{R}^n \to \mathbb{R}$ (or $\operatorname{graph} k$) is a cone if k is positive homogeneous of degree one. We call it a smooth cone if k is smooth outside the origin.

Let u be convex, differentiable and asymptotic to a smooth convex cone k in the sense that

$$\lim_{r\to\infty} \sup_{\mathbb{R}^n \setminus B_r(0)} |u-k| = 0.$$

Then u fulfills the ν-condition.

This is also true if $u_0 - k$ growth at most sublinearly at infinity.

(iii) A function u close to the cone $k\colon \mathbb{R}^2 \to \mathbb{R}$ with $k(x,y) := \max\{\pm x, \pm y\}$ does not fulfill the ν-condition.

We prove an existence result for initial data that fulfill the ν-condition.

Theorem 6.5. *Let $0 < \alpha < \frac{1}{n-1}$. Let $u_0 \in C^{2,\beta}_{loc}(\mathbb{R}^n)$ be strictly convex for some $0 < \beta < 1$. Assume that u_0 fulfills the ν-condition. Then there exists*

$$u \in C^{2;1}_{loc}\left(\mathbb{R}^n \times (0,\infty)\right) \cap C^0_{loc}\left(\mathbb{R}^n \times [0,\infty)\right)$$

solving (6.2)*.*

If $\sup\limits_{\mathbb{R}^n} |Du_0| < \infty$, u is unique.

Remark 6.6. Corresponding to the convergence results for mean curvature flow [6, 27], we get:

(i) If u_0 is asymptotic to a smooth convex cone k at infinity, then

$$\sup_{\mathbb{R}^n} |u(\cdot,t) - U(\cdot,t)| \to 0 \quad \text{as } t \to \infty,$$

where U is the homothetically expanding solution with $U(\cdot,0) = k$, see [32].

(ii) If u_0 deviates at most sublinearly from k near infinity, then the graph of $u(\cdot,t)$ converges to the graph of $U(\cdot,t)$ after suitable rescaling.

The proofs follow along the lines of the corresponding results for mean curvature flow. Observe that $u_0 \ge k$ if u is asymptotic to k.

The crucial local C^2 a priori estimates in the proof of Theorem 6.5 are contained in

Theorem 6.7. *Let $0<\alpha<\frac{1}{n-1}$, $T>0$ and $(M_t)_{t\in[0,T]}$ be a family of complete convex C^2-hypersurfaces solving* (6.1). *Pick a coordinate system such that each $M_t^-:=M_t\cap\{x^{n+1}\le 0\}$ can be written as $\operatorname{graph} u(\cdot,t)$ in some domain with $|Du(\cdot,t)|\le G$ in M_t^-. If M_0^- is bounded, $\beta=\beta(n,\alpha)\ge 1$ is large and $G=G(n,\alpha,\beta)>0$ is small enough, then*

$$(-\eta_\alpha X^\alpha)\cdot\max_{|\xi|=1}\frac{h_{ij}\xi^i\xi^j}{g_{ij}\xi^i\xi^j}\cdot e^{\beta v}$$

is bounded in M_t^- by the maximum of $c=c\left(n,\alpha,\beta,G,\max -\eta_\alpha X^\alpha\right)$ and its value at $t=0$.

We will prove this theorem below in a model situation.

Main steps of the proof of Theorem 6.5.

(i) We approximate our initial data with closed strictly convex surfaces and prove a priori estimates that allow to pass to a limit. Using spheres as barriers, we see that we can derive the a priori estimates for entire prospective solutions.
(ii) Spheres, used as barriers, and the convexity of u imply local C^0- and C^1-bounds.
(iii) The ν condition is preserved during the flow: Use spheres as barriers and the convexity of u.
(iv) There exist (R,T,G,H)-coordinate systems, where $R,T,G>0$ are given and $H>0$ depends on u_0 and on these parameters:
   - The coordinate systems with coordinates $(\tilde{x}^\gamma)$ differ by a translation and rotation from the original coordinate systems.
   - $\operatorname{graph} u\cap\{\tilde{x}^{n+1}<0\}$ can be written as a convex graph of a function $\tilde{u}$ with $|\nabla\tilde{u}|\le G$.
   - Those coordinate systems exist for a set of points C of distance at least R from the origin such that $\tilde{u}\le -H$ for at least one of these coordinate systems for each point in C. C surrounds the origin, i. e. a bounded component of $M_t\setminus C$ contains $M_t\cap B_R(0)$.
   - Those coordinate systems exist for $t\in[0,T]$.
(v) Theorem 6.7 implies local upper C^2 a priori estimates.
(vi) The ν-condition allows to control the normal image near infinity. Hence a Harnack inequality [2] and an explicit barrier imply local lower bounds on K for positive times.
(vii) Those estimates suffice to pass to a limit of the evolving approximates solutions and to obtain a solution for all $t>0$. ☐

Instead of Theorem 6.7, we consider a model situation. This is a partial differential equation for which we will prove similar estimates. It is not motivated geometrically.

Theorem 6.8. *Let $u\colon\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ be convex (in space). Define $\Omega_t:=\{x\colon u(x,t)<0\}$. Assume that $\bigcup\limits_{t\in I}\Omega_t$ is bounded for any compact time interval and that u is of class $C^{4;2}$ in a neighborhood of these sets. Let $0<\alpha<\frac{1}{n-1}$. If $|Du|\le G$ in each Ω_t for G sufficiently small, then for any such u solving*

$$\dot{u}=\tfrac{1}{\alpha}\cdot(\det D^2u)^\alpha,$$
$$W:=(-u)\cdot\max_{|\xi|=1}u_{\xi\xi}\cdot e^{\frac12|Du|^2},$$

is bounded above in Ω_t in terms of n,α,G and its value at $t=0$.

Proof. We differentiate the evolution equation and obtain

$$\dot{u}=\tfrac{1}{\alpha}\cdot(\det D^2u)^\alpha\equiv\tfrac{1}{\alpha}\cdot\mathcal{D},$$

$$\begin{aligned}\dot{u}_i =& \mathcal{D}u^{kl}u_{kli},\\ \dot{u}_{11} =& \mathcal{D}u^{ij}u_{ij11} - \mathcal{D}u^{ik}u^{jl}u_{ij1}u_{kl1} + \alpha\mathcal{D}u^{ij}u_{ij1}u^{kl}u_{kl1}.\end{aligned}$$

Instead of bounding

$$W = (-u)\cdot\max_{|\xi|=1} u_{\xi\xi}\cdot e^{\frac{1}{2}|Du|^2},$$

we will consider the test function

$$w = \log(-u) + \log u_{11} + \tfrac{1}{2}|Du|^2.$$

It suffices to bound this function: Without loss of generality we may assume that $\max\limits_{|\xi|=1} u_{\xi\xi} = u_{11}$ at the point, where W attains its maximum. Then $W = e^w$ there and $W \le e^w$ elsewhere. Hence it suffices too bound w.

In an increasing maximum of w, we obtain

$$\begin{aligned}0 \le \dot{w} =& \frac{-\dot{u}}{-u} + \frac{\dot{u}_{11}}{u_{11}} + u^k\dot{u}_k,\\ 0 = w_i =& \frac{-u_i}{-u} + \frac{u_{11i}}{u_{11}} + u^ku_{ki},\\ 0 \ge w_{ij} =& \frac{-u_{ij}}{-u} + \frac{u_{11ij}}{u_{11}} + u^ku_{kij}\\ &- \frac{u_iu_j}{(-u)^2} - \frac{u_{11i}u_{11j}}{u_{11}^2} + u^k_ju_{ki}.\end{aligned}$$

We get there

$$\begin{aligned}0 \le& \dot{w} - \mathcal{D}u^{ij}w_{ij}\\ =& \frac{-\dot{u}}{-u} + \frac{\dot{u}_{11}}{u_{11}} + u^k\dot{u}_k\\ &- \mathcal{D}u^{ij}\left(\frac{-u_{ij}}{-u} + \frac{u_{11ij}}{u_{11}} + u^ku_{kij} - \frac{u_iu_j}{(-u)^2} - \frac{u_{11i}u_{11j}}{u_{11}^2} + u^k_ju_{ki}\right)\\ =& \frac{1}{u_{11}}\left(\dot{u}_{11} - \mathcal{D}u^{ij}u_{ij11}\right) + u^k\left(\dot{u}_k - \mathcal{D}u^{ij}u_{ijk}\right)\\ &+ \mathcal{D}\frac{u^{ij}u_iu_j}{(-u)^2} + \mathcal{D}\frac{1}{u_{11}^2}u^{ij}u_{11i}u_{11j} - \frac{\mathcal{D}}{\alpha(-u)} + \mathcal{D}\frac{n}{(-u)} - \mathcal{D}\Delta u.\end{aligned}$$

We use the differentiated evolution equation and obtain in the maximum considered

$$\begin{aligned}0 \le& \frac{\mathcal{D}}{u_{11}}\left(-u^{ik}u^{jl}u_{ij1}u_{kl1} + \alpha u^{ij}u_{ij1}u^{kl}u_{kl1}\right) + 0\\ &+ \mathcal{D}\frac{u^{ij}u_iu_j}{(-u)^2} + \frac{\mathcal{D}}{u_{11}^2}u^{ij}u_{11i}u_{11j} + \mathcal{D}\frac{n}{-u} - \mathcal{D}\Delta u.\end{aligned}$$

Let us assume that u_{ij} is diagonal at the point considered. We obtain there

$$0 = w_i = \frac{-u_i}{-u} + \frac{u_{11i}}{u_{11}} + u_iu_{ii},$$

$$\begin{aligned}
\frac{u^{ij}u_iu_j}{(-u)^2} &= \sum_{i=1}^n u^{ii}\left(\frac{u_{11i}}{u_{11}}+u_iu_{ii}\right)^2\\
&= \sum_{i=1}^n \left(u^{ii}\frac{u_{11i}^2}{u_{11}^2}+2u_i\frac{u_{11i}}{u_{11}}+u_i^2u_{ii}\right)\\
&= \sum_{i=1}^n \left(u^{ii}\frac{u_{11i}^2}{u_{11}^2}+2u_i\left(\frac{u_i}{-u}-u_iu_{ii}\right)+u_i^2u_{ii}\right)\\
&\le \sum_{i=1}^n u^{ii}\frac{u_{11i}^2}{u_{11}^2}+2\frac{|Du|^2}{-u}.
\end{aligned}$$

We use this estimate in our previous inequality, divide by $\mathcal{D}$ and obtain

$$\begin{aligned}
0 \le{}& \frac{1}{u_{11}}\left(-u^{ik}u^{jl}u_{ij1}u_{kl1}+\alpha u^{ij}u_{ij1}u^{kl}u_{kl1}\right)\\
&+\sum_{i=1}^n u^{ii}\frac{u_{11i}^2}{u_{11}^2}+\frac{1}{u_{11}^2}u^{ij}u_{11i}u_{11j}+2\frac{|Du|^2}{-u}+\frac{n}{-u}-\Delta u.
\end{aligned}$$

Let us assume that u_{ij} (and hence u^{ij}) is diagonal at the point considered. We obtain for the 3^{rd}-order terms

$$\begin{aligned}
u_{11}\cdot 3^{rd}\text{o.t.} \le{}& \sum_{i,j=1}^n \left(-u^{ii}u^{jj}u_{ij1}^2+\alpha u^{ii}u_{ii1}u^{jj}u_{jj1}\right)+2\frac{1}{u_{11}}\sum_{i=1}^n u^{ii}u_{11i}^2\\
={}& -\frac{u_{111}^2}{u_{11}^2}-2\sum_{r=2}^n\frac{u^{rr}}{u_{11}}u_{11r}^2-\sum_{r,s=2}^n u^{rr}u^{ss}u_{rs1}^2\\
&+\alpha\frac{u_{111}^2}{u_{11}^2}+2\alpha\frac{u_{111}}{u_{11}}\sum_{r=2}^n u^{rr}u_{rr1}+\alpha\sum_{r,s=2}^n u^{rr}u_{rr1}u^{ss}u_{ss1}\\
&+2\frac{u_{111}^2}{u_{11}^2}+2\sum_{r=2}^n\frac{u^{rr}}{u_{11}}u_{11r}^2\\
\le{}& (1+\alpha)\frac{u_{111}^2}{u_{11}^2}-\sum_{r=2}^n\left(u^{rr}u_{rr1}\right)^2+\alpha(n-1)\sum_{r=2}^n\left(u^{rr}u_{rr1}\right)^2\\
&+\varepsilon\sum_{r=2}^n\left(u^{rr}u_{rr1}\right)^2+\frac{\alpha^2(n-1)}{\varepsilon}\frac{u_{111}^2}{u_{11}^2}
\end{aligned}$$

where $\varepsilon>0$

$$\begin{aligned}
&\le \left(1+\alpha+\frac{\alpha^2(n-1)}{\varepsilon}\right)\cdot\frac{u_{111}^2}{u_{11}^2}+\underbrace{(-1+\alpha(n-1)+\varepsilon)}_{=0\text{ if }\varepsilon=1-\alpha(n-1)>0}\cdot\sum_{r=2}^n\left(u^{rr}u_{rr1}\right)^2\\
&\equiv c(n,\alpha)\cdot\frac{u_{111}^2}{u_{11}^2}\\
&= c(n,\alpha)\cdot\left(\frac{-u_1}{-u}+u_1u_{11}\right)^2\\
&= c(n,\alpha)\cdot\left(\frac{u_1^2}{(-u)^2}-2\frac{u_1^2u_{11}}{-u}+u_1^2u_{11}^2\right)\\
&\le u_{11}\cdot c(n,\alpha)\cdot\left(\frac{u_1^2}{u_{11}(-u)^2}+u_1^2u_{11}\right).
\end{aligned}$$

This implies at the increasing maximum considered

$$0 \le c(n,\alpha)\cdot\left(\frac{u_1^2}{u_{11}(-u)^2}+u_1^2u_{11}\right)+2\frac{|Du|^2}{-u}+\frac{n}{-u}-u_{11}.$$

We multiply this bound by $(-u)$, add $(-u)\cdot u_{11}$ and obtain

$$\begin{aligned}(-u)\cdot u_{11} \le{}& c(n,\alpha)\left(\frac{|Du|^2}{u_{11}(-u)}+|Du|^2(-u)u_{11}\right)+2|Du|^2+n,\\ e^w ={}&(-u)\cdot u_{11}\cdot e^{\frac12|Du|^2}\le c(\max w(\cdot,0),n,\alpha,\sup|Du|).\end{aligned}$$

The claim follows. □

7. Mean Curvature Flow of Complete Graphs

The material in this section is based on joint work with M. Sáez, see [22]. Have a look at the article for illustrations.

Intuition.

Remark 7.1.

(i) Long time existence for entire graphs was shown before by K. Ecker and G. Huisken [9], see Theorem 6.1.
(ii) We wish to study the evolution of complete graphs defined on subsets of Euclidean space $\mathbb{R}^{n+1}$. The additional dimension is related to Theorem 7.3.
(iii) We assume for the moment that such initial data have smooth solutions. Then the following pictures (only on the blackboard, not in these notes) should give an intuition about the behavior of these solutions.
  a) A rotationally symmetric solution defined on a ball.
  b) A solution initially defined on a domain that will form a neck-pinch under mean curvature flow.
  c) A solution initially defined on an annulus.
  d) A solution defined on a domain in the plane bounded by disjoint curves.

Results. Let us consider mean curvature flow for graphs defined on a relatively open set

$$\Omega\equiv\bigcup_{t\ge0}\Omega_t\times\{t\}\subset\mathbb{R}^{n+1}\times[0,\infty). \tag{7.1}$$

Our existence result for bounded domains is

Theorem 7.2 (Existence)**.** *Let $A\subset\mathbb{R}^{n+1}$ be a bounded open set and $u_0\colon A\to\mathbb{R}$ a locally Lipschitz continuous function with $u_0(x)\to\infty$ for $x\to x_0\in\partial A$.*

Then there exists (Ω,u), where $\Omega\subset\mathbb{R}^{n+1}\times[0,\infty)$ is relatively open, such that u solves graphical mean curvature flow

$$\dot u=\sqrt{1+|Du|^2}\cdot\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\quad \text{in } \Omega\cap\{t>0\},$$

u is smooth for $t>0$ and continuous up to $t=0$, $\Omega_0=A$, $u(\cdot,0)=u_0$ in A and $u(x,t)\to\infty$ as $(x,t)\to(x_0,t_0)\in\partial\Omega$, where $\partial\Omega$ is the relative boundary of Ω in $\mathbb{R}^{n+1}\times[0,\infty)$.

Such smooth solutions yield weak solutions to mean curvature flow. We have

Theorem 7.3 (Weak flow)**.** *Let (A,u_0) and (Ω,u) be as in Theorem 7.2. Let $\partial\mathcal{D}_t$ be the level set evolution of $\partial\Omega_0$ with $\mathcal{D}_0=\Omega_0$. If $\partial\mathcal{D}_t$ does not fatten, the measure theoretic boundaries of Ω_t and $\mathcal{D}_t$ coincide for every $t\ge0$.*

Strategy of Proof.

Strategy of the proof of Theorem 7.2.

(i) Fix $L > 0$. Then there exists a solution with initial value $\min\{u_0, L\}$ for all $t \in [0, \infty]$, see [9].

(ii) If $L_1 < L$, we prove a priori estimates for the part of the evolving graphs which is below L_1. This is done in Theorem 7.6 for the (spatial) first order derivatives of u. See Theorem 7.11 for the second derivative bounds.

(iii) We let $L \to \infty$ and use a variant of the Theorem of Arzelà-Ascoli to pass to a subsequence which is our solution. □

Sketch of the strategy of the proof of Theorem 7.3.
In the following sketch of a proof we try to give an idea of the argument without mentioning technical details, e. g. approximations or fattening. None of the steps works exactly as described below.

(i) The constructed solution corresponds to a level-set solution.

(ii) The level-set solution starting from $\partial A \times \mathbb{R}$ is an outer barrier to the graphical solution graph $u(\cdot, t)$. Observe that Ω_t is the projection of the evolving graph at time t to $\mathbb{R}^{n+1}$. Hence Ω_t is contained in the level-set evolution of A.

(iii) By shifting downwards the level set solution, we obtain convergence to the level set solution starting with the cylinder $\partial A \times \mathbb{R}$. This prevents graph $u(\cdot, t)$ from detaching near infinity from the evolution of the cylinder. □

The a Priori Estimates.

Let $\eta := (\eta_\alpha) = (0, \ldots, 0, 1)$ and define $u := X^\alpha \eta_\alpha$. Here, we consider u as a function on the evolving hypersurfaces rather than as a function depending on $(x, t) \in \mathbb{R}^{n+1} \times [0, \infty)$. Whenever we use quantities like v or $|A|^2$, we also use this meaning of u.

Theorem 7.4. *Let X be a solution to mean curvature flow. Then we have the following evolution equations.*

$$\begin{aligned}
\left(\tfrac{d}{dt} - \Delta\right) u &= 0,\\
\left(\tfrac{d}{dt} - \Delta\right) v &= -|A|^2 v - \tfrac{2}{v}|\nabla v|^2,\\
\left(\tfrac{d}{dt} - \Delta\right) |A|^2 &= -2|\nabla A|^2 + 2|A|^4,\\
\left(\tfrac{d}{dt} - \Delta\right) g &\le -2kg^2 - 2\varphi v^{-3}\langle \nabla v, \nabla g\rangle,
\end{aligned}$$

where $g = \varphi|A|^2 \equiv \frac{v^2}{1-kv^2}|A|^2$ and $k > 0$ is chosen so that $2v^2 \le k$ in the domain considered.

Proof. The first equation follows from $\left(\frac{d}{dt} - \Delta\right) X = 0$. For the remaining claims see [8, 9].

More details: For mean curvature flow, we have $F^{ij} = g^{ij}$. Hence $F^{ij}h_{ij} = H$ and (4.8) implies the evolution equation for u.

For the evolution equation of $w := |A|^2$, we calculate

$$\begin{aligned}
\left(\tfrac{d}{dt} - \Delta\right) g_{ij} &= -2Hh_{ij}, \qquad \text{see (4.2)},\\
\left(\tfrac{d}{dt} - \Delta\right) h_{ij} &= |A|^2 h_{ij} - 2Hh_i^a h_{aj}, \qquad \text{see (4.7)},\\
w &= g^{ik}h_{ij}g^{jl}h_{kl},\\
\dot w &= 2g^{ik}\dot h_{ij}g^{jl}h_{kl} - 2g^{ir}g^{sk}h_{ij}g^{jl}h_{kl}\dot g_{rs},\\
w_r &= 2g^{ik}h_{ij;r}g^{jl}h_{kl},\\
w_{rs} &= 2g^{ik}h_{ij;rs}g^{jl}h_{kl} + 2g^{ik}h_{ij;r}g^{jl}h_{kl;s},\\
\left(\tfrac{d}{dt} - \Delta\right) |A|^2 &= 2g^{ik}(|A|^2 h_{ij} - 2Hh_i^a h_{aj})g^{jl}h_{kl} + 4H \operatorname{tr} A^3 - 2|\nabla A|^2\\
&= 2|A|^4 - 2|\nabla A|^2.
\end{aligned}$$

□

Assumption 7.5. For the proof of the a priori estimates, we will assume that $u\colon \mathbb{R}^{n+1}\times[0,\infty)$ is a smooth solution to mean curvature flow such that

$$\{x\colon u(x)\le 0\}\subset B_R(0)$$

for some $R>0$. In order to be able to consider smooth solutions, a few extra constructions are necessary.

Theorem 7.6 (C^1-estimates)**.** *Let u be as in Assumption 7.5. Then*

$$vu^2\le \max_{\substack{t=0\\ \{u<0\}}} vu^2$$

at points where $u<0$.

Here and in the following, it is often possible to increase the exponent of u.

Proof. According to Theorem 7.4, $w:=vu^2$ fulfills

$$\begin{aligned}
\dot w &= \dot v u^2+2vu\dot u,\\
w_i &= v_iu^2+2vuu_i,\\
w_{ij} &= v_{ij}u^2+2vuu_{ij}+2vu_iu_j+2u(v_iu_j+v_ju_i),\\
\left(\tfrac{d}{dt}-\Delta\right)w &= u^2\left(\tfrac{d}{dt}-\Delta\right)v-2v|\nabla u|^2-4u\langle\nabla v,\nabla u\rangle\\
&= u^2\left(-v|A|^2-\tfrac{2}{v}|\nabla v|^2\right)-2v|\nabla u|^2-4\left\langle\tfrac{u}{\sqrt v}\nabla v,\sqrt v\nabla u\right\rangle\\
&\le -u^2v|A|^2\le 0.
\end{aligned}$$

The estimate follows from the maximum principle applied to w in the domain where $u<0$. ☐

Remark 7.7. We recommend to consider Theorem 7.6 as an estimate for $v(-u)^2$.

Corollary 7.8. *Let u be as in Assumption 7.5. Then*

$$v\le \max_{\substack{t=0\\ \{u<0\}}} vu^2$$

at points where $u\le -1$.

Remark 7.9. Corollaries similar to Corollary 7.8 also hold for the following a priori estimates for points with $u\le-\varepsilon<0$ or $t\ge\varepsilon>0$. We do not write them down explicitly.

In Theorem 7.6 and later, we may replace every u by $u-h$ for any constant h.

Remark 7.10. For later use, we estimate derivatives of u and v,

$$|\nabla u|^2=\eta_\alpha X^\alpha_i g^{ij}X^\beta_j\eta_\beta=\eta_\alpha\left(\delta^{\alpha\beta}-\nu^\alpha\nu^\beta\right)\eta_\beta=1-v^{-2}\le 1$$

and, according to (2.3),

$$\begin{aligned}
|\nabla v|^2 &= \left(\left(-\eta_\alpha\nu^\alpha\right)^{-1}\right)_i g^{ij}\left(\left(-\eta_\beta\nu^\beta\right)^{-1}\right)_j=v^4\eta_\alpha X^\alpha_k h^k_i g^{ij}h^l_j X^\beta_l\eta_\beta\le v^4|A|^2\\
&\le v^2\varphi|A|^2=v^2g.
\end{aligned}$$

So we get

$$|\langle\nabla u,\nabla v\rangle|\le|\nabla u|\cdot|\nabla v|\le v^2|A|\le v\sqrt g.$$

Theorem 7.11 (C^2-estimates)**.** *Let u be as in Assumption 7.5.*

(i) Then there exist $\lambda > 0$, $c > 0$ and $k > 0$ (the constant in φ and implicitly in g), depending on the C^1-estimates, such that

$$tu^4g + \lambda u^2v^2 \le \sup_{\substack{t=0\\ \{u<0\}}} \lambda u^2v^2 + ct$$

at points where $u < 0$ and $0 < t \le 1$.

(ii) Moreover, if u is in C^2 initially, we get C^2-estimates up to $t = 0$: Then there exists $c > 0$, depending only on the C^1-estimates, such that

$$u^4g \le \sup_{\substack{t=0\\ \{u<0\}}} u^4g + ct$$

at points where $u < 0$.

Proof. In order to prove both parts simultaneously, we underline terms and factors that can be dropped everywhere. We get the first part if we consider the underlined terms and the second part if we drop those and set $\lambda = 0$.

We remark that in this proof $g_{ij} = g_{;ij}$ always denotes second derivatives of g as introduced in [9].

We set

$$w := \underline{t}u^4g + \lambda u^2v^2$$

and obtain

$$\begin{aligned}
\dot w =& \underline{u^4g} + 4\underline{t}u^3g\dot u + \underline{t}u^4\dot g + 2\lambda v^2u\dot u + 2\lambda u^2v\dot v,\\
w_i =& 4\underline{t}u^3gu_i + \underline{t}u^4g_i + 2\lambda v^2uu_i + 2\lambda u^2vv_i,\\
w_{ij} =& 4\underline{t}u^3gu_{ij} + \underline{t}u^4g_{;ij} + 2\lambda v^2uu_{ij} + 2\lambda u^2vv_{ij} + 12\underline{t}u^2gu_iu_j\\
&+ 4\underline{t}u^3(g_iu_j + g_ju_i) + 2\lambda v^2u_iu_j + 2\lambda u^2v_iv_j + 4\lambda vu(u_iv_j + u_jv_i),\\
\underline{t}u^3\nabla g =& \frac{1}{u}\nabla w - 4\underline{t}u^2g\nabla u - 2\lambda v^2\nabla u - 2\lambda uv\nabla v,\\
\left(\tfrac{d}{dt} - \Delta\right) w \le& \underline{u^4g} + \underline{t}u^4\left(-2kg^2 - 2\varphi v^{-3}\langle\nabla v, \nabla g\rangle\right) + 2\lambda u^2v\left(-|A|^2v - \tfrac{2}{v}|\nabla v|^2\right)\\
&- 12\underline{t}u^2g|\nabla u|^2 - 8\underline{t}u^3\langle\nabla g, \nabla u\rangle - 2\lambda v^2|\nabla u|^2 - 2\lambda u^2|\nabla v|^2\\
&- 8\lambda uv\langle\nabla u, \nabla v\rangle.
\end{aligned}$$

In the following, we will use the notation $\langle\nabla w, b\rangle$ for generic gradient terms for the test function w. The constants c are allowed to depend on $\sup\{|u| \colon u < 0\}$ (which does not exceed its initial value) and the C^1-estimates. It may also depend on an upper bound for t, but we assume that $0 < t \le 1$. I. e. we suppress dependence on already estimated quantities.

We estimate the terms involving ∇g separately. Let $\varepsilon > 0$ be a constant. We fix its value blow. Using Remark 7.10 for estimating terms, we get

$$\begin{aligned}
-2\varphi\underline{t}u^4v^{-3}\langle\nabla v, \nabla g\rangle =& -2\frac{\varphi u}{v^3}\left\langle\nabla v, \frac{1}{u}\nabla w - 4\underline{t}u^2g\nabla u - 2\lambda v^2\nabla u - 2\lambda uv\nabla v\right\rangle\\
\le& \langle\nabla w, b\rangle + 8\underline{t}\frac{\varphi u^3}{v}g|A| + 4\lambda\varphi uv|A| + 4\frac{\lambda\varphi u^2}{v^2}|\nabla v|^2\\
\le& \langle\nabla w, b\rangle + \varepsilon\underline{t}u^4g^2 + \varepsilon\lambda u^2v^2|A|^2 + \lambda u^2|\nabla v|^2\cdot 4\frac{\varphi}{v^2} + c(\varepsilon, \lambda),\\
-8\underline{t}u^3\langle\nabla g, \nabla u\rangle =& -8\left\langle\nabla u, \frac{1}{u}\nabla w - 4\underline{t}u^2g\nabla u - 2\lambda v^2\nabla u - 2\lambda uv\nabla v\right\rangle\\
\le& \langle\nabla w, b\rangle + 32\underline{t}u^2g + 16\lambda v^2 + 16\lambda|u|v^3|A|\\
\le& \langle\nabla w, b\rangle + \varepsilon\underline{t}u^4g^2 + \varepsilon\lambda u^2v^2|A|^2 + c(\varepsilon, \lambda).
\end{aligned}$$

We obtain

$$\begin{aligned}\left(\tfrac{d}{dt}-\Delta\right)w \le{}& \underline{u^4 g}+\underline{t}u^4g^2(-2k+2\varepsilon)+\langle\nabla w,b\rangle\\ &+\lambda u^2v^2|A|^2(-2+3\varepsilon)+\lambda u^2|\nabla v|^2\left(4\frac{\varphi}{v^2}-6\right)+c(\varepsilon,\lambda).\end{aligned}$$

Let us assume that $k>0$ is chosen so small that $kv^2\le\frac13$ in $\{u<0\}$. This implies $\varphi\le 2v^2$. We may assume that $\lambda\ge 2u^2$ in $\{u<0\}$ and get $\underline{u^4g}\le\frac12\lambda u^2\varphi|A|^2\le\lambda u^2v^2|A|^2$. We get

$$4\frac{\varphi}{v^2}-6=\frac{4}{1-kv^2}-6\le 0.$$

Finally, fixing $\varepsilon>0$ sufficiently small, we obtain

$$\left(\tfrac{d}{dt}-\Delta\right)w\le\langle\nabla w,b\rangle+c.$$

Now, both claims follow from the maximum principle. □

Appendix A. Parabolic Maximum Principles

The following maximum principle is fairly standard. For non-compact, strict or other maximum principles, we refer to [9] or [21], respectively.

We will use $C^{2,1}$ for the space of functions that have two spatial and one time derivative, if all these derivatives are continuous.

Theorem A.1 (Weak parabolic maximum principle)**.** *Let $\Omega\subset\mathbb{R}^n$ be open and bounded and $T>0$. Let a^{ij}, $b^i\in L^\infty(\Omega\times[0,T])$. Let a^{ij} be strictly elliptic, i. e. $a^{ij}(x,t)>0$ in the sense of matrices. Let $u\in C^{2,1}(\Omega\times[0,T))\times C^0\left(\overline{\Omega}\times[0,T]\right)$ fulfill*

$$\dot u\le a^{ij}u_{ij}+b^iu_i\quad\text{in }\Omega\times(0,T).$$

Then we get for $(x,t)\in\Omega\times(0,T)$

$$u(x,t)\le\sup_{\mathcal{P}(\Omega\times(0,T))}u,$$

where $\mathcal{P}(\Omega\times(0,T)):=(\Omega\times\{0\})\cup(\partial\Omega\times(0,T))$.

Proof.

(i) Let us assume first that $\dot u<a^{ij}u_{ij}+b^iu_i$ in $\Omega\times(0,T)$. If there exists a point $(x_0,t_0)\in\Omega\times(0,T)$ such that $u(x_0,t_0)>\sup\limits_{\mathcal{P}(\Omega\times(0,T))}u$, we find $(x_1,t_1)\in\Omega\times(0,T)$ and t_1 minimal such that $u(x_1,t_1)=u(x_0,t_0)$. At (x_1,t_1), we have $\dot u\ge 0$, $u_i=0$ for all $1\le i\le n$, and $u_{ij}\le 0$ (in the sense of matrices). This, however, is impossible in view of the evolution equation.

(ii) Define for $0<\varepsilon$ the function $v:=u-\varepsilon t$. It fulfills the differential inequality

$$\dot v=\dot u-\varepsilon<\dot u\le a^{ij}u_{ij}+b^iu_i=a^{ij}v_{ij}+b^iv_i.$$

Hence, by the previous considerations,

$$u(x,t)-\varepsilon t=v(x,t)\le\sup_{\mathcal{P}(\Omega\times(0,T))}v=\sup_{\mathcal{P}(\Omega\times(0,T))}u-\varepsilon t$$

and the result follows as $\varepsilon\searrow 0$.

□

There is also a parabolic maximum principle for tensors, see [16, Theorem 9.1]. (See the AMS-Review for a small correction of the proof.)

A tensor N_{ij} depending smoothly on M_{ij} and g_{ij}, involving contractions with the metric, is said to fulfill the null-eigenvector condition, if $N_{ij}v^iv^j\ge 0$ for all null-eigenvectors of M_{ij}.

Theorem A.2. *Let M_{ij} be a tensor, defined on a closed Riemannian manifold $(M, g(t))$, fulfilling*

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + b^k \nabla_k M_{ij} + N_{ij}$$

on a time interval $[0, T)$, where b is a smooth vector field and N_{ij} fulfills the null-eigenvector condition. If $M_{ij} \geq 0$ at $t = 0$, then $M_{ij} \geq 0$ for $0 \leq t < T$.

Appendix B. Some Linear Algebra

Lemma B.1. *We have*

$$\frac{\partial}{\partial a_{ij}} \det(a_{kl}) = \det(a_{kl}) a^{ji},$$

if a_{ij} is invertible with inverse a^{ij}, i. e. if $a^{ij} a_{jk} = \delta^i_k$.

Proof. It suffices to prove that the claimed inequality holds when we multiply it with a_{ik} and sum over i. Hence, we have to show that

$$\frac{\partial}{\partial a_{ij}} \det(a_{kl}) a_{ik} = \det(a_{kl}) \delta^j_k.$$

We get

$$\frac{\partial}{\partial a_{ij}} \det(a_{kl}) = \det \begin{pmatrix} a_{1\,1} & \dots & a_{1\,j-1} & 0 & a_{1\,j+1} & \dots & a_{1\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1\,1} & \dots & a_{i-1\,j-1} & 0 & a_{i-1\,j+1} & \dots & a_{i-1\,n} \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ a_{i+1\,1} & \dots & a_{i+1\,j-1} & 0 & a_{i+1\,j+1} & \dots & a_{i+1\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n\,1} & \dots & a_{n\,j-1} & 0 & a_{n\,j+1} & \dots & a_{n\,n} \end{pmatrix}.$$

and thus

$$\begin{aligned} \frac{\partial}{\partial a_{ij}} \det(a_{kl}) \cdot a_{ik} = &\det \begin{pmatrix} 0 & \dots & 0 & a_{1\,k} & 0 & \dots & 0 \\ a_{2\,1} & \dots & a_{2\,j-1} & 0 & a_{2\,j+1} & \dots & a_{2\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n\,1} & \dots & a_{n\,j-1} & 0 & a_{n\,j+1} & \dots & a_{n\,n} \end{pmatrix} \\ &+ \det \begin{pmatrix} a_{1\,1} & \dots & a_{1\,j-1} & 0 & a_{1\,j+1} & \dots & a_{1\,n} \\ 0 & \dots & 0 & a_{2\,k} & 0 & \dots & 0 \\ a_{3\,1} & \dots & a_{3\,j-1} & 0 & a_{3\,j+1} & \dots & a_{3\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n\,1} & \dots & a_{n\,j-1} & 0 & a_{n\,j+1} & \dots & a_{n\,n} \end{pmatrix} \\ &+ \dots \\ = &\det \begin{pmatrix} a_{1\,1} & \dots & a_{1\,j-1} & a_{1\,k} & a_{1\,j+1} & \dots & a_{1\,n} \\ a_{2\,1} & \dots & a_{2\,j-1} & 0 & a_{2\,j+1} & \dots & a_{2\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n\,1} & \dots & a_{n\,j-1} & 0 & a_{n\,j+1} & \dots & a_{n\,n} \end{pmatrix} \\ &+ \det \begin{pmatrix} a_{1\,1} & \dots & a_{1\,j-1} & 0 & a_{1\,j+1} & \dots & a_{1\,n} \\ a_{2\,1} & \dots & a_{2\,j-1} & a_{1\,k} & a_{2\,j+1} & \dots & a_{2\,n} \\ a_{3\,1} & \dots & a_{3\,j-1} & 0 & a_{3\,j+1} & \dots & a_{3\,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n\,1} & \dots & a_{n\,j-1} & 0 & a_{n\,j+1} & \dots & a_{n\,n} \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&+\dots\\
=&\det\begin{pmatrix} a_{1\,1} & \dots & a_{1\,j-1} & a_{1\,k} & a_{1\,j+1} & \dots & a_{1\,n}\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ a_{n\,1} & \dots & a_{n\,j-1} & a_{n\,k} & a_{n\,j+1} & \dots & a_{n\,n}\end{pmatrix}\\
=&\delta^j_k \det(a_{rs}).
\end{aligned}
$$

□

Lemma B.2. *Let $a_{ij}(t)$ be differentiable in t with inverse $a^{ij}(t)$. Then*

$$\frac{d}{dt}a^{ij} = -a^{ik}a^{lj}\frac{d}{dt}a_{kl}.$$

Proof. We have

$$a^{ik}a_{kj} = \delta^i_j.$$

Assume that there exists $\tilde a^{ij}$ such that

$$a_{ik}\tilde a^{kj} = \delta^j_i.$$

Then $a^{ij} = \tilde a^{ij}$, as

$$a^{ij} = a^{ik}\delta^j_k = a^{ik}\left(a_{kl}\tilde a^{lj}\right) = \left(a^{ik}a_{kl}\right)\tilde a^{lj} = \tilde a^{ij}.$$

We differentiate and obtain

$$0 = \frac{d}{dt}\delta^i_j = \frac{d}{dt}\left(a^{ik}a_{kj}\right) = \frac{d}{dt}a^{ik}a_{kj} + a^{ik}\frac{d}{dt}a_{kj}.$$

Hence

$$\frac{d}{dt}a^{il} = \frac{d}{dt}a^{ik}\delta^l_k = \frac{d}{dt}a^{ik}a_{kj}a^{jl} = -a^{ik}\frac{d}{dt}a_{kj}a^{jl}.$$

□

References

1. Ben Andrews, *Contraction of convex hypersurfaces in Euclidean space*, Calc. Var. Partial Differential Equations **2** (1994), no. 2, 151–171.
2. Ben Andrews, *Harnack inequalities for evolving hypersurfaces*, Math. Z. **217** (1994), no. 2, 179–197.
3. Ben Andrews, *Gauss curvature flow: the fate of the rolling stones*, Invent. Math. **138** (1999), no. 1, 151–161.
4. Bennett Chow, *Deforming convex hypersurfaces by the nth root of the Gaussian curvature*, J. Differential Geom. **22** (1985), no. 1, 117–138.
5. Julie Clutterbuck, *Parabolic equations with continuous initial data*, `arXiv:math.AP/0504455`.
6. Julie Clutterbuck, Oliver C. Schnürer, and Felix Schulze, *Stability of translating solutions to mean curvature flow*, Calc. Var. Partial Differential Equations **29** (2007), no. 3, 281–293.
7. Tobias H. Colding and William P. Minicozzi, II, *Sharp estimates for mean curvature flow of graphs*, J. Reine Angew. Math. **574** (2004), 187–195.
8. Klaus Ecker, *Regularity theory for mean curvature flow*, Progress in Nonlinear Differential Equations and their Applications, 57, Birkhäuser Boston Inc., Boston, MA, 2004.
9. Klaus Ecker and Gerhard Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), no. 3, 547–569.
10. Mark E. Feighn, *Separation properties of codimension-*1 *immersions*, Topology **27** (1988), no. 3, 319–321.
11. William J. Firey, *Shapes of worn stones*, Mathematika **21** (1974), 1–11.
12. Michael Gage and Richard S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom. **23** (1986), no. 1, 69–96.
13. Claus Gerhardt, *Curvature problems*, Series in Geometry and Topology, vol. 39, International Press, Somerville, MA, 2006.
14. Claus Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geom. **32** (1990), no. 1, 299–314.
15. Matthew A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom. **26** (1987), no. 2, 285–314.

16. Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.
17. Gerhard Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984), no. 1, 237–266.
18. Gerhard Huisken and Tom Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437.
19. Gerhard Huisken and Alexander Polden, *Geometric evolution equations for hypersurfaces*, Calculus of variations and geometric evolution problems (Cetraro, 1996), Lecture Notes in Math., vol. 1713, Springer, Berlin, 1999, pp. 45–84.
20. James A. McCoy, *The surface area preserving mean curvature flow*, Asian J. Math. **7** (2003), no. 1, 7–30.
21. Murray H. Protter and Hans F. Weinberger, *Maximum principles in differential equations*, Springer-Verlag, New York, 1984, Corrected reprint of the 1967 original.
22. Mariel Sáez Trumper and Oliver C. Schnürer, *Mean curvature flow without singularities*, 2012, `arXiv:1210.6007 [math.DG]`.
23. Oliver C. Schnürer, *Surfaces contracting with speed* $|A|^2$, J. Differential Geom. **71** (2005), no. 3, 347–363.
24. Oliver C. Schnürer, *Surfaces expanding by the inverse Gauß curvature flow*, J. Reine Angew. Math. **600** (2006), 117–134.
25. Oliver C. Schnürer, *The Dirichlet problem for Weingarten hypersurfaces in Lorentz manifolds*, Math. Z. **242** (2002), no. 1, 159–181.
26. Felix Schulze, *Nonlinear evolution by mean curvature and isoperimetric inequalities*, `arXiv:math.DG/0606675`.
27. Nikolaos Stavrou, *Selfsimilar solutions to the mean curvature flow*, J. Reine Angew. Math. **499** (1998), 189–198.
28. Charles Sturm, *Mémoire sur la résolution des équations numeriques*, Bull. Sci. Math. Ferussac **11** (1829), 419–422.
29. Peter Topping, *Mean curvature flow and geometric inequalities*, J. Reine Angew. Math. **503** (1998), 47–61.
30. Kaising Tso, *Deforming a hypersurface by its Gauss-Kronecker curvature*, Comm. Pure Appl. Math. **38** (1985), no. 6, 867–882.
31. John I. E. Urbas, *An expansion of convex hypersurfaces*, J. Differential Geom. **33** (1991), no. 1, 91–125.
32. John Urbas, *Complete noncompact self-similar solutions of Gauss curvature flows. I. Positive powers*, Math. Ann. **311** (1998), no. 2, 251–274.

OLIVER C. SCHNÜRER, FB MATHEMATIK UND STATISTIK, UNIVERSITÄT KONSTANZ, GERMANY
E-mail address: `Oliver.Schnuerer@uni-konstanz.de`