Hyperbolic polynomials and the generalized Lax conjecture

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More information at https://easychair.org/cfp/POEMA-19-22.

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Scientific context. The Generalized Lax Conjecture [7, Conjecture 1.5] (initially stated in different terminology [9, Section 6]) says that every hyperbolicity cone is spectrahedral. The conjecture has a completely algebraic reformulation in terms of linear determinantal representations [7, Remark 1.6]. Only special cases are solved (see [7, Remark 1.6] and the references therein), e.g., by the deep Helton-Vinnikov theorem [9, 10] or by algebraic combinatorics [6, 7]. During a Banff workshop in 2010 [8], Petter Brändén refuted a stronger version of the Generalized Lax Conjecture [5]. There are several interesting connections to sums of squares in commutative rings [13, 14, 15] as well as in non-commutative rings [16].

Related to that is a less explored connection to *moments* which comes down to investigate the effect of additional algebraic constraints on positive semidefinite generalized Hankel matrices. This will be one of the subjects to be investigated in this doctoral project.

Another line of attack will be through *free convexity* [17] trying to prove that every hyperbolicity cone comes from a matrix-convex cone with a "sufficiently nice" algebraic description [13, Theorem 4.2] where it would already be a great step to prove versions with weaker algebraic descriptions that are still insufficient to prove the conjecture.

A third possibility of approaching the problem, a priori perhaps the most promising one, will be through finite analogs of *free probability* that have been developed mainly by Adam W. Marcus [1] in the aftermath of the positive solution to the Kadison-Singer problem [2, 3, 5].

Finally, we will also study certain properties of positive semidefinite matrices and of mixed discriminants which are not known to hold for hyperbolic polynomials although they can be formulated in one way or the other in this more general setting (see for example [12, Section 3] or [11]).

Rather than working on all of these approaches to the conjecture, the candidate should concentrate on those that fit particularly well his background and his interests. It is in no way expected that the conjecture will be solved but some significant progress towards a possible solution (whether it is positive or negative) should be made.

Working Context. The PhD candidate will be hosted by the Real Geometry and Algebra group [https://www.mathematik.uni-konstanz.de/en/rag/] within the Department of Mathematics and Statistics of the University of Konstanz. The town of

Konstanz is located at Lake Constance in the south of Germany, bordering Switzerland. The University of Konstanz is German's southernmost university and has been successful in all three funding lines of the German Excellence Initiative, both in 2007 and in 2012, and is therefore considered one of Germany's elite universities. The local Real Geometry and Algebra group has a strong expertise in real algebraic geometry and its applications to optimization, especially moment problems.

Planned secondments. The PhD candidate will have a research stay (secondments) at Università di Firenze, Florence, Italy (Giorgio Ottaviani) and at Sorbonne Université, Paris, France (Mohab Safey El Din).

Required Skills. Motivated candidates should hold — at the date of recruitment — a master's degree in mathematics or a similar diploma. The applicant should have a solid background in several of the following fields: algebra, combinatorics, real algebraic geometry, optimization, functional analysis, probability theory or theoretical computer science. Good programming skills are also a plus. Knowledge of German does not constitute a pre-requisite. The candidates are kindly asked to send an e-mail with "POEMA candidate" in the title, a CV and motivation letter to

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and to submit their documents at:

https://easychair.org/cfp/POEMA-19-22

References

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[https://arxiv.org/abs/1108.4380]

[16] T. Netzer, A. Thom: Polynomials with and without determinantal representations
[https://arxiv.org/abs/1008.1931]

[17] T. Kriel: An introduction to matrix convex sets and free spectrahedra
[https://arxiv.org/abs/1611.03103]