Introduction to Polynomial Optimization

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RWTH Aachen
Experimental and constructive algebra
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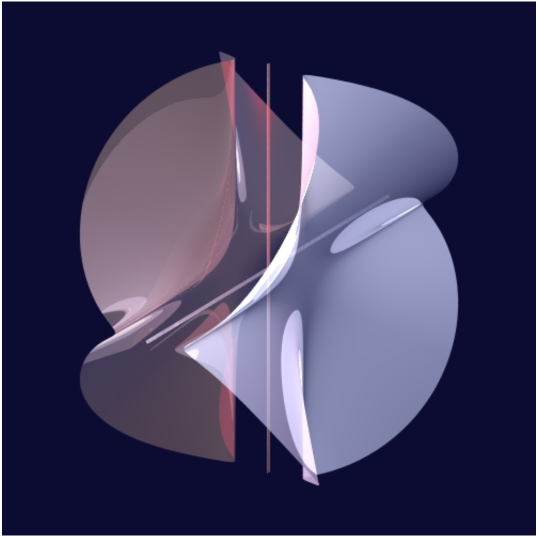
Systems of polynomial inequalities

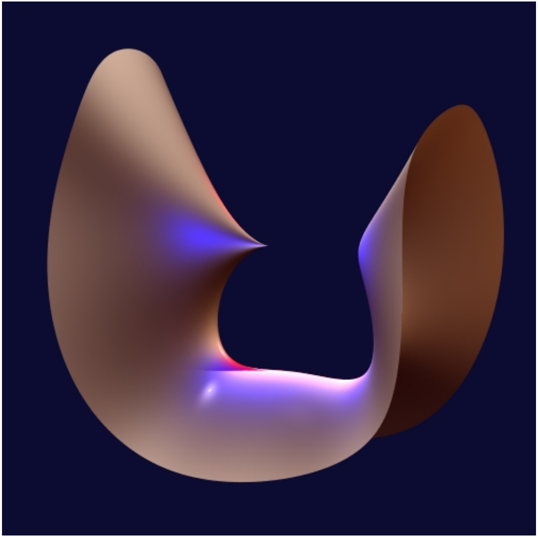
$$x_1^4 - x_1^2 - x_2^2 x_3^2 = 0,\ x_1^2 + x_2^2 + x_3^2 \leq 25$$



http://www1-c703.uibk.ac.at/mathematik/project/bildergalerie/gallery.html

Systems of polynomial inequalities

$$x_1^2 - x_1^3 + x_2^2 + x_2^4 + x_3^3 - x_3^4 = 0,\ x_1^2 + x_2^2 + x_3^2 \leq 4$$



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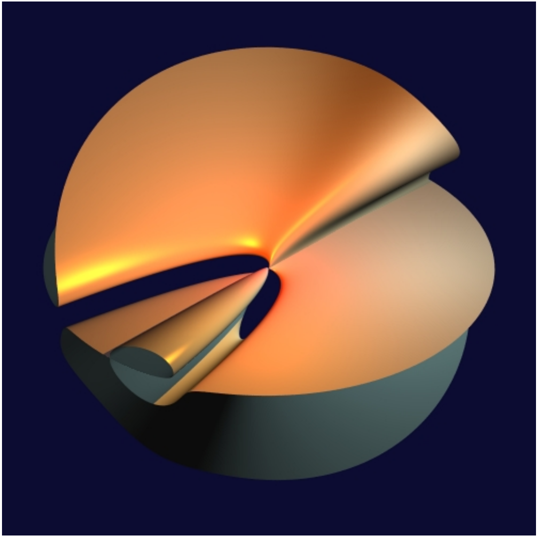
$$x_1^3 x_3 + x_1^2 + x_2 x_3^3 + x_3^4 = 3x_1 x_2 x_3, \; x_1^2 + x_2^2 + x_3^2 \leq 4$$



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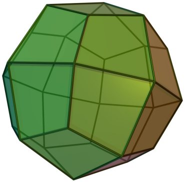

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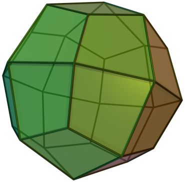

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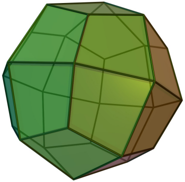

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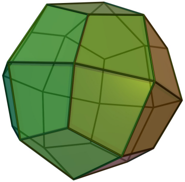

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Can a POP be reduced to an LP?

Linearizing a POP in a very naive way

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\text{minimize} & x_1x_2^2 & + & x_1x_2x_3^5 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\[1em]
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\[1em]
\text{subject to} & 2x_1x_2 & + & 6x_2^2 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 \\
& -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 \\
& 7x_1x_2x_4^2 & - & \frac{1}{2}x_2^2 & + & x_1x_3^2 & - & 2x_1x_2^2 & \geq & 0 \\
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\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\
\text{over} & \multicolumn{8}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcrl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcc}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & x_2^3x_4^2 & \geq & 0 \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\[1em]
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\[1em]
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & x_2^3x_4^2 & \geq & 0 \\
& -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 \\
& 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & 2y_1 & \geq & 0 \\
& 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\[1em]
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\[1em]
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\[2ex]
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\[2ex]
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
 & -x_1 & - & {\color{red}x_1x_2^6} & - & x_3 & + & 7x_4 & \geq & 0 \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\
\\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
& -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 \\
& 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & 2y_1 & \geq & 0 \\
& 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\
 & & & & & & & & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
 & & & & & & & & & \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & \color{red}{7x_1x_2x_4^2} & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcrl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & y_9 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & y_9 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$\begin{array}{rrcrcrcrcrcr}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0
\end{array}$$

Linearizing a POP in a very naive way

$$\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3{\color{red}x_1 x_2^8 x_3} & + & 7 & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0
\end{array}$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3y_{12} & + & 7 & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcc}
\text{minimize} & y_1 & + & y_4 & - & 3y_{12} & + & 7 & & \\[2ex]
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\[2ex]
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
& -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 \\
& y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & 2y_1 & \geq & 0 \\
& 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a very naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3y_{12} & + & 7 & & \\
\text{over} & \multicolumn{9}{l}{x \in \mathbb{R}^4, y \in \mathbb{R}^{12}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & 2y_1 & \geq & 0 \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcrl}
\text{minimize} & x_1x_2^2 & + & x_1x_2x_3^5 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\[2ex]
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\[2ex]
\text{subject to} & 2x_1x_2 & + & 6x_2^2 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & A \\
& -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
& 7x_1x_2x_4^2 & - & \frac{1}{2}x_2^2 & + & x_1x_3^2 & - & \ldots & \geq & 0 & D \\
& 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq & 0 & E
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrl}
\text{minimize} & x_1x_2^2 & + & x_1x_2x_3^5 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2x_1x_2 & + & 6x_2^2 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}x_2^2 & + & x_1x_3^2 & - & \ldots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6x_1x_2^2 & - & 2x_1^2x_2^7 & - & \ldots & \geq & 0 & F
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcccl}
\text{minimize} & x_1x_2^2 & + & x_1x_2x_3^5 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\\
\text{over} & \multicolumn{7}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & & & & \\
\\
\text{subject to} & 2x_1x_2 & + & 6x_2^2 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}x_2^2 & + & x_1x_3^2 & - & \ldots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6x_1x_2^2 & - & 2x_1^2x_2^7 & - & \ldots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & x_1x_2 & + & \ldots & \geq & 0 & G
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & x_1x_2^2 & + & x_1x_2x_3^5 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2x_1x_2 & + & 6x_2^2 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}x_2^2 & + & x_1x_3^2 & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6x_1x_2^2 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & x_1x_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & x_1 x_2^2 & + & x_1 x_2 x_3^5 & - & 3x_1 x_2^8 x_3 & + & 7 & & & & \\
\\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2x_1 x_2 & + & 6x_2^2 & - & x_2 x_3^3 & + & x_2^3 x_4^2 & \geq & 0 & & \color{red}{A} \\
 & -x_1 & - & x_1 x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & & \color{red}{B} \\
 & 7x_1 x_2 x_4^2 & - & \frac{1}{2} x_2^2 & + & x_1 x_3^2 & - & \dots & \geq & 0 & & \color{red}{D} \\
 & 3x_1 & + & 6x_2 x_3^4 & + & x_1 x_3 & - & 3 & \geq & 0 & & \color{red}{E} \\
\color{blue}{A \cdot B} & -2x_1^2 x_2 & - & 6x_1 x_2^2 & - & 2x_1^2 x_2^7 & - & \dots & \geq & 0 & & \color{blue}{F} \\
\color{blue}{C^2} & 2x_1^2 & + & x_2^4 & - & x_1 x_2 & + & \dots & \geq & 0 & & \color{blue}{G} \\
\color{blue}{C^2 \cdot D} & x_1^2 x_2^2 & + & \frac{1}{2} x_1 x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & & \color{blue}{H} \\
\color{blue}{A \cdot B \cdot E} & 6x_1^2 x_2 & - & 6x_1^3 x_2 & + & 18x_1 x_2^2 & + & \dots & \geq & 0 & & \color{blue}{I}
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & x_1x_2^2 & + & x_1x_2x_3^5 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2x_1x_2 & + & 6x_2^2 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}x_2^2 & + & x_1x_3^2 & - & \dots & \geq & 0 & & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq & 0 & & E \\
A \cdot B & -2x_1^2x_2 & - & 6x_1x_2^2 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & x_1x_2 & + & \dots & \geq & 0 & & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18x_1x_2^2 & + & \dots & \geq & 0 & & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & x_1x_2x_3^5 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2x_1x_2 & + & 6x_2^2 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & A \\
& -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
& 7x_1x_2x_4^2 & - & \frac{1}{2}x_2^2 & + & x_1x_3^2 & - & \dots & \geq & 0 & D \\
& 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & x_1x_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcll}
\text{minimize} & y_1 & + & x_1x_2x_3^5 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\\
\text{over} & \multicolumn{7}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & & & \\
\\
\text{subject to} & 2x_1x_2 & + & 6x_2^2 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}x_2^2 & + & x_1x_3^2 & - & \ldots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \ldots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & x_1x_2 & + & \ldots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \ldots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrccl}
\text{minimize} & y_1 & + & x_1x_2x_3^5 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\\
\text{over} & \multicolumn{10}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6x_2^2 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}x_2^2 & + & x_1x_3^2 & - & \ldots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \ldots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \ldots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \ldots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & x_1x_2x_3^5 & - & 3x_1x_2^8x_3 & + & 7 & & \\
\\
\text{over} & \multicolumn{8}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6\textcolor{red}{x_2^2} & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq 0 & \textcolor{red}{A} \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq 0 & \textcolor{red}{B} \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}\textcolor{red}{x_2^2} & + & x_1x_3^2 & - & \ldots & \geq 0 & \textcolor{red}{D} \\
 & 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq 0 & \textcolor{red}{E} \\
\textcolor{blue}{A \cdot B} & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \ldots & \geq 0 & \textcolor{blue}{F} \\
\textcolor{blue}{C^2} & 2x_1^2 & + & x_2^4 & - & y_2 & + & \ldots & \geq 0 & \textcolor{blue}{G} \\
\textcolor{blue}{C^2 \cdot D} & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq 0 & \textcolor{blue}{H} \\
\textcolor{blue}{A \cdot B \cdot E} & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \ldots & \geq 0 & \textcolor{blue}{I}
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rcccccccccccc}
\text{minimize} & y_1 & + & x_1x_2x_3^5 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\\
\text{over} & \multicolumn{7}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & & & \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \ldots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \ldots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \ldots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \ldots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & x_1x_2x_3^5 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\[2ex]
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\[2ex]
\text{subject to} & 2y_2 & + & 6y_3 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & \\
\text{subject to} & 2y_2 & + & 6y_3 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & \color{red}{A} \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & \color{red}{B} \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \ldots & \geq & 0 & \color{red}{D} \\
 & 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq & 0 & \color{red}{E} \\
\color{blue}{A \cdot B} & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \ldots & \geq & 0 & \color{blue}{F} \\
\color{blue}{C^2} & 2x_1^2 & + & x_2^4 & - & y_2 & + & \ldots & \geq & 0 & \color{blue}{G} \\
\color{blue}{C^2 \cdot D} & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq & 0 & \color{blue}{H} \\
\color{blue}{A \cdot B \cdot E} & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \ldots & \geq & 0 & \color{blue}{I}
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcll}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\[1em]
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\[1em]
\text{subject to} & 2y_2 & + & 6y_3 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & x_1x_3 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\text{over} & \multicolumn{5}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & & & & & \\
\text{subject to} & 2y_2 & + & 6y_3 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & x_2x_3^3 & + & x_2^3x_4^2 & \geq & 0 & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\\
\text{over} & \multicolumn{7}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & & & & \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & x_2^3x_4^2 & \geq & 0 & A \\
& -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
& 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \ldots & \geq & 0 & D \\
& 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \ldots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \ldots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \ldots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$\begin{array}{rccccccccccl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\\
\text{over} & \multicolumn{7}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & & & & \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & x_2^3x_4^2 & \geq & 0 & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}$$

Linearizing a POP in a less naive way

$$\begin{array}{rrcrcrcrcll}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcll}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
& -x_1 & - & x_1x_2^6 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
& 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \dots & \geq & 0 & D \\
& 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\\
\text{over} & \multicolumn{3}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & & & & & & & & \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \dots & \geq & 0 & & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0 & & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcll}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\\
\text{over} & \multicolumn{7}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & & & \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & 7x_1x_2x_4^2 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcll}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\text{over} & \multicolumn{4}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & & & & & & \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcll}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & x_1x_3^2 & - & \ldots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \ldots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \ldots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \ldots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
& & & & & & & & & & & \\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
& & & & & & & & & & & \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
& -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
& y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \dots & \geq & 0 & D \\
& 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\\
\text{over} & \multicolumn{10}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6x_2x_3^4 & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & \\
\\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \ldots & \geq 0 & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \ldots & \geq 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \ldots & \geq 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \ldots & \geq 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcrl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \ldots & \geq & 0 & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2x_1^2x_2 & - & 6y_1 & - & 2x_1^2x_2^7 & - & \ldots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \ldots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq & 0 & H \\
A \cdot B \cdot E & 6x_1^2x_2 & - & 6x_1^3x_2 & + & 18y_1 & + & \ldots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\[1em]
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\[1em]
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6y_{12} & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6y_{12} & - & 6x_1^3x_2 & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

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\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \ldots & \geq & 0 & & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2x_1^2x_2^7 & - & \ldots & \geq & 0 & & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \ldots & \geq & 0 & & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq & 0 & & H \\
A \cdot B \cdot E & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \ldots & \geq & 0 & & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2x_1^2x_2^7 & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \dots & \geq & 0 & & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2y_{14} & - & \dots & \geq & 0 & & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & & H \\
A \cdot B \cdot E & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \dots & \geq & 0 & & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rccccccccccc}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
& -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
& y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \dots & \geq & 0 & D \\
& 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2y_{14} & - & \dots & \geq & 0 & F \\
C^2 & 2x_1^2 & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \ldots & \geq & 0 & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2y_{14} & - & \ldots & \geq & 0 & F \\
C^2 & 2y_{15} & + & x_2^4 & - & y_2 & + & \ldots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq & 0 & H \\
A \cdot B \cdot E & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \ldots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rcccccccccccc}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\\
\text{over} & \multicolumn{7}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & & & \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
& -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
& y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \dots & \geq & 0 & D \\
& 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2y_{14} & - & \dots & \geq & 0 & F \\
C^2 & 2y_{15} & + & x_2^4 & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \ldots & \geq & 0 & & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2y_{14} & - & \ldots & \geq & 0 & & F \\
C^2 & 2y_{15} & + & y_{16} & - & y_2 & + & \ldots & \geq & 0 & & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq & 0 & & H \\
A \cdot B \cdot E & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \ldots & \geq & 0 & & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcll}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2y_{14} & - & \dots & \geq & 0 & F \\
C^2 & 2y_{15} & + & y_{16} & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & x_1^2x_2^2 & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \ldots & \geq & 0 & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2y_{14} & - & \ldots & \geq & 0 & F \\
C^2 & 2y_{15} & + & y_{16} & - & y_2 & + & \ldots & \geq & 0 & G \\
C^2 \cdot D & y_{17} & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \ldots & \geq & 0 & H \\
A \cdot B \cdot E & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \ldots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
 & & & & & & & & & & & \\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
 & & & & & & & & & & & \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \dots & \geq & 0 & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2y_{14} & - & \dots & \geq & 0 & F \\
C^2 & 2y_{15} & + & y_{16} & - & y_2 & + & \dots & \geq & 0 & G \\
C^2 \cdot D & y_{17} & + & \frac{1}{2}x_1x_2^3 & - & 3x_2^6 & + & \dots & \geq & 0 & H \\
A \cdot B \cdot E & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \dots & \geq & 0 & I
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcccl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\\
\text{over} & \multicolumn{7}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} & & & \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & {\color{red}A} \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & {\color{red}B} \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \ldots & \geq & 0 & {\color{red}D} \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & {\color{red}E} \\
{\color{blue}A \cdot B} & -2y_{12} & - & 6y_1 & - & 2y_{14} & - & \ldots & \geq & 0 & {\color{blue}F} \\
{\color{blue}C^2} & 2y_{15} & + & y_{16} & - & y_2 & + & \ldots & \geq & 0 & {\color{blue}G} \\
{\color{blue}C^2 \cdot D} & y_{17} & + & \frac{1}{2}y_{18} & - & 3x_2^6 & + & \ldots & \geq & 0 & {\color{blue}H} \\
{\color{blue}A \cdot B \cdot E} & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \ldots & \geq & 0 & {\color{blue}I}
\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \dots & \geq & 0 & & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq & 0 & & E \\
A \cdot B & -2y_{12} & - & 6y_1 & - & 2y_{14} & - & \dots & \geq & 0 & & F \\
C^2 & 2y_{15} & + & y_{16} & - & y_2 & + & \dots & \geq & 0 & & G \\
C^2 \cdot D & y_{17} & + & \frac{1}{2}y_{18} & - & 3x_2^6 & + & \dots & \geq & 0 & & H \\
A \cdot B \cdot E & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \dots & \geq & 0 & & I
\end{array}
$$

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$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & & \\
\\
\text{over} & \multicolumn{11}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
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\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3x_1x_2^8x_3 & + & 7 & & & \\
\text{over} & \multicolumn{10}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \ldots & \geq & 0 & D \\
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$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcl}
\text{minimize} & y_1 & + & y_4 & - & 3y_{20} & + & 7 & & \\
\\
\text{over} & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \in \mathbb{R}} \\
\\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq 0 & B \\
 & y_9 & - & \frac{1}{2}y_3 & + & y_{10} & - & \dots & \geq 0 & D \\
 & 3x_1 & + & 6y_{11} & + & y_5 & - & 3 & \geq 0 & E \\
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\end{array}
$$

Linearizing a POP in a less naive way

$$
\begin{array}{rrcrcrcrcll}
\text{minimize} & y_1 & + & y_4 & - & 3y_{20} & + & 7 & & & \\
\text{over} & \multicolumn{9}{l}{x \in \mathbb{R}^4, y \in \mathbb{R}^{20}} \\
\text{subject to} & 2y_2 & + & 6y_3 & - & y_6 & + & y_7 & \geq & 0 & A \\
 & -x_1 & - & y_8 & - & x_3 & + & 7x_4 & \geq & 0 & B \\
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\end{array}
$$

Idea: Blue inequalities are a priori redundant

Linearizing a POP in a less naive way

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\text{minimize} & y_1 & + & y_4 & - & 3y_{20} & + & 7 & & \\
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A \cdot B \cdot E & 6y_{12} & - & 6y_{13} & + & 18y_1 & + & \dots & \geq 0 & I
\end{array}
$$

Idea: Blue inequalities are a priori redundant but yield valuable information after linearization.

Linearizing a POP in a less naive way

The Positivstellensatz from Real Algebraic Geometry

In his very first and fulminant work Anneaux préordonnés, Jean-Louis Krivine [*1939] proved in 1964 the so-called Positivstellensatz

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Linearizing a POP in a less naive way

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To each given infeasible POP, in the just described (less naive) linearization procedure, you can always add finitely many blue inequalities such that the resulting LP is infeasible.

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[1] Every nonnegative polynomial in several variables is a sum of squares of rational functions.

Linearizing a POP in a less naive way

Schmüdgen's Positivstellensatz

An other major breakthrough was Konrad Schmüdgen's [*1947] Positivstellensatz from 1991 which is essentially equivalent to the following theorem:

Linearizing a POP in a less naive way

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To each given POP with compact feasible region, in the just described (less naive) linearization procedure, if you add all blue inequalities,

Linearizing a POP in a less naive way

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To each given POP with compact feasible region, in the just described (less naive) linearization procedure, if you add all blue inequalities, then the optimal values of the original POP and of the resulting "infinite LP" coincide.

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All proofs use the Positivstellensatz (from Krivine). Schmüdgen's original proof uses functional analysis. The first algebraic proof was found by Wörmann in 1998.

Linearizing a POP in a less naive way

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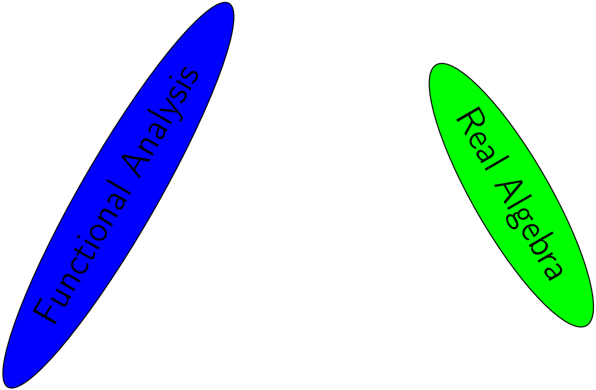
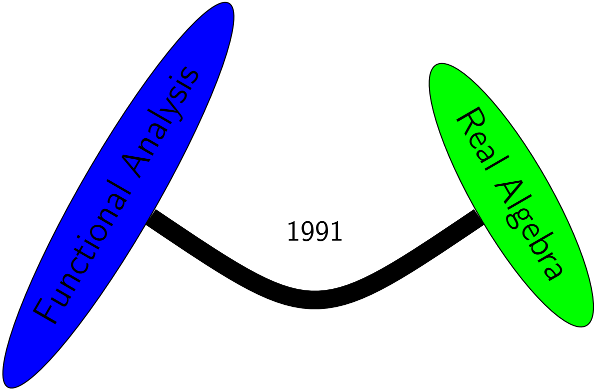
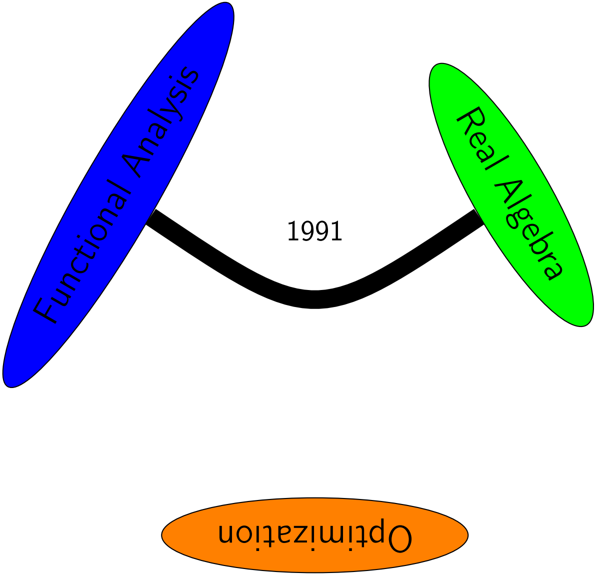
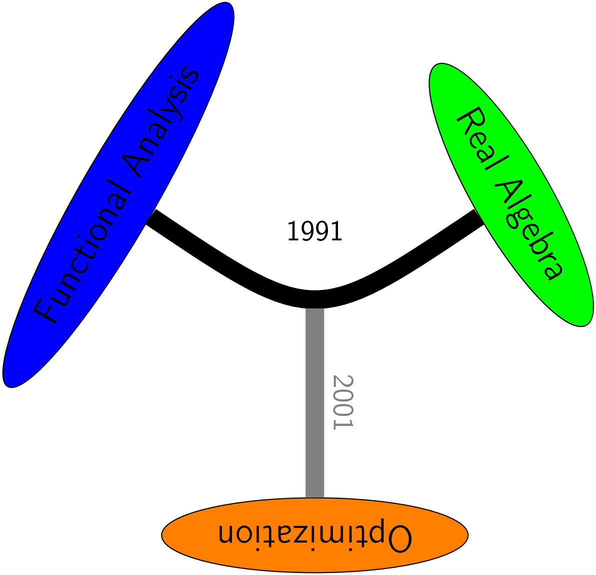
All proofs use the Positivstellensatz (from Krivine). Schmüdgen's original proof uses functional analysis. The first algebraic proof was found by Wörmann in 1998.

In 1993, Mihai Putinar [*1955] showed that products like $A \cdot B$ are not needed in Schmüdgen's theorem when the compactness assumption is replaced by a stronger technical assumption, namely the archimedean condition, which is for practical purposes not far from compactness.


Functional Analysis
Real Algebra


Functional Analysis
1991
Real Algebra


Functional Analysis
Real Algebra
1991
Optimization


Functional Analysis
Real Algebra
1991
2001
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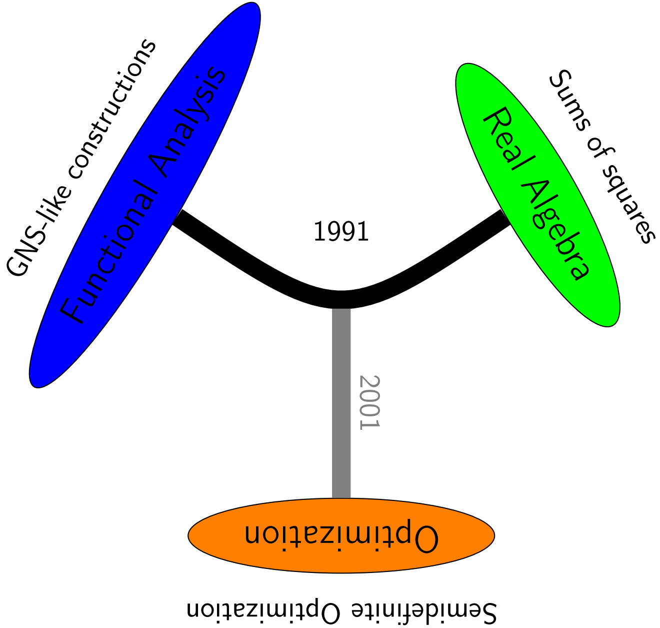
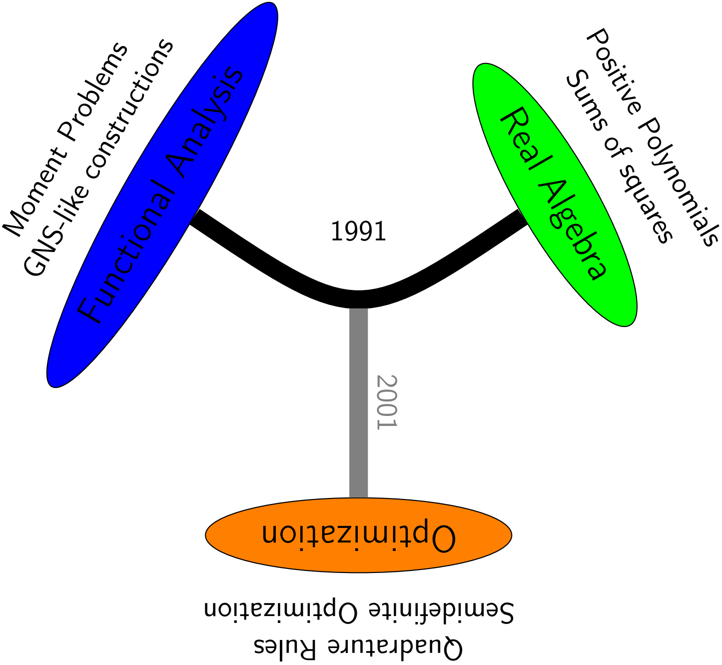

GNS-like constructions
Functional Analysis
1991
Real Algebra
Sums of squares
2001
Optimization
Semidefinite Optimization


Moment Problems
GNS-like constructions
Functional Analysis
Positive Polynomials
Sums of squares
Real Algebra
1991
2001
Optimization
Semidefinite Optimization
Quadrature Rules

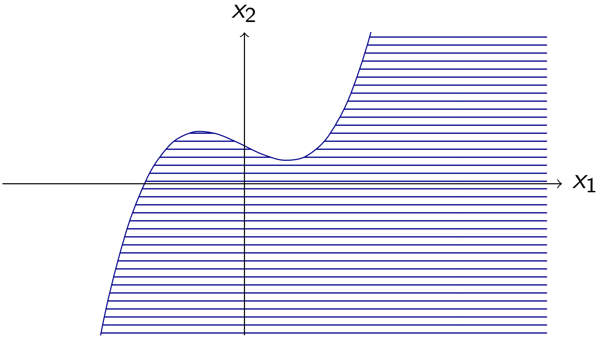
System of polynomial inequalities

$$
\begin{array}{rrrrrrrrrrr}
 & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
- & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
 & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

System of polynomial inequalities

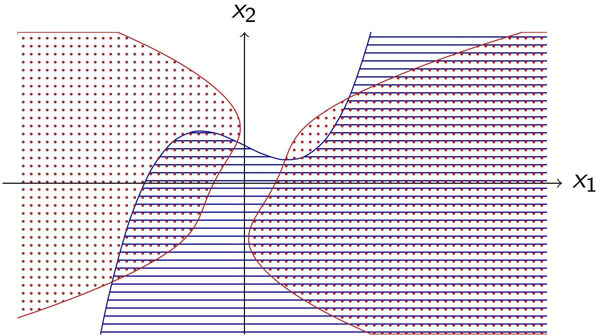
A

$$\begin{array}{rrrrrrrrrrrrr}
 & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
- & x_2^4 & + & 2x_1^2 & & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
 & & - & x_1^2 & & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$



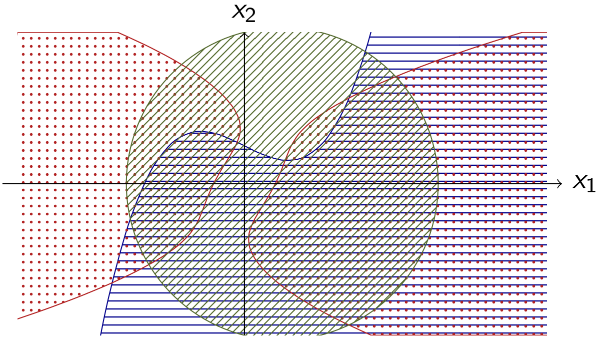
System of polynomial inequalities

$$
\begin{array}{lrrrrrrl}
A & & x_1^3 & - \; x_1 & - \; 2x_2 & + \; 1 & \geq & 0 \\
B & -\; x_2^4 & + \; 2x_1^2 & - \; 2x_1x_2 & + \; x_2^2 & - \; \frac{1}{3} & \geq & 0 \\
 & & - \; x_1^2 & - \; x_2^2 & + \; x_1 & + \; 4 & \geq & 0
\end{array}
$$



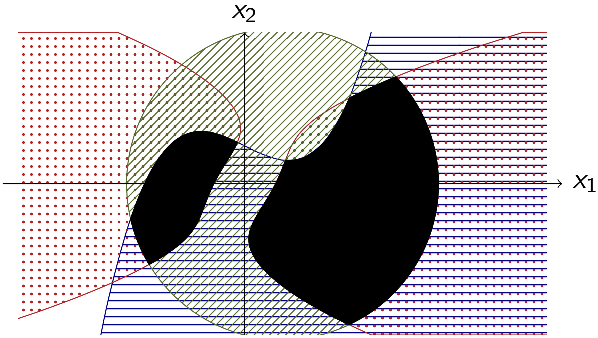
System of polynomial inequalities

$$\begin{array}{lrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$



System of polynomial inequalities

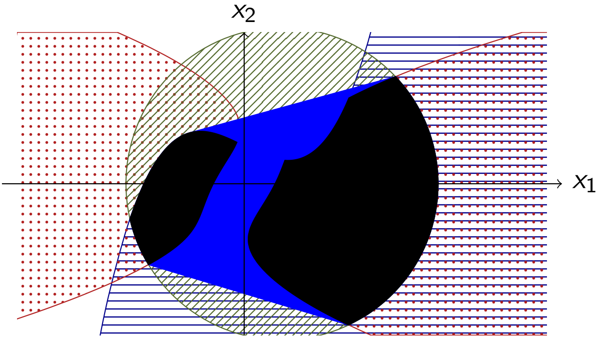
$$\begin{array}{lrcl}
A & x_1^3 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}$$



S

System of polynomial inequalities

$$\begin{array}{lrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

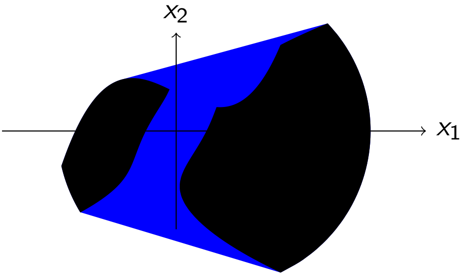


S

conv S

System of polynomial inequalities

$$
\begin{array}{ll}
A & x_1^3 - x_1 - 2x_2 + 1 \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$



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$$\begin{array}{lcccccccccccccc}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
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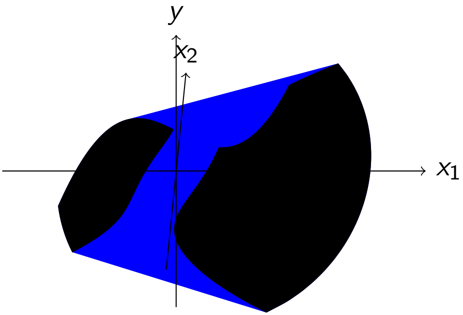
\end{array}$$



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C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
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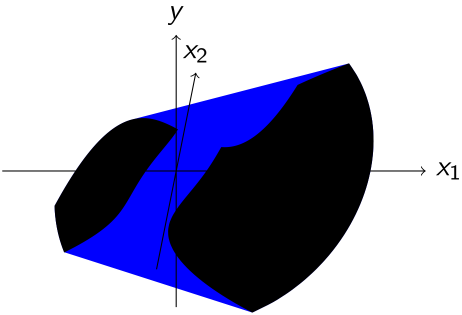
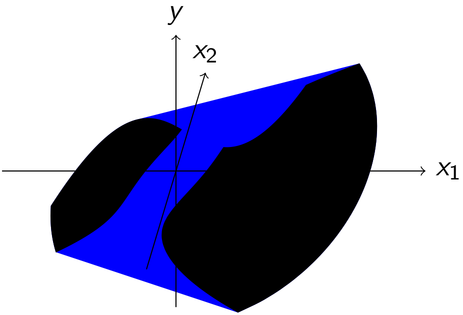


S

conv S

System of polynomial inequalities

$$
\begin{array}{lrrrrrrrrrrrr}
\color{blue}{A} & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
\color{red}{B} & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
\color{green}{C} & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



S

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System of polynomial inequalities

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\begin{array}{llrrrrrrrrr}
A & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq 0 \\
B & -\ x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq 0 \\
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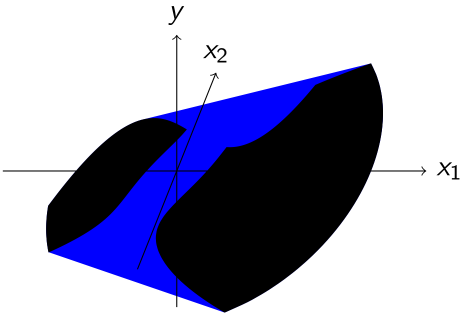
\end{array}
$$

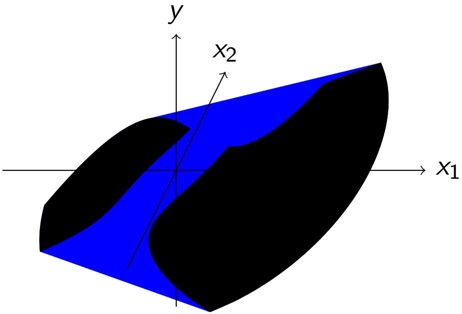


S

conv S

System of polynomial inequalities

$$\begin{array}{lrcl}
A & x_1^3 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}$$

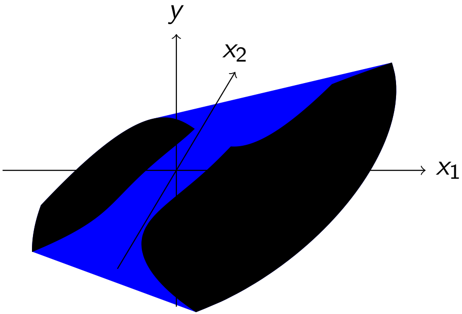


S

conv S

System of polynomial inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

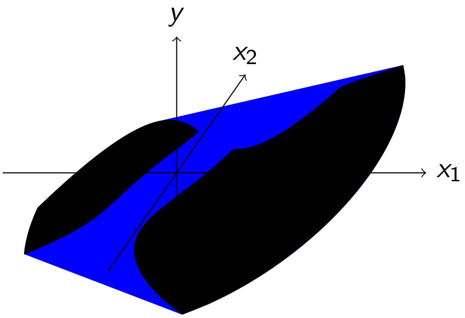


S

conv S

System of polynomial inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & & \geq & 0
\end{array}
$$

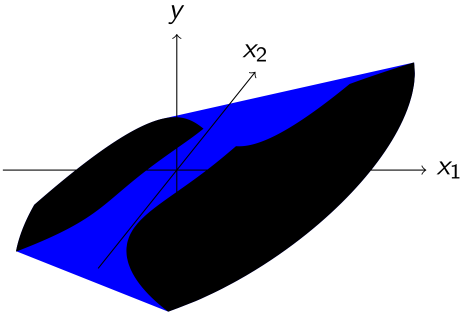


S

conv S

System of polynomial inequalities

$$
\begin{array}{lrrrrrrl}
A & & x_1^3 & - \; x_1 & - \; 2x_2 & + \; 1 & \geq & 0 \\
B & -\; x_2^4 & + \; 2x_1^2 & - \; 2x_1x_2 & + \; x_2^2 & - \; \frac{1}{3} & \geq & 0 \\
C & & - \; x_1^2 & - \; x_2^2 & + \; x_1 & + \; 4 & \geq & 0
\end{array}
$$

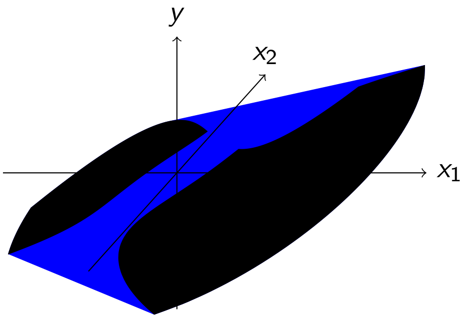


S

conv S

System of polynomial inequalities

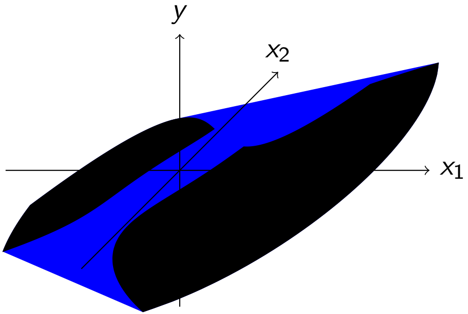
$$
\begin{array}{lrcl}
\color{blue}{A} & x_1^3 - x_1 - 2x_2 + 1 & \geq & 0 \\
\color{red}{B} & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
\color{green}{C} & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

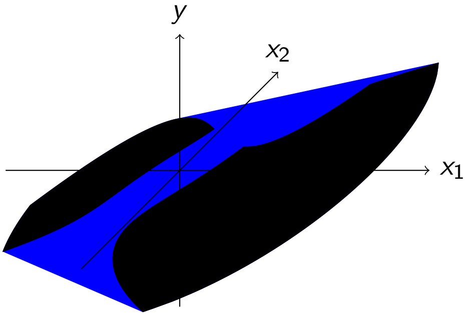
$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

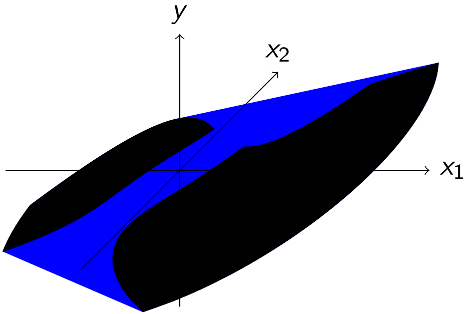
$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$



conv S

System of polynomial inequalities

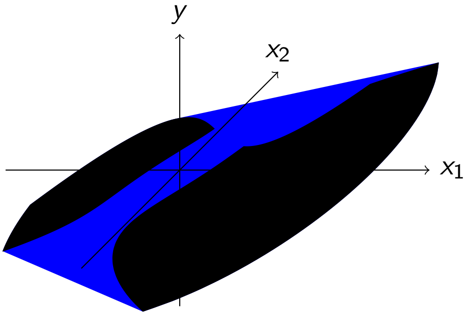
$$
\begin{array}{llcccccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

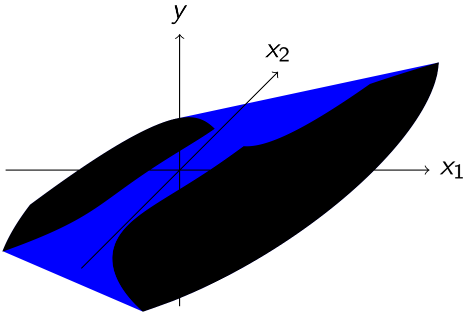
$$
\begin{array}{lrcl}
A & y_1 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -y_2 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

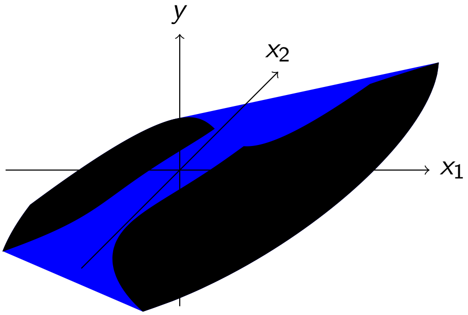
$$
\begin{array}{lrcl}
A & y_1 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -y_2 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

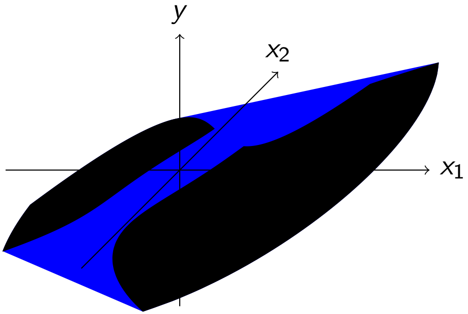
$$\begin{array}{llrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$



conv S

System of polynomial inequalities

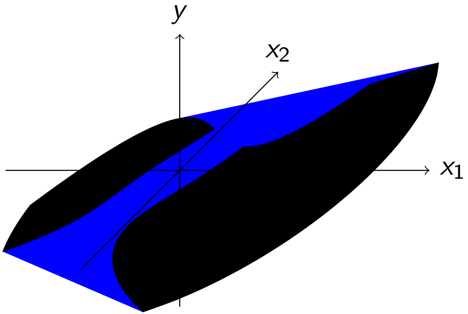
$$
\begin{array}{lrcl}
A & y_1 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -y_2 + 2y_3 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -y_3 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

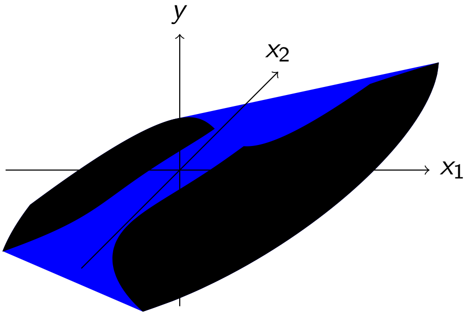
$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

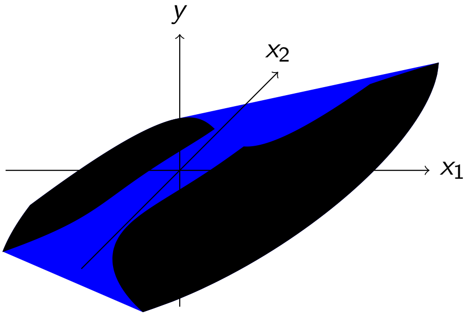
$$
\begin{array}{lrrrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



conv S

System of linear inequalities

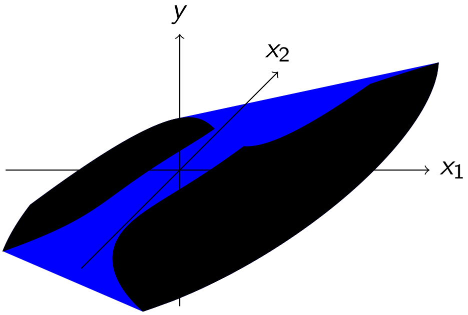
$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & \geq & 0
\end{array}
$$



conv S

System of linear inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 & & \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 & &
\end{array}
$$



conv S

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \ldots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant:

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \ldots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant:

$$\begin{array}{lrrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \ldots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0
\end{array}$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrl}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \ldots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant:

$$\begin{array}{lrrrrrrrrrrrrl}
AB & - & x_1^3 x_2^4 & + & \ldots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \ldots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0
\end{array}$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \ldots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} & & & & & & & & & & & & & \\
AB & - & x_1^3 x_2^4 & + & \ldots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \ldots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & & x_1 & + & 8x_2 & - & 4 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} & & & & & & & & & & & & \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \ldots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} & & & & & & & & & & & & & \\
AB & - & x_1^3x_2^4 & + & \ldots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \ldots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrl}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & & & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & & & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & & & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & & y_3 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & & 4y_3 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant:

$$\begin{array}{lrrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & & y_3 & - & 2y_4 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & & 4y_3 & + & 4y_4 & + & 4x_2^2 & \geq & 0
\end{array}$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrl}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \ldots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrrl}
AB & - & x_1^3x_2^4 & + & \ldots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \ldots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & & y_3 & - & 2y_4 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & & 4y_3 & + & 4y_4 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & - & x_1^3x_2^4 & + & \dots & + & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & - & y_6 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & y_6 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & y_6 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & y_6 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \ldots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} & & & & & & & & & & & & \\
AB & - & y_6 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & y_6 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \ldots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & y_{13} & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrl}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrl}
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & y_{13} & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

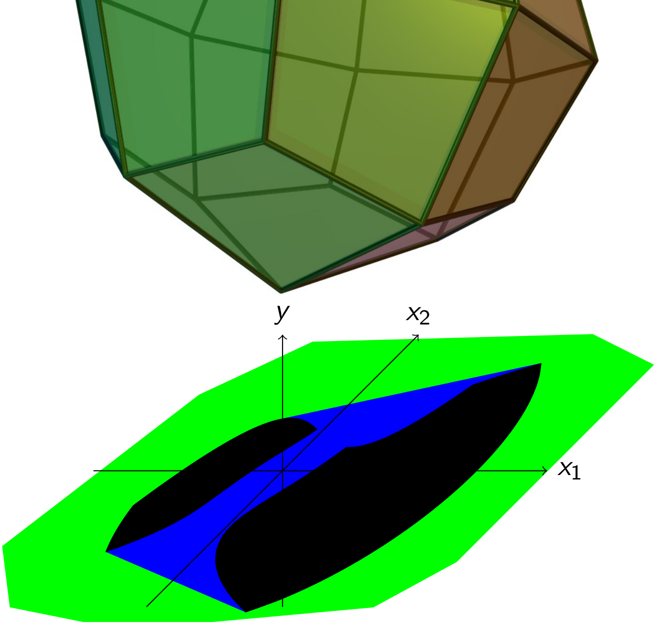
System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \ldots & + & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & - & y_6 & + & \ldots & + & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AB & - & y_6 & + & \ldots & + & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \ldots & - & & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & y_{13} & + & \ldots & - & & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & y_{18} & + & \ldots & + & & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$



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System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0
\end{array}$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \ldots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcrcrcrcrcrcrcc}
A & & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & & - & & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + f x_2^2)^2 \geq 0 \qquad \Longleftrightarrow$$

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix}
\begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1 x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \dots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & x_1x_2 & x_2^2 \\ x_1 & y_3 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{llcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & x_1x_2 & x_2^2 \\
x_1 & y_3 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & x_2^2 \\
x_1 & y_3 & y_4 & y_1 & x_1^2 x_2 & x_1 x_2^2 \\
x_2 & y_4 & x_2^2 & x_1^2 x_2 & x_1 x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2 x_2 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & x_1^2 x_2 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
x_2^2 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & x_2^2 \\ x_1 & y_3 & y_4 & y_1 & x_1^2 x_2 & x_1 x_2^2 \\ x_2 & y_4 & x_2^2 & x_1^2 x_2 & x_1 x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2 x_2 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & x_1^2 x_2 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ x_2^2 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{llccccccccccccc}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & y_4 & y_5 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
y_4 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
y_5 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & y_4 & y_5 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
y_4 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
y_5 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{llcrcrcrcrcrcl}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & x_1x_2^2 \\ x_2 & y_4 & y_5 & y_6 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & y_6 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ y_4 & y_6 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ y_5 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & x_1x_2^2 \\ x_2 & y_4 & y_5 & y_6 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & y_6 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ y_4 & y_6 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ y_5 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\
y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\ y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{llcccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\ y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\ y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & x_1^3x_2 & x_1^2x_2^2 \\
y_4 & y_6 & y_7 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
y_5 & y_7 & y_9 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{llllllllllllll}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & x_1^2x_2^2 \\ y_4 & y_6 & y_7 & y_{10} & x_1^2x_2^2 & x_1x_2^3 \\ y_5 & y_7 & y_9 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & y_{10} & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\
y_4 & y_6 & y_7 & y_{10} & y_{11} & x_1x_2^3 \\
y_5 & y_7 & y_9 & y_{11} & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{llllllllllllll}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\
y_4 & y_6 & y_7 & y_{10} & y_{11} & x_1 x_2^3 \\
y_5 & y_7 & y_9 & y_{11} & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccccl}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\ y_4 & y_6 & y_7 & y_{10} & y_{11} & y_{12} \\ y_5 & y_7 & y_9 & y_{11} & y_{12} & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\ y_4 & y_6 & y_7 & y_{10} & y_{11} & y_{12} \\ y_5 & y_7 & y_9 & y_{11} & y_{12} & y_2 \end{pmatrix} \succeq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lccccccccccccccc}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrl}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \ldots & + & & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{llrcrcrcrcrcrcl}
A & & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$(-x_1^2 - x_2^2 + x_1 + 4)\begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}\begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c \end{pmatrix} (-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \dots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c \end{pmatrix} (-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccc}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \dots & \dots \\
-x_1^3 - x_1x_2^2 + x_1^2 + 4x_1 & \dots & \dots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \dots & \dots \\
-x_1^3 - x_1 x_2^2 + x_1^2 + 4x_1 & \dots & \dots \\
-x_1^2 x_2 - x_2^3 + x_1 x_2 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{llcccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \dots & \dots \\
-y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \dots & \dots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \dots & \dots \\
-y_1 - x_1 x_2^2 + x_1^2 + 4x_1 & \dots & \dots \\
-x_1^2 x_2 - x_2^3 + x_1 x_2 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lccccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \dots & \dots \\
-y_1 - x_1 x_2^2 + x_1^2 + 4x_1 & \dots & \dots \\
-x_1^2 x_2 - x_2^3 + x_1 x_2 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \dots & \dots \\
-y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \dots & \dots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lccccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - x_2^2 + x_1 + 4 & \dots & \dots \\ -y_1 - x_1x_2^2 + y_3 + 4x_1 & \dots & \dots \\ -x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \dots & \dots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - x_2^2 + x_1 + 4 & \dots & \dots \\
-y_1 - x_1x_2^2 + y_3 + 4x_1 & \dots & \dots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \ldots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - x_2^2 + x_1 + 4 & \ldots & \ldots \\
-y_1 - x_1x_2^2 + y_3 + 4x_1 & \ldots & \ldots \\
-x_1^2x_2 - x_2^3 + y_4 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - x_2^2 + x_1 + 4 & \dots & \dots \\
-y_1 - x_1 x_2^2 + y_3 + 4x_1 & \dots & \dots \\
-x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3x_2^4 & + & \dots & + & & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & & & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & & & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - y_5 + x_1 + 4 & \dots & \dots \\
-y_1 - x_1x_2^2 + y_3 + 4x_1 & \dots & \dots \\
-x_1^2x_2 - x_2^3 + y_4 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & x_1^3 x_2^4 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - y_5 + x_1 + 4 & \dots & \dots \\
-y_1 - x_1 x_2^2 + y_3 + 4x_1 & \dots & \dots \\
-x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - y_5 + x_1 + 4 & \dots & \dots \\
-y_1 - y_6 + y_3 + 4x_1 & \dots & \dots \\
-x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & & & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - y_5 + x_1 + 4 & \dots & \dots \\
-y_1 - y_6 + y_3 + 4x_1 & \dots & \dots \\
-x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - y_5 + x_1 + 4 & \dots & \dots \\ -y_1 - y_6 + y_3 + 4x_1 & \dots & \dots \\ -y_7 - x_2^3 + y_4 + 4x_2 & \dots & \dots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrl}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - y_5 + x_1 + 4 & \dots & \dots \\
-y_1 - y_6 + y_3 + 4x_1 & \dots & \dots \\
-y_7 - x_2^3 + y_4 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lccccccccccccccc}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & & & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & & & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & & & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - y_5 + x_1 + 4 & \dots & \dots \\
-y_1 - y_6 + y_3 + 4x_1 & \dots & \dots \\
-y_7 - y_8 + y_4 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

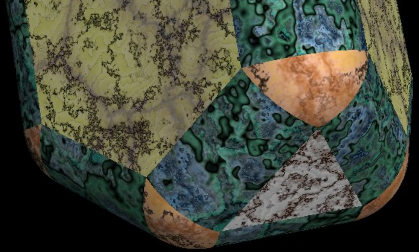
System of polynomial inequalities

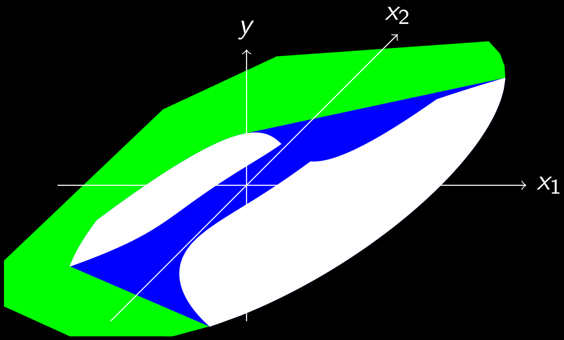
Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

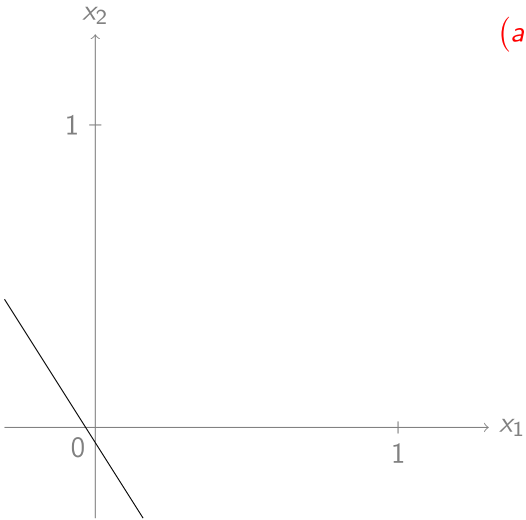
$$\begin{pmatrix} -y_3 - y_5 + x_1 + 4 & \dots & \dots \\ -y_1 - y_6 + y_3 + 4x_1 & \dots & \dots \\ -y_7 - y_8 + y_4 + 4x_2 & \dots & \dots \end{pmatrix} \succeq 0$$





conv S

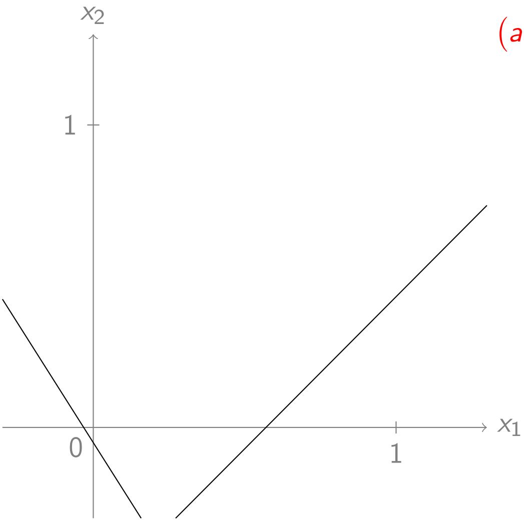
Describing convex semialgebraic sets by LMIs



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_1 \\ x_2 & 1 & x_1 \\ x_1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

Describing convex semialgebraic sets by LMIs



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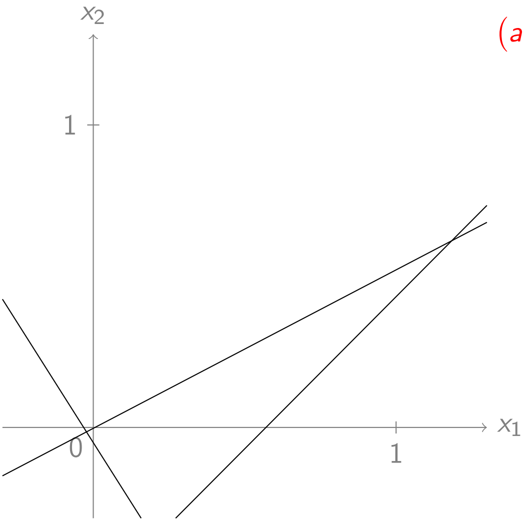

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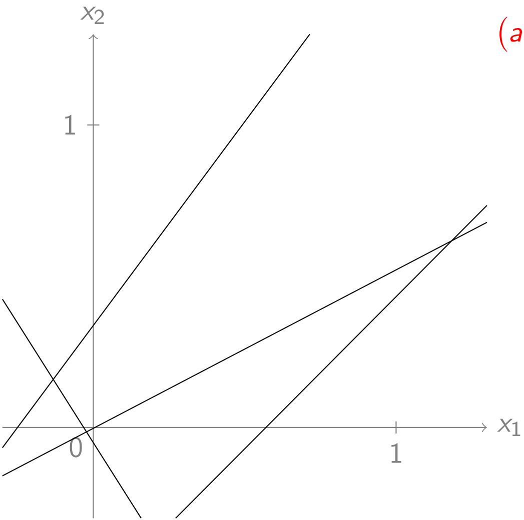

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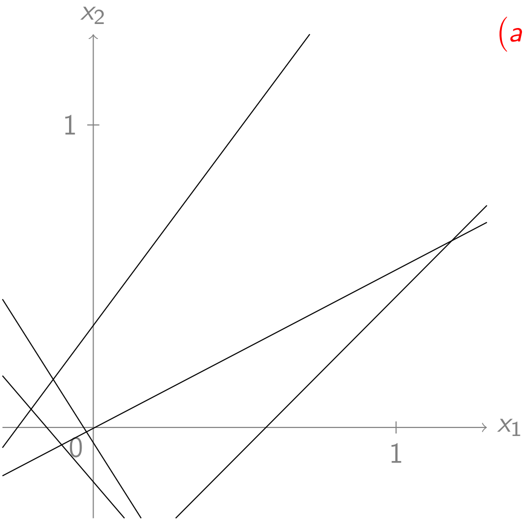

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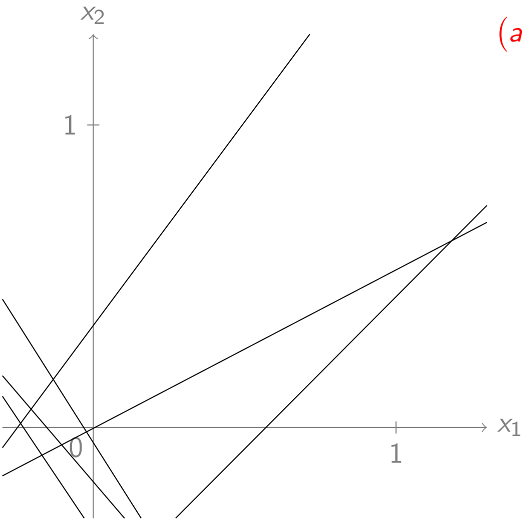

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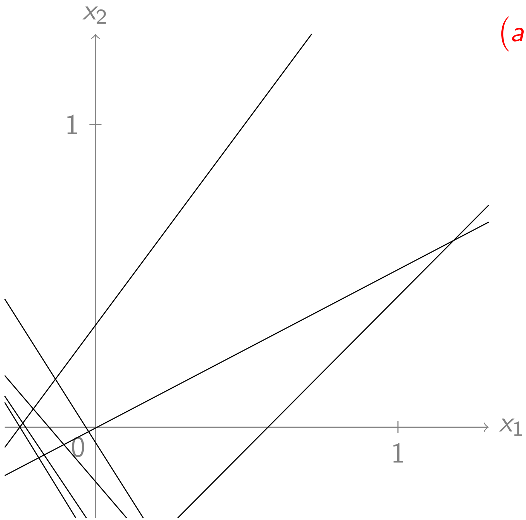

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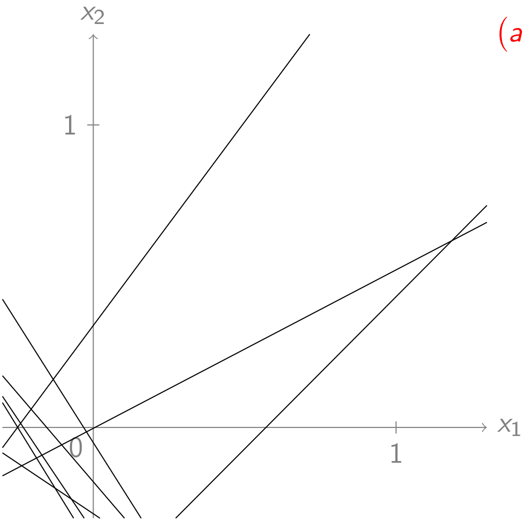

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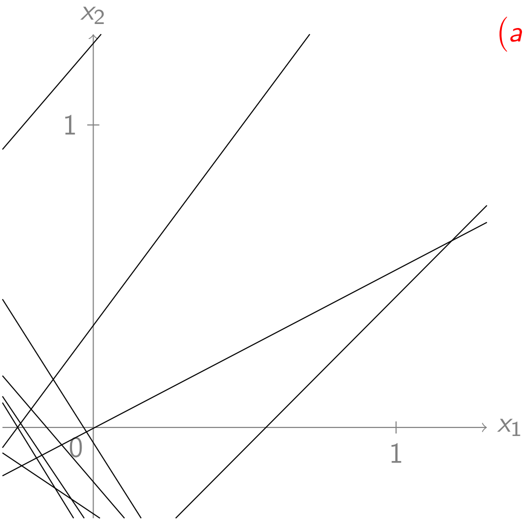

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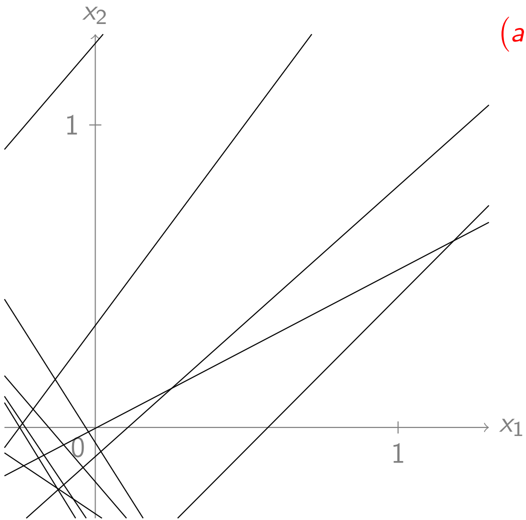

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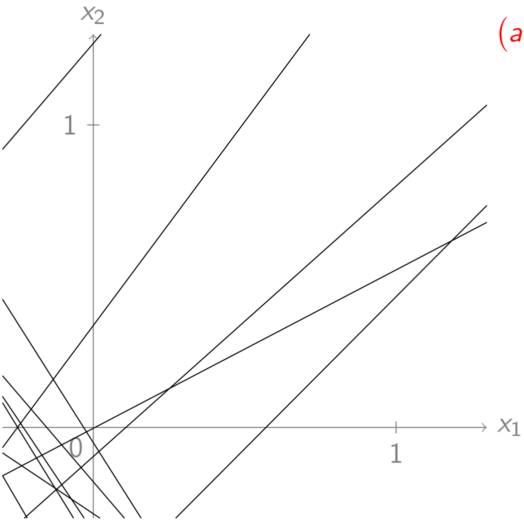

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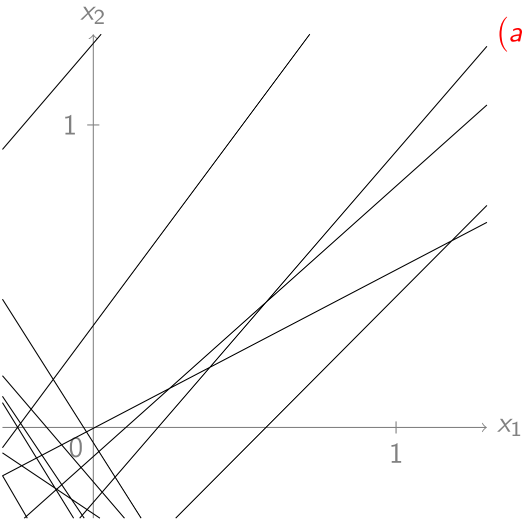

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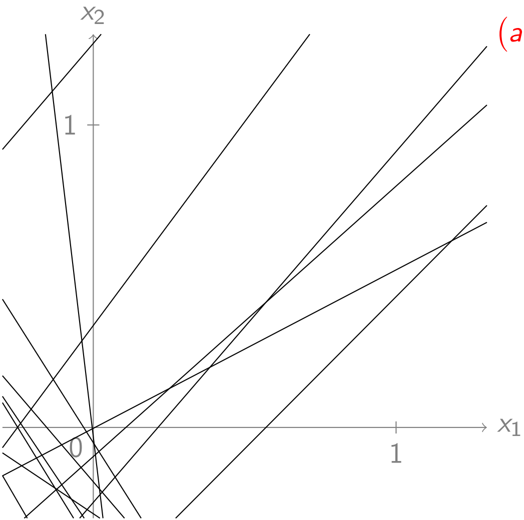

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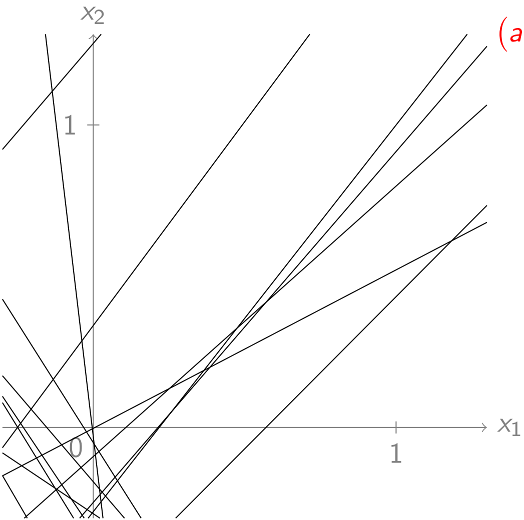

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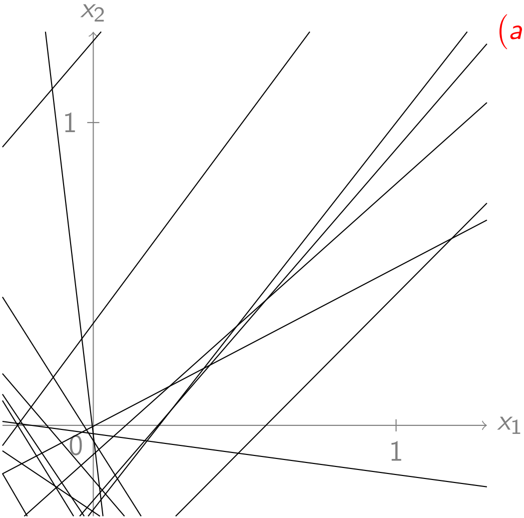

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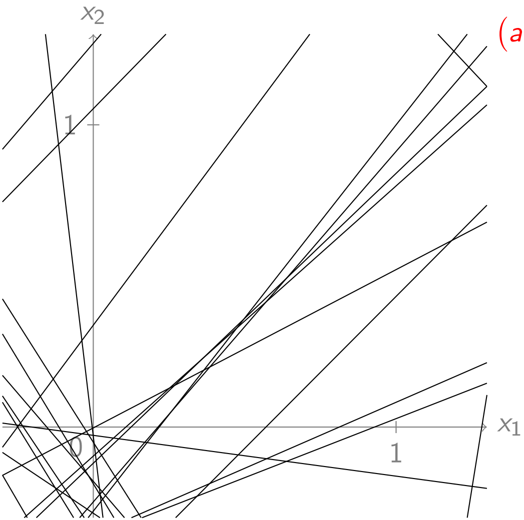

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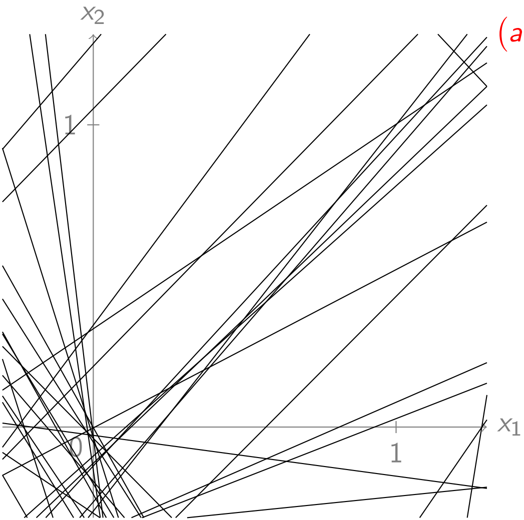

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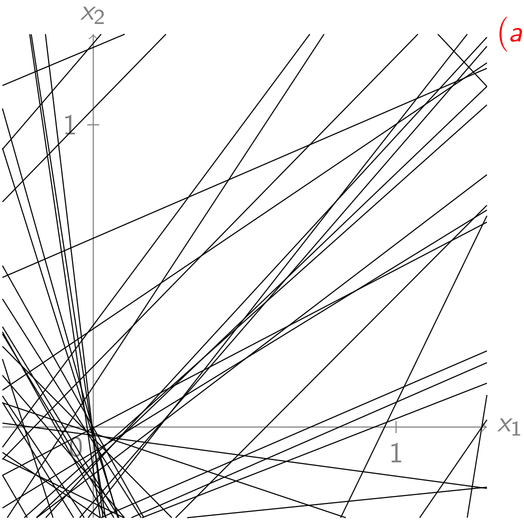

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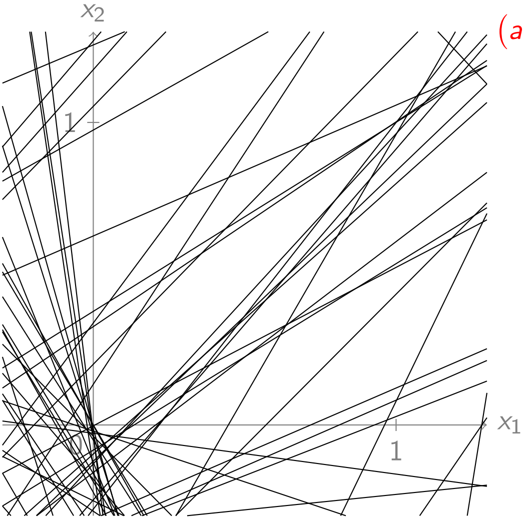

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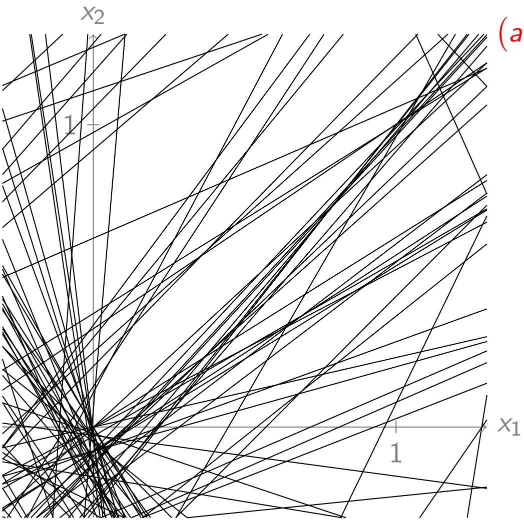

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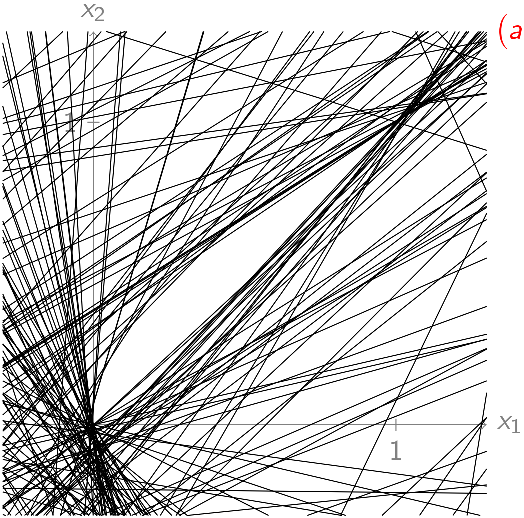

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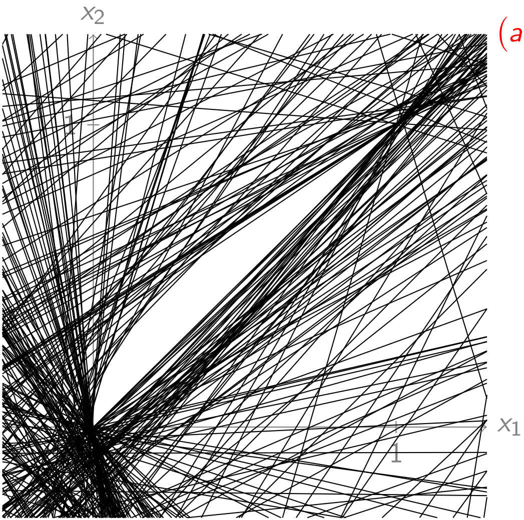

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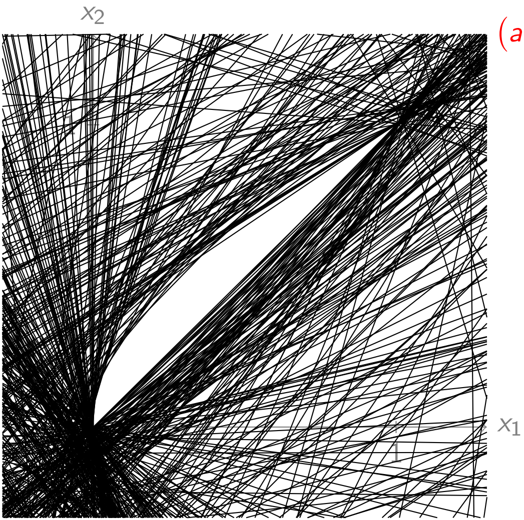

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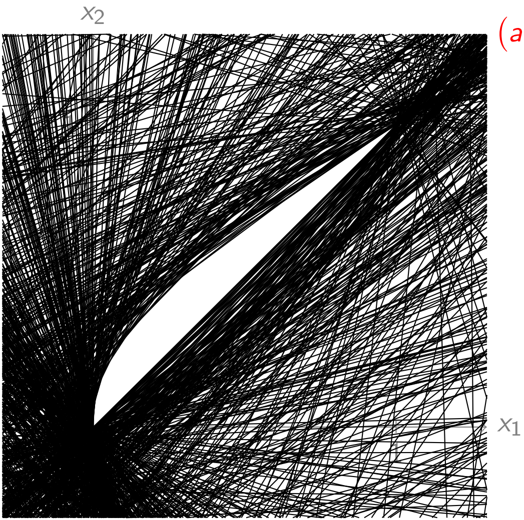

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The notion of (truncated) quadratic module will capture the generation of redundant constraints by multiplying with squares and adding:

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More generally, the quadratic module generated by $p_1, \dots, p_m \in A$ is

$$T(p_1, \dots, p_m) := \sum A^2 + \sum A^2 p_1 + \dots + \sum A^2 p_m$$

where $\sum A^2 p := \{sp \mid s \in \sum A^2\}$ for $p \in A$.

Truncated quadratic modules

Definition: Let $m \in \mathbb{N}_0$, $p_1, \dots, p_m \in \mathbb{R}[\underline{X}]$ and $k \in \mathbb{N}_0$. In the vector space $\mathbb{R}[\underline{X}]_k$ of polynomials of degree at most k, we denote the convex cone

$$(\mathbb{R}[\underline{X}]_k \cap \sum \mathbb{R}[\underline{X}]^2) + (\mathbb{R}[\underline{X}]_k \cap \sum \mathbb{R}[\underline{X}]^2 p_1) + \dots + (\mathbb{R}[\underline{X}]_k \cap \sum \mathbb{R}[\underline{X}]^2 p_m)$$

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The higher we choose the degree of relaxation k, the more redundant constraints we will add before linearization. More redundant constraints improve the linearization but enlarge also the resulting SDP.

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Let a POP

$$(P) \quad \text{minimize } f(x) \text{ over } x \in \mathbb{R}^n \text{ subject to } p_1(x) \geq 0, \ldots, p_m(x) \geq 0$$

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Let $k \in \mathbb{N}_0$ be the relaxation degree such that $f, p_1, \dots, p_m \in \mathbb{R}[\underline{X}]_k$.

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be given $(m, n \in \mathbb{N}_0,\ f, p_1, \dots, p_m \in \mathbb{R}[\underline{X}])$.

Let $k \in \mathbb{N}_0$ be the relaxation degree such that $f, p_1, \dots, p_m \in \mathbb{R}[\underline{X}]_k$.

The degree k moment relaxation (or Lasserre relaxation) of (P) is the SDP given by

$$\begin{array}{llrl} (P_k) & \text{minimize} & L(f) \qquad \text{over} & L \in \mathbb{R}[\underline{X}]_k^* \\ & & \text{subject to} & L(T_k(p_1, \dots, p_m)) \subseteq \mathbb{R}_{\geq 0} \text{ and} \\ & & & L(1) = 1 \end{array}$$

Moment relaxation

Let a POP

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Here $\mathbb{R}[\underline{X}]_k^*$ is the dual space of $\mathbb{R}[\underline{X}]_k$ and (P_k) can be written as an SDP by implementing the $L(\underline{X}^\alpha)$ ($\alpha \in \mathbb{N}_0^n, |\alpha| \leq k$) with new variables y_i as before.

Trivial properties of the moment relaxation

Proposition: Let $m, n, k \in \mathbb{N}_0$, $f, p_1, \dots, p_m \in \mathbb{R}[\underline{X}]_k$ and let (P_k) be the k-th moment relaxation of the POP

$$(P) \qquad \text{minimize } f(x) \text{ over } x \in \mathbb{R}^n \text{ subject to } x \in S$$

where $S := \{x \in \mathbb{R}^n \mid p_1(x) \geq 0, \dots, p_m(x) \geq 0\}$.

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(b) If L is integration with respect to a probability measure on S, then L is a feasible solution of (P_k) with $L(f) \geq P^*$.

(c) If (P_k) has an optimal solution L^* whose restriction to $\mathbb{R}[\underline{X}]_\ell$ is integration with respect to a measure μ on S for some $\ell \in \{1, \dots, k\}$ with $f \in \mathbb{R}[\underline{X}]_\ell$, then equality holds everywhere in (a) and each element of the support of μ is an optimal solution of (P).

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(d) In the situation of (c), if (P) has moreover a unique optimal solution x^*, then $x^* = (L^*(X_1), \dots, L^*(X_n))$ and moreover $L^*(p) = p(x^*)$ for all $p \in \mathbb{R}[\underline{X}]^*_\ell$.

Truncated moment problem

Strategy to solve a polynomial optimization problem (P):

(1) Choose relatively small relaxation degree k.

Truncated moment problem

Strategy to solve a polynomial optimization problem (P):

(1) Choose relatively small relaxation degree k.

(2) Solve moment relaxation (P_k) using an SDP solver. If it takes too long, then go to another summer school to learn more.

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(4) Compute the support of such a μ and check whether it is contained in S. If not, increase k and restart.

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(4) Compute the support of such a μ and check whether it is contained in S. If not, increase k and restart.

(5) The optimal values of (P) and (P_k) coincide and each element of the support of μ is an optimal solution x^* of (P).

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$\rightsquigarrow$Truncated moment problem: Need criteria when $L \in \mathbb{R}[\underline{X}]_k^*$ restricted to some subspace comes from a measure. Need to compute the support of the measure.

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$\rightsquigarrow$Truncated moment problem: Need criteria when $L \in \mathbb{R}[\underline{X}]_k^*$ restricted to some subspace comes from a measure. Need to compute the support of the measure.

Fact (existence of quadrature rules, Bayer and Teichmann 2006): If $L \in \mathbb{R}[\underline{X}]_k^*$ comes from a measure, it also comes from a finitely supported measure.

Truncated moment problem

Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$.

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Task: If possible, try to find a finitely supported measure μ such that L (or at least $L|_{\mathbb{R}[\underline{X}]_\ell}$ for some $\ell \leq 2d$ not too small) is integration with respect to μ.

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Would like to obtain:
nodes $x_1, \ldots, x_r \in \mathbb{R}^n$ and weights $\lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0}$ such that

$$L(p) = \sum_{i=1}^{r} \lambda_i p(x_i) \qquad \text{for all } p \in \mathbb{R}[\underline{X}]_\ell.$$

Truncated moment problem

Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$.

Task: If possible, try to find a finitely supported measure μ such that L (or at least $L|_{\mathbb{R}[\underline{X}]_\ell}$ for some $\ell \leq 2d$ not too small) is integration with respect to μ.

Would like to obtain:

$x_{11}, \dots, x_{1n}, \dots, x_{r1}, \dots, x_{rn} \in \mathbb{R}$ and $\lambda_1, \dots, \lambda_r \in \mathbb{R}_{\geq 0}$ such that

$$L(p) = \sum_{i=1}^{r} \lambda_i p(x_{i1}, \dots, x_{in}) \qquad \text{for all } p \in \mathbb{R}[\underline{X}]_\ell.$$

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Would like to obtain:

$x_{11}, \dots, x_{1n}, \dots, x_{r1}, \dots, x_{rn} \in \mathbb{R}$ and $a_1, \dots, a_r \in \mathbb{R}$ such that

$$L(p) = \sum_{i=1}^{r} a_i^2 p(x_{i1}, \dots, x_{in}) \qquad \text{for all } p \in \mathbb{R}[\underline{X}]_\ell.$$

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$x_{11}, \dots, x_{1n}, \dots, x_{r1}, \dots, x_{rn} \in \mathbb{R}$ and $a_1, \dots, a_r \in \mathbb{R}$ such that

$$L(p) = \left\langle \begin{pmatrix} p(x_{11},\dots,x_{1n}) & & \\ & \ddots & \\ & & p(x_{r1},\dots,x_{rn}) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}, \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} \right\rangle \text{ for all } p \in \mathbb{R}[\underline{X}]_\ell.$$

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$x_{11}, \dots, x_{1n}, \dots, x_{r1}, \dots, x_{rn} \in \mathbb{R}$ and $a \in \mathbb{R}^r$ such that

$$L(p) = \left\langle p\left(\begin{pmatrix} x_{11} & & \\ & \ddots & \\ & & x_{r1} \end{pmatrix}, \dots, \begin{pmatrix} x_{1n} & & \\ & \ddots & \\ & & x_{rn} \end{pmatrix}\right) a, a \right\rangle \text{ for all } p \in \mathbb{R}[\underline{X}]_\ell.$$

Truncated moment problem

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Task: If possible, try to find a finitely supported measure μ such that L (or at least $L|_{\mathbb{R}[\underline{X}]_\ell}$ for some $\ell \leq 2d$ not too small) is integration with respect to μ.

Would like to obtain:
diagonal matrices $D_1, \dots, D_n \in \mathbb{R}^{r \times r}$ and $a \in \mathbb{R}^r$ such that

$$L(p) = \langle p(D_1, \dots, D_n)a, a \rangle \text{ for all } p \in \mathbb{R}[\underline{X}]_\ell.$$

Truncated moment problem

Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$.

Task: If possible, try to find a finitely supported measure μ such that L (or at least $L|_{\mathbb{R}[\underline{X}]_\ell}$ for some $\ell \leq 2d$ not too small) is integration with respect to μ.

Would like to obtain:
commuting matrices $M_1, \dots, M_n \in S\mathbb{R}^{r \times r}$ and $a \in \mathbb{R}^r$ such that

$$L(p) = \langle p(M_1, \dots, M_n)a, a \rangle \text{ for all } p \in \mathbb{R}[\underline{X}]_\ell.$$

Truncated moment problem

Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$.

Task: If possible, try to find a finitely supported measure μ such that L (or at least $L|_{\mathbb{R}[\underline{X}]_\ell}$ for some $\ell \leq 2d$ not too small) is integration with respect to μ.

Would like to obtain:
a finite-dimensional euclidean vector space V, commuting self-adjoint endomorphisms $M_1, \ldots, M_n$ of V and $a \in V$ such that

$$L(p) = \langle p(M_1, \ldots, M_n)(a), a\rangle \text{ for all } p \in \mathbb{R}[\underline{X}]_\ell.$$

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Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$.

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a finite-dimensional euclidean vector space V, commuting self-adjoint endomorphisms $M_1, \dots, M_n$ of V and $a \in V$ such that

$$L(p) = \langle p(M_1, \dots, M_n)(a), a \rangle \text{ for all } p \in \mathbb{R}[\underline{X}]_\ell.$$

Idea: If we had $L \in \mathbb{R}[\underline{X}]^*$, $L(p^2) > 0$ for all $p \in \mathbb{R}[\underline{X}] \setminus \{0\}$ and if V were allowed to be infinite-dimensional, then we could take $V := \mathbb{R}[\underline{X}]$ with the scalar product defined by $\langle p, q \rangle := L(pq)$ for all $p, q \in \mathbb{R}[\underline{X}]$, $M_i \colon \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{X}],\ p \mapsto X_i p$ for all $i \in \{1, \dots, n\}$ and $a := 1 \in \mathbb{R}[\underline{X}]$.

Truncated GNS like construction

Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$.

Truncated GNS like construction

Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$. Then

$$U_L := \{p \in \mathbb{R}[\underline{X}]_d \mid \forall q \in \mathbb{R}[\underline{X}]_d : L(pq) = 0\}$$

is a subspace of $\mathbb{R}[\underline{X}]_d$ which we call the GNS kernel of L.

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We call the quotient vector space $V_L := \mathbb{R}[\underline{X}]_d / U_L$ the GNS representation space of L.

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We call the quotient vector space $V_L := \mathbb{R}[\underline{X}]_d / U_L$ the GNS representation space of L. It is easy to show that

$$\langle \overline{p}, \overline{q} \rangle_L := L(pq) \qquad (p, q \in \mathbb{R}[\underline{X}]_d)$$

defines a scalar product on V_L which we call the GNS scalar product on V_L.

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Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$. Then

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defines a scalar product on V_L which we call the GNS scalar product on V_L. This makes V_L into a finite-dimensional euclidian vector space.

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Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$. Then

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defines a scalar product on V_L which we call the GNS scalar product on V_L. This makes V_L into a finite-dimensional euclidian vector space.
We call the orthogonal projection π_L from V_L to its subspace $\{\overline{p} \mid p \in \mathbb{R}[\underline{X}]_{d-1}\}$ the GNS truncation of L.

Truncated GNS like construction

Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$. Then

$$U_L := \{p \in \mathbb{R}[\underline{X}]_d \mid \forall q \in \mathbb{R}[\underline{X}]_d : L(pq) = 0\}$$

is a subspace of $\mathbb{R}[\underline{X}]_d$ which we call the GNS kernel of L.
We call the quotient vector space $V_L := \mathbb{R}[\underline{X}]_d / U_L$ the GNS representation space of L. It is easy to show that

$$\langle \overline{p}, \overline{q} \rangle_L := L(pq) \qquad (p, q \in \mathbb{R}[\underline{X}]_d)$$

defines a scalar product on V_L which we call the GNS scalar product on V_L. This makes V_L into a finite-dimensional euclidian vector space.
We call the orthogonal projection π_L from V_L to its subspace $\{\overline{p} \mid p \in \mathbb{R}[\underline{X}]_{d-1}\}$ the GNS truncation of L.
Then for each $i \in \{1, \ldots, n\}$,

$$M_{L,i} \colon \pi_L V_L \to \pi_L V_L,\ \overline{p} \mapsto \pi_L(\overline{X_i p}) \qquad (p \in \mathbb{R}[\underline{X}]_{d-1})$$

is a self-adjoint endomorphism of $\pi_L V_L$ which we call the i-th truncated multiplication operator of L.

Truncated GNS like construction

Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$. To show that

$$M_{L,i} \colon \pi_L V_L \to \pi_L V_L,\ \overline{p} \mapsto \pi_L(\overline{X_i p}) \qquad (p \in \mathbb{R}[\underline{X}]_{d-1})$$

is indeed well-defined,

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is indeed well-defined, note that $\pi_L(\overline{X_i p}) = 0$ for all $p \in \mathbb{R}[\underline{X}]_{d-1} \cap U_L$,

Truncated GNS like construction

Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$. To show that

$$M_{L,i} \colon \pi_L V_L \to \pi_L V_L,\ \overline{p} \mapsto \pi_L(\overline{X_i p}) \qquad (p \in \mathbb{R}[\underline{X}]_{d-1})$$

is indeed well-defined, note that $\pi_L(\overline{X_i p}) = 0$ for all $p \in \mathbb{R}[\underline{X}]_{d-1} \cap U_L$, for if you choose $q \in \mathbb{R}[\underline{X}]_{d-1}$ with $\overline{q} = \pi_L(\overline{X_i p})$, then

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$$\begin{aligned} \langle \overline{q}, \overline{q} \rangle_L &= \langle \pi_L(\overline{X_i p}), \overline{q} \rangle_L = \langle \overline{X_i p}, \pi_L(\overline{q}) \rangle_L \\ &\overset{\overline{q} \in \pi_L V_L}{=} \langle \overline{X_i p}, \overline{q} \rangle_L = L(X_i p q) = L(p(X_i q)) = \langle \overline{p}, \overline{X_i q} \rangle_L \overset{p \in U_L}{=} 0. \end{aligned}$$

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To confirm that $M_{L,i}$ is self-adjoint, let $p, q \in \mathbb{R}[\underline{X}]_{d-1}$ and observe that

$$\begin{aligned}\langle M_{L,i}(\overline{p}), \overline{q} \rangle_L &= \langle \pi_L(\overline{X_i p}), \overline{q} \rangle_L = \langle \overline{X_i p}, \pi_L(\overline{q}) \rangle_L \overset{\overline{q} \in \pi_L V_L}{=} \langle \overline{X_i p}, \overline{q} \rangle_L \\ &= L(X_i pq) = L(p(X_i q)) = \langle \overline{p}, \overline{X_i q} \rangle_L \overset{\overline{p} \in \pi_L V_L}{=} \langle \pi_L(\overline{p}), \overline{X_i q} \rangle_L \\ &= \langle \overline{p}, \pi_L(\overline{X_i q}) \rangle_L = \langle \overline{p}, M_{L,i}(\overline{q}) \rangle_L\end{aligned}$$

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Theorem. Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$. Suppose that the truncated GNS multiplication operators of L commute and set

$$W_L := \left\{ \sum_{i=1}^{m} p_i q_i \mid m \in \mathbb{N}_0, p_i, q_i \in \mathbb{R}[\underline{X}]_{d-1} + U_L \right\} \supseteq \mathbb{R}[\underline{X}]_{2(d-1)}.$$

Then $L|_{W_L}$ is integration with respect to a finitely supported measure.

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Corollary. Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$. Suppose that the truncated GNS multiplication operators of L commute. Then $L|_{\mathbb{R}[\underline{X}]_{2(d-1)}}$ is integration with respect to a finitely supported measure.

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Corollary. Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$. Suppose that the truncated GNS multiplication operators of L commute. Then $L|_{\mathbb{R}[\underline{X}]_{2(d-1)}}$ is integration with respect to a finitely supported measure.

Corollary. Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[X]_{2d}^*$ (one variable) with $L(\sum \mathbb{R}[X]_d^2) \subseteq \mathbb{R}_{\geq 0}$. Then $L|_{\mathbb{R}[X]_{2(d-1)}}$ is integration with respect to a finitely supported measure.

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Flatness condition

Let $d \in \mathbb{N}$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$, $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$ and $L' := L|_{\mathbb{R}[\underline{X}]_{2(d-1)}}$.
Then the following are equivalent:

(a) $\mathbb{R}[\underline{X}]_{d-1} + U_L = \mathbb{R}[\underline{X}]_d$

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both maps are isomorphisms.

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(f) $(L(\underline{X}^{\alpha+\beta}))_{|\alpha|,|\beta| \leq d-1}$ and $(L(\underline{X}^{\alpha+\beta}))_{|\alpha|,|\beta| \leq d}$ have the same rank.

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both maps are isomorphisms.

(e) $\dim V_{L'} = \dim V_L$

(f) $(L(\underline{X}^{\alpha+\beta}))_{|\alpha|,|\beta| \leq d-1}$ and $(L(\underline{X}^{\alpha+\beta}))_{|\alpha|,|\beta| \leq d}$ have the same rank.

If (one of) these conditions is satisfied, then we call L flat.

Flatness condition

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If (one of) these conditions is satisfied, then we call L flat. One can show that in this case, the truncated GNS multiplication operators of L commute. So the following theorem of Curto and Fialkow (1996) is a corollary: If L is flat, then L comes from a finitely supported measure.

The Gram matrix method

Let $m \in \mathbb{N}_0$, $p_1, \dots, p_m \in \mathbb{R}[\underline{X}]$ and $k \in \mathbb{N}_0$. Remember that the k-truncated quadratic module associated to $p_1, \dots, p_m$ is

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$$\mathbb{R}[\underline{X}]_k \cap \sum \mathbb{R}[\underline{X}]^2 p = \left\{ \sum_{i=1}^{s} h_i^2 p \mid h_i \in \mathbb{R}[\underline{X}]_d \right\} = \{u^* G u p \mid G \in \mathbb{R}_{\succeq 0}^{s \times s}\}.$$

The dual of the moment relaxation

Let a POP

$$(P) \quad \text{minimize } f(x) \text{ over } x \in \mathbb{R}^n \text{ subject to } p_1(x) \geq 0, \ldots, p_m(x) \geq 0$$

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One can show that the dual SDP can be written as

$$
\begin{array}{llrl}
(D_k) & \text{maximize} & a \qquad \text{over} & a \in \mathbb{R} \\
& & \text{subject to} & f - a \in T_k(p_1, \ldots, p_m)
\end{array}
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Nontrivial properties of the moment relaxation

Let $m, n, k \in \mathbb{N}_0$ such that $f, p_1, \dots, p_m \in \mathbb{R}[\underline{X}]_k$.

$S := \{x \in \mathbb{R}^n \mid p_1(x) \geq 0, \dots, p_m(x) \geq 0\}$

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Remark (relaxation hierarchy). $(D_k^*)_k$ and $(P_k^*)_k$ are increasing sequences not exceeding P^*.

Remark (weak duality). $P_k^* \geq D_k^*$

Lemma. If S has non-empty interior in $\mathbb{R}^n$, then T_k is closed in $\mathbb{R}[\underline{X}]_k$.

Proposition (strong duality). If S has non-empty interior in $\mathbb{R}^n$, then $P_k^* = D_k^*$ and, unless $P_k^* = D_k^* = -\infty$, (D_k) has an optimal solution.

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Theorem (Krivine 1964, Schmüdgen 1991, Putinar 1993). Equivalent:

(a) $\exists t \in \mathbb{N}_0 : \exists q_1, \ldots, q_t \in \mathbb{R}[\underline{X}] : (\forall I \subseteq \{1, \ldots, t\} : \prod_{i \in I} q_i \in T$ & $\{x \in \mathbb{R}^n \mid q_1(x) \geq 0, \ldots, q_t(x) \geq 0\}$ compact$)$

(b) $\exists q \in T : \{x \in \mathbb{R}^n \mid q(x) \geq 0\}$ compact

(c) $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in T$

(d) $\forall p \in \mathbb{R}[\underline{X}] : \exists N \in \mathbb{N} : N + p \in T$

(e) S compact & $\forall p \in \mathbb{R}[\underline{X}] : (p > 0$ on $S \implies p \in T)$

Proof. $(e) \implies (d) \implies (c) \implies (b) \implies (a)$ are trivial.

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Proof. $(a) \implies (c)$ very hard (proved by Schmüdgen 1991, simplified by Wörmann 1998, all proofs use Krivine's 1964 Positivstellensatz)

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Proof. $(c) \implies (d)$ three (tricky) lines

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Proof. $(d) \implies (e)$ Putinar 1993 (via duality and functional analysis), Jacobi 2001 (algebraic, tricky, several pages), Marshall 2008 (three lines)

Nontrivial properties of the moment relaxation

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If these equivalent conditions are satisfied, we call T archimedean.

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(D_k) maximize a over $a \in \mathbb{R}$ subject to $f - a \in T_k$

Theorem (Krivine 1964, Schmüdgen 1991, Putinar 1993). Equivalent:

(a) $\exists t \in \mathbb{N}_0 : \exists q_1, \dots, q_t \in \mathbb{R}[\underline{X}] : (\forall I \subseteq \{1, \dots, t\} : \prod_{i \in I} q_i \in T$ & $\{x \in \mathbb{R}^n \mid q_1(x) \geq 0, \dots, q_t(x) \geq 0\}$ compact$)$

(b) $\exists q \in T : \{x \in \mathbb{R}^n \mid q(x) \geq 0\}$ compact

(c) $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in T$

(d) $\forall p \in \mathbb{R}[\underline{X}] : \exists N \in \mathbb{N} : N + p \in T$

(e) S compact & $\forall p \in \mathbb{R}[\underline{X}] : (p > 0 \text{ on } S \implies p \in T)$

If S is compact, you can establish (a) by augmenting the p_i by their products (too expensive when m is large!).

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If S is compact and you know a ball around 0 containing S, you can establish (c) easily by adding a redundant p_i.

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Corollary. If T is archimedean, then $(D_k^*)_k$ and $(P_k^*)_k$ converge to P^*.

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The following are equivalent:

(a) $L \in \mathbb{R}[\underline{X}]^*$, $L(T) \subseteq \mathbb{R}_{\geq 0}$ and $L(1) = 1$.

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This is dual to the previous result of Schmüdgen and Putinar.

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Here $L_\mu \colon \mathbb{R}[\underline{X}]_d \to \mathbb{R},\ p \mapsto \int p(x) d\mu(x)$.

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Corollary. If (P) has exactly one optimal solution x^*, then $\lim_{k \to \infty}(L_k(X_1), \dots, L_k(X_n)) = x^*$.

Implementation

- YALMIP, Linköping
  Löfberg
  http://users.isy.liu.se/johanl/yalmip/
- GloptiPoly, LAAS Toulouse
  Henrion, Lasserre, Löfberg
  http://www.laas.fr/~henrion/software/gloptipoly3/
- SparsePOP, Tokyo Institut of Technology
  Waki, Kim, Kojima, Muramatsu, Sugimoto, Yamashita
  http://www.is.titech.ac.jp/~kojima/SparsePOP/
- SOSTOOLS, Caltech
  Prajna, Papachristodoulou, Seiler, Parrilo
  http://www.cds.caltech.edu/sostools/
- YouPOP, Your Institute
  You and your colleagues
  http://www.rwth-aachen.de/~you/

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  http://users.isy.liu.se/johanl/yalmip/
- GloptiPoly, LAAS Toulouse
  Henrion, Lasserre, Löfberg
  http://www.laas.fr/~henrion/software/gloptipoly3/
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  Waki, Kim, Kojima, Muramatsu, Sugimoto, Yamashita
  http://www.is.titech.ac.jp/~kojima/SparsePOP/
- SOSTOOLS, Caltech
  Prajna, Papachristodoulou, Seiler, Parrilo
  http://www.cds.caltech.edu/sostools/
- YouPOP, Your Institute
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