The sums of squares dual of a semidefinite program

(joint work with Igor Klep)

Markus Schweighofer

Universität Konstanz

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Nonlinear Programming
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Standard SDP duality

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(P) & \text{minimize} & c_1 x_1 + \cdots + c_n x_n \\
 & \text{subject to} & x \in \mathbb{R}^n \\
 & & A_0 + x_1 A_1 + \cdots + x_n A_n \succeq 0
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Weak duality: If x is feasible in (P) and (S, a) is feasible in (D), then $\ell(x) \geq a$. Indeed, $\ell(x) - a = \operatorname{tr}(L(x)S) \geq 0$ since the trace of the product of two positive semidefinite matrices is nonnegative.

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Strong duality: Denote by $P^*, D^* \in \{-\infty\}\cup\mathbb{R}\cup\{\infty\}$ the optimal values of (P) and (D) respectively. Suppose that the feasible set of (P) has non-empty interior.

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Theorem: For any pencil $L \in \mathbb{R}[\underline{X}]^{m\times m}$, the following are equivalent:

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For each $d \in \mathbb{N}_0$, let $m_d := \binom{d+n}{n}$ denote the number of monomials of degree at most d in n variables and $\overrightarrow{x_d} \in \mathbb{R}[\underline{X}]^{m_d}$ the column vector

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- matrices $U_1, \dots, U_n \in S\mathbb{R}^{m \times m}$, $W_1, \dots, W_n \in \mathbb{R}^{k \times m}$ and
- a real number $a \geq 0$

such that

$$\begin{aligned}
&\overrightarrow{x_1}^* U_i \overrightarrow{x_1} + \overrightarrow{x_2}^* W_{i-1} \overrightarrow{x_1} + \operatorname{tr}(LS_i) = 0 && (i \in \{1, \dots, n\}),\\
&U_i \succeq W_i^* W_i && (i \in \{1, \dots, n\}),\\
&\ell + \overrightarrow{x_2}^* W_n \overrightarrow{x_1} = a + \operatorname{tr}(LS)
\end{aligned}$$

where $W_0 := 0 \in \mathbb{R}^{k \times m}$.

An exact duality theory for SDP based on sums of squares

This provides a duality theory for semidefinite programming where strong duality (zero gap & dual attainment) always holds and the size of the dual is polynomial in the size of the primal. Based on other ideas, such a duality theory has also been given by Matt Ramana:

M. Ramana: An exact duality theory for semidefinite programming and its complexity implications
Math. Programming **77** (1997), no. 2, Ser. B, 129–162
http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.47.8540&rep=rep1&type=pdf
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(ii) There are an sos-polynomial $s \in \mathbb{R}[\underline{X}]$ and an sos-matrix $S \in \mathbb{R}[\underline{X}]^{m \times m}$ both of degree at most $\min\{m-1, n\}$ such that

$$-1 = s + \operatorname{tr}(LS).$$

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Thank you!