ITERATED RINGS OF BOUNDED ELEMENTS: ERRATUM

MARKUS SCHWEIGHOFER

Abstract. We close a gap in the author’s thesis [S1, S2].

The author’s proof of [S1, S2, Lemma 4.10] is not correct. In this note, we show that this does not affect the validity of any other statement in [S1, S2]. We will observe that the lemma in question holds in any of the following important special cases:

(a) $T = \sum A^2$

(b) $A$ contains a f.g. subalgebra $C$ such that $T$ is as a preordering generated by $T \cap C$.

(c) $T$ is as a preordering finitely generated (this is just a special case of (b)).

(d) $A$ is a reduced ring.

Unfortunately, we don’t know whether the lemma holds without any such additional hypothesis.

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1. The error

Recall the situation in the proof of the lemma. We have an extension $B \subseteq A$ of preordered rings, i.e., an extension $B \subseteq A$ of rings such that $A$ is equipped with a preordering $T$ and $B$ with $T \cap B$. We have $p \in \text{Spec } A$ and $q = p \cap B$. Then $B/q$ can be viewed as a subring of $A/p$ but perhaps not as a preordered subring, contrary to what is said in [S1, S2]. Here $A/p$ and $B/q$ are equipped with the preorderings

$T_{A/p} = \{ t + p \mid t \in T \}$ and $T_{B/q} = \{ t + q \mid t \in T \cap B \}$,

respectively. The problem is that for some $b \in B$ it could happen that $b - t$ lies in $p$ for some $t \in T$ but not in $q$ for any $t \in T$. Consequently, it is not guaranteed whether the preorderings on the quotient field $qf(A/p) = qf(B/q)$ induced (or generated) by $T_{A/p}$ and $T_{B/q}$ coincide. Hence it is not clear whether $T \subseteq P$ when $P$ is chosen like in the proof under review.

2. Cases where the proof still works

In case (a), $T \subseteq P$ holds trivially. In case (b), we may assume that $C \subseteq B$. This implies $T \cap C \subseteq T \cap B \subseteq Q \subseteq P$ and therefore $T \subseteq P$.

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4. Proof of the Lemma for Reduced Rings

In this section, we will prove that [S1, S2, Lemma 4.10] holds under the additional assumption (d) that $A$ is a reduced ring, i.e., contains no nonzero nilpotent elements. This is done in Lemma 7 below.

We need some well-known facts from commutative ring theory whose proofs we include for the convenience of the reader. Let $A$ always denote a commutative ring (with unity, of course).

Lemma 1. Let $p_1, \ldots, p_n$ be prime ideals of a commutative ring $A$ satisfying $p_i \not\subseteq p_j$ for $i \neq j$. Then $p_i \nsubseteq \bigcup_{j \neq i} p_j$ for all $i \in \{1, \ldots, n\}$.

Proof. For every $(i, j)$ with $j \neq i$, choose $a_{ij} \in p_i \setminus p_j$. Then we have for instance

$$\sum_{i=2}^{n} \prod_{j \neq i} a_{ij} \in p_1 \setminus \bigcup_{j \neq i} p_j.$$

Lemma 2. Let $A$ be a reduced ring with only finitely many pairwise distinct minimal prime ideals $p_1, \ldots, p_n$. Then the zero divisors in $A$ (i.e., the elements $a \in A$ for which there is some $0 \neq b \in A$ with $ab = 0$) are exactly the elements of $p_1 \cup \ldots \cup p_n$.

Proof. Consider an element of $p_1 \cup \ldots \cup p_n$, say $a \in p_1$. By the preceding lemma, we can choose $b_i \in p_i \setminus \bigcup_{j \neq i} p_j$ for $i \in \{2, \ldots, n\}$. Then $a b_2 \cdots b_n \in p_1 \cap \ldots \cap p_n = 0$ but $b_2 \cdots b_n \notin p_1$, in particular $b_2 \cdots b_n \neq 0$. Thus $a$ is a zero divisor.

Conversely, suppose $a \in A \setminus (p_1 \cup \ldots \cup p_n)$. We show that $a$ is not a zero divisor. Suppose therefore that $b \in A$ and $ab = 0$. From $ab = 0 \in p_i$ and $a \notin p_i$ it follows that $b \in p_i$ for all $i \in \{1, \ldots, n\}$. Hence $b$ lies in $p_1 \cap \ldots \cap p_n = 0$. \qed

Definition 3. The total quotient ring of $A$ is the ring

$$\text{Quot}(A) := S^{-1}A$$

where $S$ is the set non zero divisors of $A$ (note that $1 \in S$ and $SS \subseteq S$).

Lemma 4. The canonical homomorphism $A \to \text{Quot}(A)$ is an embedding.

Proof. Suppose $a \in A$ and $a/1 = 0 \in \text{Quot}(A)$. Then there is some non zero divisor $s$ of $A$ such that $as = 0$ in $A$. But then, of course, $a = 0$. \qed

Lemma 5. Let $A$ be reduced with only finitely many pairwise distinct minimal prime ideals $p_1, \ldots, p_n$. Then there is a canonical isomorphism

$$\text{Quot}(A) \cong \text{qf}(A/p_1) \times \cdots \times \text{qf}(A/p_n).$$

Proof. Let $S$ denote the set of non zero divisors of $A$. Then $\text{Quot}(A) = S^{-1}A$ by Definition 3. The prime ideals of $S^{-1}A$ correspond to the prime ideals of $A$ contained in $A \setminus S = p_1 \cup \ldots \cup p_n$ (Lemma 2). But by Lemma 1, the only prime ideals of $A$ contained in $p_1 \cup \ldots \cup p_n$ are $p_1, \ldots, p_n$ themselves. Therefore $S^{-1}p_1, \ldots, S^{-1}p_n$ are (all) pairwise distinct maximal ideals of $S^{-1}A$. In particular, these ideals are pairwise coprime. Moreover it is easy to see that

$$S^{-1}p_1 \cap \ldots \cap S^{-1}p_n = 0$$

and that there is a canonical isomorphism

$$S^{-1}A/S^{-1}p_i \cong \text{qf}(A/p_i).$$
Lemma 6. Let \( A \) be reduced with only finitely many pairwise distinct minimal prime ideals \( p_1, \ldots, p_n \). Let \( B \) be a subring of \( A \) such that the following conditions hold:

1. The canonical embedding \( qf(B/(p_i \cap B)) \hookrightarrow qf(A/p_i) \) is a field isomorphism for all \( i \).
2. \( p_i \cap B \not\subseteq p_j \cap B \) for all \( i \neq j \).

Then there is a canonical isomorphism \( \text{Quot}(B) \cong \text{Quot}(A) \).

Proof. Every minimal prime ideal of \( B \) is of the form \( p_i \cap B \) for some \( i \in \{1, \ldots, n\} \) (confer \[S1, S2, \text{Remark 4.8}\]). Conversely, we argue that every \( p_i \cap B \) is actually a minimal prime ideal of \( B \). To see this, observe that \( p_i \cap B \) contains in any case a minimal prime ideal. Hence \( p_j \cap B \subseteq p_i \cap B \) for some minimal prime ideal \( p_j \cap B \) of \( B \). Condition (2) forces \( i = j \) showing that \( p_i \cap B \) is itself a minimal prime ideal. Our claim follows now from condition (1), the preceding lemma and the fact that \( p_1 \cap B, \ldots, p_n \cap B \) are exactly the pairwise distinct minimal prime ideals of \( B \).  

Lemma 7. Suppose \( A \) is f.f., almost archimedean and reduced. Then

\[
A = \bigcup B, \quad \text{where } B \text{ ranges over f.g., almost archimedean algebras.}
\]

Proof. Denote the finitely many minimal pairwise distinct prime ideals of \( A \) by \( p_1, \ldots, p_n \). Since \( A \) is f.f., we can choose \( a_1, \ldots, a_m \in A \) such that \( qf(A/p_i) \) is for each \( i \in \{1, \ldots, n\} \) generated by \( a_1 + p_i, \ldots, a_m + p_i \) as a field over \( K \). Choose moreover \( b_{ij} \in p_i \setminus p_j \) for all \( i \neq j \). Clearly, \( A = \bigcup B \) where \( B \) ranges over all f.g. subalgebras of \( A \) containing all of the finitely many \( a_i \) and \( b_{ij} \).

Now we fix such a \( B \). It remains to show that \( B \) is almost archimedean. Fix an arbitrary \( Q \in \text{Sper} B \) such that \( Q \cap -Q \) is a minimal prime ideal of \( B \). We have to show that \( Q \) is archimedean. There is \( i \) such that \( Q \cap -Q = p_i \cap B \) by \[S1, S2, \text{Remark 4.8}\]. Since \( B \) contains all \( a_i \) and \( b_{ij} \), conditions (1) and (2) from Lemma 6 are satisfied. From (1) we see that \( qf(A/p_i) = qf((B/(p_i \cap B)) \) as fields (not necessarily as preordered fields!). Consequently, there is some ordering \( P \) of the ring \( A \) such that \( Q = P \cap B \) and \( P \cap -P = p_i \). It remains to show that \( P \in \text{Sper} A \) or, in more explicit words, \( T \subseteq P \). Once we have shown this, it follows that \( P \) is archimedean since \( A \) is almost archimedean. But then \( Q \) must of course be archimedean, too.

To show \( T \subseteq P \), we use Lemma 5 saying that there is a canonical isomorphism \( \text{Quot}(B) \cong \text{Quot}(A) \). Consider an arbitrary \( t \in T \). Since \( t/1 \) lies in the image of this isomorphism, there is some \( b \in B \) and some non zero divisor \( s \) in \( B \) such that \( b/s = t/1 \) holds in \( \text{Quot}(A) \). This implies \( b/1 = st/1 \) in \( \text{Quot}(A) \) and a fortiori \( b = st \) in \( A \) by Lemma 4. Hence \( s^2t \in T \cap B \subseteq Q \subseteq P \). If \( s \) were an element of \( P \cap -P = p_i \), then it would lie in the minimal prime ideal \( p_i \cap B = Q \cap -Q \) of \( B \) which is impossible by Lemma 2. From \( s \notin P \cap -P \) it follows now that \( t \in T \).  

5. CLOSING THE GAP

Finally, we show that Lemma 7 is enough to ensure the validity of all results of \[S1, S2\], with the only possible exception of \[S1, S2, \text{Lemma 4.10}\], of course. In
fact, [S1, S2, Lemma 4.10] is only applied once in [S1, S2], namely in the proof of [S1, S2, Theorem 4.13] (and its variant for quadratic modules and semiordeings in [S1, S2, Subsection 6.2] which can be treated completely analogously).

Hence it suffices to show that we can restrict ourselves in the proof of [S1, S2, Theorem 4.13] to reduced $A$ since then we can apply Lemma 7 instead of [S1, S2, Lemma 4.10]. Assume therefore that [S1, S2, Theorem 4.13] has already been shown for reduced $A$.

Now for general $A$, denote the nilradical of $A$ by $\text{Nil}(A)$. Suppose $A = H(A)$. Then $A/\text{Nil}(A) = H(A/\text{Nil}(A))$ and therefore $A/\text{Nil}(A) = H'(A/\text{Nil}(A))$ since $A/\text{Nil}(A)$ is reduced. Suppose $a \in A$. We show that $a \in H'(A)$. From $a + \text{Nil}(A) \in A/\text{Nil}(A) = H'(A/\text{Nil}(A))$ we obtain $\nu \in \mathbb{N}$ and $b \in \text{Nil}(A)$ such that $\nu - a + b \in T$. Clearly $b \in H'(A)$ by [S1, S2, Lemma 4.1]. This supplies us with a $\nu' \in \mathbb{N}$ such that $\nu' - b \in T$. Finally, we see that

$$(\nu + \nu') - a = (\nu - a + b) + (\nu' - b) \in T + T \subseteq T.$$ 

Since $a \in A$ was arbitrary, we see that $A = H'(A)$ as desired.

References


Universität Konstanz, Fachbereich Mathematik und Statistik, 78457 Konstanz, Allemagne

E-mail address: Markus.Schweighofer@uni-konstanz.de