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Compact seminar on Profinite Groups
Introduction

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1. Topological groups

Definition 1.1. A topological group is a group G with a topology with respect to which the maps

$$G \times G \to G,\ (g,h) \mapsto gh \quad G \to G,\ g \mapsto g^{-1}$$

are continuous (on $G \times G$ we take the product topology).

Examples 1.2.
(1) Every group can be made into a topological group by equipping it with the discrete topology.
(2) The real numbers with addition $(\mathbb{R}, +)$.
(3) More generally $(\mathbb{R}^n, +)$.
(4) The General Linear Group $\mathrm{GL}_n(\mathbb{R})$ with the induced topology, when seen as a subspace of $\mathbb{R}^{n\times n}$. Recall that multiplication and inversion of matrices are polynomial maps on the coefficients. Hence continuous.
(5) $(\mathbb{Q}, +)$ when we put on $\mathbb{Q}$ the topology induced by $\mathbb{R}$.

Definition 1.3. A *morphism of topological groups* is a group homomorphism which is continuous. An *isomorphism* is a group isomorphism which is also a homeomorphism.

2. Projective systems and projective limits

Definition 2.1 (Directed partially ordered set)**.** A set I with a binary relation $\leq$ is called a *directed partially ordered set* or a *directed poset* if

(a) for all $i \in I$ we have $i \leq i$;
(b) for all $i, j \in I$, if $i \leq j$ and $j \leq k$ then $i \leq k$;
(c) for all $i, j \in I$, if $i \leq j$ and $j \leq i$ then $i = j$;
(d) for all $i, j \in I$ there exists a $k \in I$ with $i, j \leq k$.

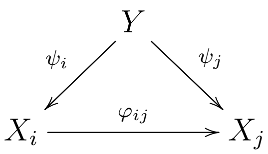
Example 2.2. The set $(\mathbb{N}^+, |)$ of positive natural numbers together with the binary relation "is divisible by" is a directed poset.

Definition 2.3 (Projective system)**.** A *projective system* of topological spaces (resp. groups) is a triple (X_i, φ_{ij}, I) where

- $(I, \leq)$ is a directed poset.
- $\{X_i,\ i \in I\}$ is a collection of topological spaces (resp. groups) indexed by I.
- $\varphi_{ij} \colon X_i \to X_j$ is a continuous map (resp. homomorphism) defined whenever $i \geq j$ and such that, whenever $k \geq j \geq i$ we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$. Moreover, for all $i \in I$, we have $\varphi_{ii} = \mathrm{Id}_{X_i}$

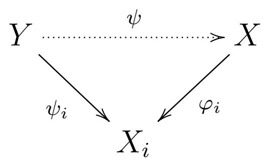
Example 2.4 (Constant projective system)**.** Fix a topologigal space X and a directed poset I. For all $i \in I$ let $X_i = X$. Then the system $\{X, \mathrm{Id}_X\}$ is a projective system called *the constant projective system on* X.

Definition 2.5 (Compatible maps)**.** Let Y be a topological space (group) and (X_i, φ_{ij}, I) a projective system of topological spaces (groups). For all $i \in I$ let $\psi_i \colon Y \to X_i$ be a continuous map (group homomorphism). We say that the ψ_i's are *compatible* if for all $i \geq j$ the following diagram commutes.



2.1. **Projective limits.**

Definition 2.6 (Projective limit)**.** Let (X_i, φ_{ij}, I) be a projective system of topologial spaces (groups). A *projective limit* of $\{X_i\}$ is a topological space (group) X together with a family $\{\varphi_i \colon X \to X_i | i \in I\}$ of continuous maps (homomorphisms) such that the following *universal property* holds:
for every topological space (group) Y and family of compatible maps (homomorphisms) $\{\psi_i \colon Y \to X_i\}$ there exists a unique continuous map (homomorphism) $\psi \colon Y \to X$, such that, for every $i \in I$ the following diagram commutes.



We say that ψ is *induced* by the compatible maps (homomorphisms) ψ_i.

Recall that if $\{X_i\}$ is a collection of topological spaces (groups) indexed by a set I then their *(Cartesian or direct) product* is the topological space (group) $\prod_{i \in I} X_i$ endowed with the product topology. In the case of groups the operation is defined coordinate-wise.

Proposition 2.7 ([RZ10, Proposition 1.1.1])**.** *Let* $\{X_i, \varphi, I\}$ *be a projective system of topological spaces (groups) over a directed poset* I. *Then*

(a) There exists a projective limit of the projective system $\{X_i\}$*;*

(b) *This limit is unique up to unique isomorphism. This means the following: if (X, φ_i) and (Y, ψ_i) are two limits of the projective system $\{X_i\}$, then there exists a unique homeomorphism (topological isomorphism) $\varphi : X \to Y$ such that, for all $i \in I$, we have $\psi_i \varphi = \varphi_i$*

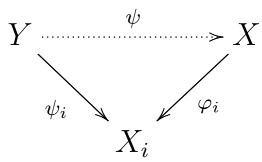
Proof. (a) Let X be the subspace (subgroup) of $\prod_{i\in I} X_i$ defined as follows

$$X = \{(x_i)_{i\in I} \text{ s.t. } i \geq j \Rightarrow \varphi_{ij}(x_i) = x_j\}$$

and for all $i \in I$ let $\varphi_i \colon X \to X_i$ be the restriction of the canonical projection $\prod_{i\in I} \to X_i$, that is

$$\forall j \in I,\ \forall (x_i)_{i\in I} \in X,\ \varphi_j((x_i)_{i\in I}) = x_j.$$

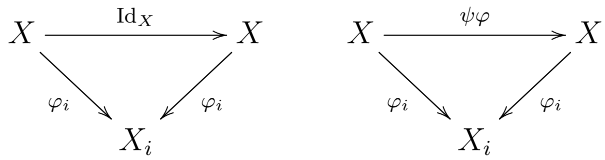
This is a continuous map (homomorphism). Now we prove that (X, φ_i) is a projective limit of the system $\{X_i, \varphi_{ij}, I\}$. To do this, let (Y, ψ_i) be a topological space (group) together with a family of compatible maps (homomorphisms). We need to show that there exists a unique continuous map (group homomorphism) $\psi \colon Y \to X$ such that, for all $i \in I$ the diagram



commutes. Let $y \in Y$. Then $\psi(y)$ is a tuple $(x_i)_{i\in I}$ and we must have $\varphi_i(\psi(y)) = x_i = \psi_i(y)$. Therefore, if ψ exists, it is unique and must be defined by $\psi(y) = (\psi_i(y))_{i\in I}$. Such a map is clearly continuous, because each of its components (the ψ_i's) is. We only need to check that it is well defined, that is, that its image is contained in X. For this, we need that for all $i \geq j$ in I, the identity $\psi_j = \varphi_{ij}\psi_i$ hold. And this is true because, by assumption, the ψ_i's are compatible maps. This completes the proof of the existence of a projective limit.

(b) The proof of uniqueness is very simple and general. In fact, any object defined by a universal property is unique up to isomorphism, provided it exists. Universal properties are a very powerful way of defining mathematical objects, and I suggest that you think about this and get acquainted with them in their general fashion, e.g, by looking at `https://en.wikipedia.org/wiki/Universal_property` and, if you get teased, at the references thereby. But let us get back to business, and let's prove the uniqueness: this is a prototype proof of uniqueness using the universal property. Suppose (X, φ_i) and (Y, ψ_i) are two projective limits. By the universal property of X there exists a unique continuous map (homomorphism) $\psi \colon Y \to X$ such that, for all $i \in I$, $\varphi_i \psi = \psi_i$. Similarly, this time by the universal property of Y, there exists a unique continuous map (homomorphism) $\varphi \colon X \to Y$ such that, for all $i \in I$,

$\psi_i\varphi = \varphi_i$. Now consider the following diagrams, for any $i \in I$:



The first one is certainly commutative. But the second one is too, indeed, if we compose the top arrow with the right one, we get

$$\varphi_i\psi\varphi = (\varphi_i\psi)\varphi = \psi_i\varphi = \varphi_i.$$

But, by the universal property of X, for each $i \in I$ there exists $\overline{\text{a unique}}$ map $X \to X$ making each of the above diagrams commutative. Therefore, $\psi\varphi = \mathrm{Id}_X$. The same exact reasoning applies to Y, thereby showing that $\varphi\psi = \mathrm{Id}_Y$. Hence φ and ψ are mutually inverse homeomorphisms (topological isomorphisms). This completes the proof.

□

In light of the previous proposition, we will talk about *the* projective limit of a projective system $\{X_i, \varphi_{ij}, I\}$ and denote it by one of the following

$$\varprojlim_{i\in I} X_i,\ \varprojlim_i X_i,\ \varprojlim_I X_i,\ \varprojlim X_i$$

Before the following Lemma, we introduce some notation that might be useful in the sequel. By the symbols $\subseteq_o,\ \subseteq_c,\ \leq_o,\ \leq_c$ we mean "open subset, closed subset, open subgroup, closed subgroup" respectively.

Lemma 2.8 ([RZ10, Lemma 1.1.2])**.** *If $\{X_i, \varphi_{ij}, I\}$ is a projective system of Hausdorff topological spaces (groups), then $\varprojlim X_i$ is a closed subspace (subgroup) of of $\prod_I X_i$.*

Proof. We want to show that $\prod X_i \setminus \varprojlim X_i$ is open. To this end, let $x = (x_i)_{i\in I} \in \prod X_i \setminus \varprojlim X_i$ and let us show that there exists an open neighbourhood W of x disjoint from $\varprojlim X_i$. Recall that

$$\varprojlim_i X_i = \{(x_i)_{i\in I} \text{ s.t. } i \geq j \Rightarrow \varphi_{ij}(x_i) = x_j\}.$$

Since $x \notin \varprojlim X_i$ there exist $r \geq s \in I$ such that $\varphi_{rs}(x_r) \neq x_s$. Now, $\varphi_{rs}(x_r), x_s \in X_s$ and X_s is Hausdorff by assumption, so there exist $U, V \subseteq_o X_s$ with $\varphi_{rs}(x_r) \in U,\ x_s \in V$ and $U \cap V = \emptyset$.

By continuity of $\varphi_{rs}\colon X_r \to X_s$ there exists an open subset U' of X_r containing x_r such that $\varphi_{rs}(U') \subseteq U$. Now, for all $i \in I$ define a topological space W_i as follows:

$$W_r = U',\ W_s = V,\ i \neq r, s \Rightarrow W_i = X_i$$

and set

$$W := \prod_{i\in I} W_i.$$

Now

- W is open in $\prod X_i$.
- $x = (x_i)_{i\in I} \in W$. Clear: $x \in U' \times V \times \prod_{i\neq r,s} W_i = W$.
- $W \cap \varprojlim X_i = \emptyset$. Indeed, if $y = (y_i)_{i\in I} \in W$, then $y_r \in U'$ so $\varphi_{rs}(y_r) \in U$ and as $U \cap V_s = \emptyset$ we have $\varphi_{rs}(y_r) \notin V_s$.

□

Definition 2.9 (Totally disconnected space)**.** A topological space X is *totally disconnected* if for all $x \in X$ the connected component of x is $\{x\}$. For example, a space with the discrete topology.

Recall Tychonoff's theorem:

Theorem 2.10 (Tychonoff)**.** *Let $\{X_i\}$ be a family of compact topological spaces. Than $\prod X_i$ is compact.* □

The following is an immediate corollary

Proposition 2.11 ([RZ10, Proposition 1.1.3])**.** *Let $\{X_i, \varphi_{ij}, I\}$ be a projective system of compact, Hausdorff, totally disconnected topological spaces (groups). Then $\varprojlim X_i$ is also a compact Hausdorff totally disconnected topological space (group).* □

2.2. **Morphisms of projective systems.** Let $\{X_i, \varphi_{ij}, I\}$ and $\{X'_i, \varphi'_{ij}, I\}$ be two projective systems of topological spaces (groups) over the same directed poset I. A *morphism* of projective systems

$$\Theta\colon \{X_i, \varphi_{ij}, I\} \longrightarrow \{X'_i, \varphi'_{ij}, I\}$$

consists of a collection of continuous maps (homomorphisms) $\vartheta_i\colon X_i \to X'_i$, for each $i \in I$ such that, whenever $i \geq j$, the following diagram is commutative:

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_{ij}} & X_j \\ \downarrow{\scriptstyle \vartheta_i} & & \downarrow{\scriptstyle \vartheta_j} \\ X'_i & \xrightarrow{\varphi'_{ij}} & X'_j \end{array}$$

The ϑ_i's are called the *components* of Θ.
Morphisms of projective systems can be composed in a natural way.
Now let $\{X_i, \varphi_{ij}, I\}$ and $\{X'_i, \varphi'_{ij}, I\}$ be two projective systems, let (X, φ_i) and (X', φ'_i) be their limits, respectively, and let $\Theta\colon \{X_i, \varphi_{ij}, I\} \longrightarrow \{X'_i, \varphi'_{ij}, I\}$ be a morphism of projective systems. Then the maps (homomorphisms)

$$\vartheta_i\varphi_i\colon X \longrightarrow X'_i$$

are compatible maps, and induce a continuous map (homomorphism)

$$\varprojlim \Theta = \varprojlim_{i\in I} \vartheta_i\colon \varprojlim_i X_i \longrightarrow \varprojlim_i X'_i$$

by using the fact that X' is a projective limit, so for all $i \in I$ there is a unique continuous map (homomorphism) making the following diagram commute.

$$\begin{array}{ccc} X & \overset{\varprojlim \Theta}{\dashrightarrow} & X' \\ & \searrow^{\vartheta_i \varphi_i} & \downarrow \varphi_i' \\ & & X_i' \end{array}$$

2.3. **Profinite spaces and groups.** We will be interested in topological spaces (groups)

$$X = \varprojlim_{i_I} X_i$$

that are projective limits of (surjective) systems of finite spaces (groups) X_i equipped with the discrete topology. Such spaces (groups) are called *profinite.*

Lemma 2.12 ([RZ10, Lemma 1.1.11])**.** *Let X be a compact Hausdorff topological space and let $x \in X$. Then the connected component C_x of x is the intersection of all the clopen (i.e., closed and open) neighbourhoods of x.*

The following theorem provides a characterisation of profinite spaces.

Theorem 2.13 ([RZ10, Theorem 1.1.12])**.** *Let X be a topological space. Then the following are equivalent.*
(a) X is profinite:
(b) X is compact Hausdorff and totally disconnected:
(c) X is compact Hausdorff and admits a basis of clopen subsets for its topology.

Proof. (a)⇒(b). Let $X = \varprojlim X_i$ where each X_i is a finite space. Then by Prop 2.11 X is compact Hausdorff and totally disconnected.
(b)⇒(c). Assume X is a compact Hausdorff totally disconnected space. We need to show that for all $x \in X$ and for all open neighbourhood W of x, W contains a clopen neighbourhood of X. Let $\{U_t,\ t \in T\}$ be the family of all clopen neighbourhoods of x. By Lemma 2.12

$$\{x\} = \bigcap_t U_t.$$

Now, $X \setminus W$ is closed and disjoint from $\bigcap_t U_t$, therefore, since X is compact, there exists a finite subset $T' \subseteq T$ such that

$$(X \setminus W) \cap \bigcap_{t \in T'} U_t = \emptyset.$$

Hence $\bigcap_{t \in T'} U_t$ is the clopen neighbourhood of x contained in W we were looking for.
(c)⇒(a). Assume X is compact, Hausdorff and admits a base of clopen subsets for its topology. Let $\mathcal{R}$ be the set of equivalence relations R on X with the property

that $[x]_R$ is a clopen subset of X for all $x \in X$. Then, for any $R \in \mathcal{R}$, the quotient X/R is finite and discrete.

(1) Finite because, since X is compact and $\{[x]_R, x \in X\}$ is an open cover then there exist $x_1, \ldots, x_n$ such that $X = \cup [x_i]_R$.
(2) Discrete because every $[x]_R$ is open in X and therefore, the preimage of any subset $U \subseteq X/R$ is a union of open subsets of X and hence open, so U is open in the quotient topology.

Moreover, $\mathcal{R}$ is a directed poset.

(1) It is ordered under $R \geq R'$ if $[x]_R \subseteq [x]_{R'}$ for all x.
(2) Is directed because given R_1 and R_2 we can find a relation $R_1 \cap R_2$ defined as the one corresponding to the partition

$$\{[x]_{R_1} \cap [y]_{R_2} | x, y \in X\}$$

So, if we define, for all $R \geq R' \in \mathcal{R}$ a map

$$\varphi_{RR'} : X/R \to X/R', \ [x]_R \mapsto [x]_{R'}$$

we get a projective system of finite discrete topological spaces over $\mathcal{R}$

$$\{X/R, \ \varphi_{RR'}, \ \mathcal{R}\}$$

of which we can take the projective limit. We claim that

$$X \simeq \varprojlim_{R \in \mathcal{R}} X/R.$$

Let $\psi : X \to \varprojlim_{R \in \mathcal{R}} X/R$ be the continuous map induced by the canonical projections $\psi_R : X \to X/R$ (we are using the universal property of the projective limit). This is continuous and surjective (see [Corollary 1.1.6][RZ10]). Since X is compact, to prove that it is a homeomorphism it is enough to show that it is injective.

Let $x \neq y \in X$. As X is Hausdorff and admits a basis consisting of clopen subsetes, there exists a clopen neighbourhood U of x that does not contain y. Let R' be the equivalence relation on X having only the two equivalent classes U and $X \setminus U$. It is clear that $R' \in \mathcal{R}$ and we have $\psi_{R'}(x) \neq \psi_{R'}(y)$. Hence $\psi(x) \neq \psi(y)$ and ψ is injective. □

Exercise 2.14. Show that the product of a family of topological spaces X_i can be expressed as a projective limit.

3. Profinite groups

3.1. **Classes of finite groups.** We'll consider now classes $\mathcal{C}$ of finite groups. For example

(1) Finite groups;
(2) Finite cyclic groups;
(3) Finite abelian groups;
(4) Finite p-groups.

they can satisfy some properties, like

i. subgroup closed;
ii. closed under taking quotients: $G \in \mathcal{C}$ and $H \triangleleft G$ then $G/H \in \mathcal{C}$;
iii. closed under forming finite direct products;
iv. if G is a finite group with normal subgroups H, K such that $G/H, G/K \in \mathcal{C}$ then $G/(H \cap K) \in \mathcal{C}$.

Finite groups respect all, finite cyclic groups respect only the first two, finite abelian groups respect all, p-groups respect all.

Definition 3.1. If $\{G_i\}$ is a surjective projective system of groups belonging to one of the class $\mathcal{C}$ their projective limit is called a *pro-$\mathcal{C}$-group.*

(1) profinite groups
(2) pro-cyclic groups
(3) pro-abelian groups
(4) pro-p-groups

Definition 3.2 (Fundamental system of open neighbourhoods)**.** If X is a topological spaces and $x \in X$ a *fundamental system of open neighbourhoods of x* is a family of open subsets $U_i \subseteq_o X$ such that, if V is any open subset containing x there exists some i such that $U_i \subseteq V$.

From now we move our attention to groups. We want to exhibit a special fundamental system of neighbourhoods of 1 of a profinite group.

Lemma 3.3 ([RZ10, Lemma 2.1.1])**.** *Let*

$$G = \varprojlim_{i \in I} G_i$$

be the projective limit of a system $\{G_i, \varphi_{ij}, I\}$ of finite groups. So a profinite group. Let

$$\varphi_i : G \to G_i$$

be the projection homomorphisms. Then

$$\{S_i | S_i = \ker \varphi_i\}$$

is a fundamental system of neighbourhoods of 1 in G.

Proof. Consider the family of neighborhoods of 1 in $\prod G_i$ of the form

$$\left(\prod_{i \neq i_1, \ldots, i_t} G_i \right) \times \{1\}_{i_1} \times \ldots \times \{1\}_{i_t}$$

for a finite collection of indices $i_1, \ldots, i_t \in I$. As we let the collection of i's vary, we obtain a fundamental system of neighbourhoods of 1 in $\prod G_i$. As I is directed, there is i_0 such that $i_0 \geq i_1, \ldots, i_t$. Then

$$G \cap [(\prod_{i \neq i_0} G_i) \times \{1\}_{i_0}] = G \cap [(\prod_{i \neq i_1, \ldots, i_t} G_i) \times \{1\}_{i_1} \times \ldots \times \{1\}_{i_t}$$

Indeed, an element on the left-hand-side is a tuple having 1 in the i_0th position, such that it satisfies the property of the projective limit. So for all the i_j we have $\varphi_{i_0 i_j}(1) = 1$ because it is a group homomorphism.

therefore the family of neighbourhoods of 1 in G of the form

$$G \cap [(\prod_{i \neq i_0} G_i) \times \{1\}_{i_0}]$$

is a fundamental system. Finally, just observe that

$$G \cap [(\prod_{i \neq i_0} G_i) \times \{1\}_{i_0}] = \ker \varphi_{i_0}.$$

□

The following is a very important and easy property characterising open subgroups in compact groups.

Lemma 3.4. *In a compact topological group G a subgroup H is open if and only if it is closed of finite index.*

Proof. Let H be closed of finite index. For a fixed $g \in G$ the map $G \to G,\ x \mapsto xg$ is a homeomorphism (check!). therefore, since H is closed, its right cosets are closed. Since H has finite index, there are finitely many cosets, hence the union of the cosets other than H is a finite union of closed, hence closed. So H, which is the complement, is open.

Let H be open in G. As the map described above is a homeomorphism, then the cosets of H are open. Hence H is closed. Moreover, the cosets are an open cover of G, which is compact, hence there are finitely many cosets.

□

Definition 3.5. Let G be a group and H a subgroup. The *core* of H in G, denoted H_G, is the largest normal subgroup of G contained in H. Equivalently, it is the intersection of all conjugates of H:

$$H_G = \bigcap_{g \in G} g^{-1} H g$$

Note that it is enough to take the intersection over a set of representatives of the right cosets of H.

Theorem 3.6 ([RZ10, Theorem 2.1.3])**.** *Let $\mathcal{C}$ be a class of finite groups satisfying properties (2) and (4). The following are equivalent for a topological group G.*

(a) G is a pro-$\mathcal{C}$-group;

(b) G is compact, Hausdorff, totally disconnected and for every open normal subgroup U of G, $G/U \in \mathcal{C}$;

(c) G is compact and the identity element 1 admits a fundamental system of neighbourhoods $\mathcal{U}$ such that $\bigcap_{\mathcal{U}} U = 1$ and each U is open and normal in G with $G/U \in \mathcal{C}$;

(d) The identity element 1 admits a fundamental system $\mathcal{U}$ of neighbourhoods U such that each U is normal in G with $G/U \in \mathcal{C}$ and

$$G = \varprojlim_{U \in \mathcal{U}} G/U.$$

Proof. (a)⇒(b). Let

$$G = \varprojlim_{i \in I} G_i,$$

for a surjective system $\{G_i, \varphi_{ij}, I\}$. Let φ_i be the projection homomorphisms. By Theorem 2.13 G is compact Hausdorff and totally disconnected. Let U be an open normal subgroup of G. By Lemma 3.3 there is some $S_i = \ker \varphi_i \leq U$. Then

$$G/U \simeq (G/S_i)/(U/S_i)$$

and $S_i \in \mathcal{C}$ because it is closed under taking subgroups and hence $G/U \in \mathcal{C}$ because it is closed under taking quotients.
(b)⇒(c). Let $\mathcal{V}$ be the set of clopen neighbourhoods of 1 in G. By Theorem 2.13 this is a fundamental system. Moreover,

$$\bigcap_{V \in \mathcal{V}} V = 1.$$

Therefore, it is enough to prove that if V is a clopen neighbourhood of 1, it contains an open normal subgroup of G.

Notation. If X is a subset of G and $n \in \mathbb{N}$ we denote by X^n the set of all products of n elements of X, and by X^{-1} the set of all inverses of elements of X.

Fix a $V \in \mathcal{V}$ and set $F := (G \setminus V) \cap V^2$. Then F is closed, and therefore compact. Now let $x \in V$. Then $x \in G \setminus F$, which is open. Since multiplication in G is continuous there exist open neighbourhoods V_x of x and S_x of 1, both contained in V such that $V_x S_x \subseteq G \setminus F$ (because $x \cdot 1 \in G \setminus F$).

Now, V is compact, so there exist finitely many $x_1, \ldots, x_n$ such that $V_{x_1}, \ldots, V_{x_n}$ cover V. Consider the corresponding S_{x_i} and take the intersection:

$$S := \bigcap_{i=1}^{n} S_{x_i}$$

and let $W = S \cap S^{-1}$. Now W is a neighbourhood of 1 and $w \in W \iff w^{-1} \in W$; $W \subseteq V$ and $VW \subseteq G \setminus F$. In particular, $VW \cap F = \emptyset$. Moreover, $VW \subseteq V^2$ hence $VW \cap (G \setminus V) = \emptyset$. Thus $VW \subseteq V$, and then

$$VW^n \subseteq V \ \forall n \in \mathbb{N}.$$

Since W is symmetric ($w \in W \iff w^{-1} \in W$), then

$$R := \bigcup_{n \in \mathbb{N}} W^n$$

is an open subgroup of G contained in V. Therefore the core

$$R_G = \bigcap_{g \in G} (g^{-1} R g)$$

is open and normal in G. Finally, $R_G \subseteq V$ because

$$R_G \leq R \subseteq VR \subseteq \bigcup_{n\in\mathbb{N}} VW^n \subseteq V.$$

So R_G is the open normal subgroup contained in V we were looking for.
(c)⇒(d). Let $\mathcal{U}$ be as in (c). Then $\mathcal{U}$ is a directed poset, if we set that for $U, V \in \mathcal{U}$, $U \preceq V \iff U \leq V$. Then we can consider the projective system over $\mathcal{U}$ given by $\{G/U, \varphi_{UV}\}$ where $\varphi_{UV} : G/U \to G/V$ is the natural surjective morphism define whenever $V \leq U$. With respect to this system, the canonical epimorphisms

$$\psi_U \colon G \to G/U$$

are compatible. Hence they induce a continuous homomorphism

$$\psi \colon G \longrightarrow \varprojlim_{U\in\mathcal{U}} G/U.$$

This is an isomorphism of topological groups. Indeed, by [RZ10, Corollary 1.1.6], since all the ψ_U's are surjective, then so is ψ. To show that ψ is a homeomorphism, since G is compact, we just need to show injectivity. Now, if $\psi(x) = 1$ then x belongs to every $U \in \mathcal{U}$. And as $\bigcap U = 1$ we have $x = 1$.
(d)⇒(a). Is obvious. □

3.2. **Completions.** Consider a formation $\mathcal{C}$ of finite groups ($\mathcal{C}$ satisfies properties (ii) and (iv)) and let G be a group (not necessarily in $\mathcal{C}$). Let

$$\mathcal{N} = \{N \triangleleft_f G | G/N \in \mathcal{C}\}$$

be the set of normal subgroups N of finite index of G such that $G/N \in \mathcal{C}$. Then $\mathcal{N}$ can be naturally made into a directed poset by setting $M \preceq N \iff N \leq M$. If $M, N \in \mathcal{N}$ and $N \succeq M$ define $\varphi_{NM} : G/N \to G/M$ to be the canonical map. In this way

$$\{G/N,\ \varphi_{NM},\ \mathcal{N}\}$$

is a projective system of groups in $\mathcal{C}$. The pro-$\mathcal{C}$-group

$$G_{\hat{\mathcal{C}}} = \varprojlim_{N\in\mathcal{N}} G/N$$

is called the *pro-$\mathcal{C}$-completion* of G.

Example 3.7. (1) Consider $\mathbb{Z}$ made into a partially ordered set by divisibility. Then for $n|m$ we have $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, so $\{\mathbb{Z}/n\mathbb{Z}\}$ is a projective system. The profinite completion is

$$\hat{\mathbb{Z}} = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$$

called the *ring of profinite integers*.
Notice that the elements of $\hat{\mathbb{Z}}$ are (equivalence classes of) sequences $(a_n) = (a_1, a_2, \ldots)$ of natural numbers such that, whenever $n|m$ we have

$$a_m \equiv a_n \mod n.$$

In particular, $\hat{\mathbb{Z}}$ is a subring of the ring $\prod_n \mathbb{Z}/n\mathbb{Z}$. Operations are defined coordinate-wise, and we have a natural embedding

$$\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}, \ a \mapsto (a, a, a, \ldots).$$

For example, if $\mathbb{F}_q$ is a finite field, than $\mathrm{Gal}(\bar{\mathbb{F}}_q \simeq \hat{\mathbb{Z}}$.

(2) Let p be a prime integer. Than $\{p^n, \ n \in \mathbb{N}\}$ is a directed poset and $\{\mathbb{Z}/p^n\mathbb{Z}, \ n \in \mathbb{N}\}$ is a projective system of finite group. Its profinite completion is denoted by

$$\mathbb{Z}_p = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$$

and called *the ring of p-adic integers*.
Its elements can be identified with (equivalence classes of) sequences $(a_n) = (a_1, a_2, \ldots)$ of natural numbers with

$$a_n \equiv a_m \mod p^m$$

whenever $m \leq n$.

References

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