Universität Konstanz
Fachbereich Mathematik und Statistik
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Exercises for the course "Linear Algebra I"

Sheet 4

Hand in your solutions on Thursday, 21. November 2019, 09:55, in the Postbox of your Tutor in F4. Please try to write your solutions clear and readable, write your name and the name of your tutor on each sheet, and staple the sheets together.

Exercise 4.1 *(4 points)*

Consider the following three systems of linear equations over $\mathbb{Q}$. Bring them into reduced form (reduzierte Zeilenstufenform) first. Then determine (justify your answer!) the solution sets of the original system of linear equations.

(a) $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \end{array}\right)$ (b) $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 6 & 16 & 42 & 4 \\ 1 & 2 & 4 & 8 & 1 \\ 1 & 3 & 9 & 27 & 2 \end{array}\right)$ (c) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right)$

Exercise 4.2 *(4 points)*

Let $n \in \mathbb{N}$ and M an $(n \times n)$-Matrix with entries in $\mathbb{Q}$. Show the following:
The system of homogeneous linear equations $Mx = 0$ has a non trivial (i.e., $x \neq 0$) solution in $\mathbb{R}$ if and only if it has a non trivial solution in $\mathbb{Q}$.

Exercise 4.3 *(3 points)*

Let the following matrices with entries in $\mathbb{Q}$ be given. Compute (your calculations should be shown):

a) $\begin{pmatrix} 1 & 3 & 2 \\ -4 & 5 & 1 \\ 6 & -4 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & -6 \\ \frac{1}{2} & -5 & 7 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 1 & 2 \\ 3 & -1 & 2 \\ 0 & 5 & -2 \\ -1 & 7 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 & -5 & 2 \\ 1 & 9 & -7 & 3 \\ 2 & 0 & 2 & 1 \end{pmatrix}$

Exercise 4.4 *(5 points)*

Let K be a field and $n \in \mathbb{N}$. We denote by $M_{n\times n}(K)$ the set of all the $(n \times n)$-Matrices with entries in K. Further, let $+$ denote the entrywise addition, i.e.,

$$(A + B)_{ij} = A_{ij} + B_{ij},$$

and $\cdot$ the row-by-column multiplication defined in class.

In the Plenumsübung on 19.11 you will see that $(M_{n\times n}(K), +)$ is an abelian group for every field K and every $n \in \mathbb{N}$. Moreover, the associativity of multiplication will be shown in class.

Your task is to decide, which of the following properties are satisfied by $(M_{n\times n}(K), +, \cdot)$ for any field K and any $n \in \mathbb{N}$, and which are not. Justify your answers.

(1) Closure with respect to multiplication
(2) Existence of a neutral element w.r.t. multiplication
(3) Distributive law.
(4) Commutativity of multiplication
(5) Integrity, i.e., there are no *zero divisors* (i.e., if $A, B \in M_{n\times n}(K)$ and $A \cdot B = 0$, then $A = 0$ or $B = 0$).

What can we say about the structure of $(M_{n\times n}(K), +, \cdot)$?