Permanence criteria for Varieties of Hilbert type

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Definitions

Let K be a field of characteristic 0.
A K-variety is a geometrically integral K-scheme of finite type.

Definition

Let X be a K-variety.

- $A \subseteq X(K)$ is thin if

$$A \subseteq X_0(K) \cup \bigcup_{i \in I} p_i(Y_i(K)) \quad \text{for} \quad \begin{cases} X_0 \subsetneq X \text{ a closed subset;} \\ I \text{ a finite set;} \\ Y_i \ K\text{-varieties}, \ \dim Y_i = \dim X; \\ p_i \text{ dominant morphisms}, \ \deg p_i \geq 2. \end{cases}$$

- X is of Hilbert type if $X(K)$ is non-thin;
- K is Hilbertian if for all $n \geq 0$, $\mathbb{A}^n$ (equiv. $\mathbb{P}^n$) is of Hilbert type.

Examples

- For $X = \mathbb{A}^1$ examples of thin sets are
  - Finite sets of closed points;
  - $K^2 \subseteq \mathbb{A}^1(K)$ with $p\colon \mathbb{A}^1 \to \mathbb{A}^1,\ x \mapsto x^2$.
- An algebraically closed field K is not Hilbertian, as $K^2 = K$;
- Number fields are Hilbertian (Hilbert, 1892);
- Let C be a smooth projective curve over a number field K. Assume $C(K) \neq \emptyset$. Let g be the genus.

$g = 0$ $C \simeq \mathbb{P}^1 \Rightarrow C$ is of Hilbert type.

$g \geq 2$ by Faltings' theorem $C(K)$ is finite, hence thin, so C is not of Hilbert type.

$g = 1$ C is an elliptic curve. By Mordell-Weil theorem $C(K)/2C(K) = \{P_1, \dots, P_r\}$. With $p_i\colon C \to C,\ P \mapsto 2P + P_i$ we have

$$C(K) = \bigcup p_i(C(K)).$$

Some properties

Remark

- Being of Hilbert type is a birational property;

Remark

The following are equivalent

- K is Hilbertian;
- there exists some positive dimensional K-variety of Hilbert type;
- $\mathbb{A}^1$ is of Hilbert type;
- for all $f \in K[T, X]$ irreducible, $\deg_X f > 1$ there exist $t \in K$ such that $f(t, X) \in K[X]$ is irreducible.

Inverse Galois Problem

Question: Is every finite group G realisable as the Galois group of some Galois extension of $\mathbb{Q}$?

Theorem (Hilbert)

Let K be a Hilbertian field. Let $f \in K[T_1, \dots, T_r, X]$ be irreducible, monic and Galois in X. There exist $t_1, \dots, t_r \in K$ such that $f(\underline{t}, X)$ is irreducible and

$$\mathrm{Gal}(f(\underline{t}, X)/K) \simeq \mathrm{Gal}(f(\underline{T}, X)/K(\underline{T})).$$

Theorem (Noether)

Let G be a finite group of order n. If $\mathbb{Q}(T_1, \dots, T_n)^G$ is rational then G is realisable over $\mathbb{Q}$.

Inverse Galois Problem

Noether's Problem: Is $\mathbb{Q}(T_1, \ldots, T_{|G|})^G$ always rational?
Equivalently, is $\mathbb{A}^{|G|}/G$ rational for every finite group G?
No: C_{47} (Swan, 1969).

Conjecture (Colliot-Thélène)

Every unirational variety over a number field is of Hilbert type.

Examples

- A curve C over a number field
  C of Hilbert type $\iff$ C rational $\iff$ C unirational (Lüroth);
- Linear algebraic groups are unirational and of Hilbert type (Sansuc, 1981).

Definitions and examples

Let K be a Hilbertian field. Let L/K be a field extension.

Definition

A permanence criterion is a sufficient condition on L/K in order for L to be Hilbertian.

Examples

L is Hilbertian if L/K is

- finitely generated;
- small;
- abelian (Kuyk, 1970).

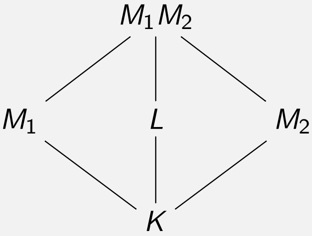
Haran's Diamond Theorem

Let K be a Hilbertian field. Let L/K be a field extension.

Theorem (Haran's Diamond Theorem, 1999)

If

- $M_1/K,\ M_2/K$ *are Galois extensions*
- $L \subsetneq M_1$ *and* $L \subsetneq M_2$
- $L \subseteq M_1 M_2$



then L is Hilbertian.

Finite abelian-simple length extensions

Let K be a Hilbertian field. Let L/K be a field extension.

Theorem (Bary-Soroker, Fehm, Wiese, 2016)

If there exist

$$K = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_m$$

such that

- K_{i+1}/K_i *is Galois, with Galois group either*
  - *abelian*
  - *a product of non-abelian finite simple groups*
- $L \subseteq K_m$

then L *is Hilbertian.*

Permanence criteria for Varieties

Recall: K is Hilbertian $\iff$ $\mathbb{A}^1_K$ is of Hilbert type.
Let X be a K-variety of Hilbert type. Let L/K be a field extension.

Definition

A permanence criterion is a sufficient condition on L/K in order for X_L to be of Hilbert type.

Examples

X_L is of Hilbert type if L/K is

- finitely generated;
- small.

Question: do the other permanence criteria also extend to varieties of Hilbert type?

Weil restriction

Definition

Let L/K be a finite field extension and X an L-scheme.
The Weil restriction is a contravariant functor $\operatorname{Res}_{L/K} X\colon \mathbf{Sch}/K \to \mathbf{Set}$ defined by

$$\operatorname{Res}_{L/K} X(S) = X(S \times_K L)$$

If representable, the K-scheme representing it is also called the Weil restriction of X and denoted by $\operatorname{Res}_{L/K} X$.

- $\dim(\operatorname{Res}_{L/K} X) = [L : K] \cdot \dim X$;
- $\operatorname{Res}_{L/K} X(K) \xleftrightarrow{1:1} X(L)$.

A generalisation

Theorem

Let X be a K-variety of Hilbert type such that for all finite extension K'/K the K-variety $\mathrm{Res}_{K'/K}(X_{K'})$ is of Hilbert type.
If an extension L of K

- *fits in a Diamond*

  or

- *is a finite abelian-simple length extension*

then X_L is of Hilbert type.

Examples

- $\mathbb{A}^1$ of Hilbert type $\iff$ $\mathrm{Res}_{K'/K}\,\mathbb{A}^1$ of Hilbert type. In particular we recover the theorems of Haran and on finite abelian-simple length extensions.
- Linear algebraic groups are stable under Weil restriction

That's all Folks!