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Stochastic II

1. Tutorial

Exercise 1

(a) Compute the characteristic function of a Cauchy distributed random variable X with density function

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}.$$

(b) Show that for iid rvs $X_1, \ldots, X_n$ with density f, $\frac{1}{n}\sum_{i=1}^n X_i$ possess the same density f.

(c) Does a law of large numbers hold for the sequence of random variables in b)?

Hint: Find the characteristic function of $Y_1 - Y_2$ where Y_1, Y_2 are independent and standard exponential distributed.

Exercise 2

(a) Let $X_1, \ldots, X_n$ be iid Cauchy distributed rvs and a_k, $k \geq 1$ a sequence of real numbers Prove that $\sum_{k=1}^n a_k X_k$ converges in distribution if and only if $\sum_{k=1}^\infty |a_k| < \infty$.

(b) Let X be a random variable distributed as $U(0,1)$ and $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with

$$P(X_n = kn^{-1}) = n^{-1} \; \forall k = 1, \ldots, n.$$

Show that $X_n \xrightarrow{d} X$.

Exercise 3

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with characteristic function $\phi_X(t)$ and $\bar{X} = n^{-1}\sum_{i=1}^n X_i$. Suppose that $\phi_X'(0)$ exists. Prove that

(a) $\phi_X'(0) = ia \quad \Longrightarrow \quad \bar{X} \xrightarrow{P} a.$

(b) $\bar{X} \xrightarrow{P} a \quad \Longrightarrow \quad \phi_X(tn^{-1})^n \to \exp(iat) \quad \Longrightarrow \quad \phi_X'(0) = ia.$

Hint : Taylor decomposition of characteristic function.

Exercise 4

(a) Suppose $X_n \xrightarrow{L_1} X$. Show that $E[X_n] \to E[X]$. Is the converse true?

(a) Suppose $X_n \xrightarrow{L_2} X$. Show that $\mathrm{var}(X_n) \to \mathrm{var}(X)$.

(c) Given a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ suppose $|X_n| \leq Z$ a.s. for all n, where $E(Z) < \infty$. Prove that

$$X_n \xrightarrow{P} X \quad \Longrightarrow \quad X_n \xrightarrow{L_1} X,$$

(d) LetX_1 and X_2 be independent and poisson distributed random variables with $X_i \sim Poi(\lambda_i)$, $i = 1, 2$. Compute the conditional expectation

$$E(X_1|X_1 + X_2).$$

Hand in **We 27.04.10 up to 16.00** in postbox 20 at F4.