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Stochastics II (Stochastic Processes I)

12. Tutorial

Exercise 1

Let $B = \{B_t : 0 \leq t \leq T\}$ be a Gaussian process with drift 0 and variance $\sigma^2 t$ with strictly positive $\sigma^2 \in \mathbb{R}^+$ on the probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)$, so that we may see B_t as σW_t for a standard Brownian motion. Let $\nu \in \mathbb{R}$ and define the rv

$$\Lambda = \exp\{\frac{\nu}{\sigma^2} B_T - \frac{\nu^2}{2\sigma^2} T\}$$

and let Q defined by $Q(A) = E[\Lambda \mathbb{1}_A] = \int \Lambda \mathbb{1}_A dP$ for all $A \in \mathfrak{F}$. Prove that Q is a probability measure.

Exercise 2

Suppose $B_t,\ t \in [0, T]$ is a Gaussian process with drift νt and variance $\sigma^2 t$ with respect to the probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, Q)$.

Consider now the deterministic process P_t and the stochastic process S_t given by

$$dP_t = rP_t dt, \quad P_0 = 1$$

and

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 = 1,$$

where W is the standard P-Brownian motion. Define a portfolio as a pair (α_t, β_t) of two bounded, continuous stochastic processes which are adapted to the filtration $\mathfrak{F}_t$ generated by the process S_t. Let the (wealth) value function for an initial wealth $V_0 > 0$ of the portfolio be

$$V_t(\alpha, \beta) = \alpha_t S_t + \beta_t P_t,\ V_0 > 0.$$

Assume moreover that the portfolio is self-financing, i.e.

$$dV_t(\alpha, \beta) = \alpha_t dS_t + \beta_t dP_t.$$

Specify a ν (as in exercise 1) such that $e^{-rt}S_t$ is a martingale under Q and prove that $e^{-rt}V_t$ is also a martingale under Q.

Hint: First, rewrite σW_t, with the help of $\tilde{W}_t$, where $\tilde{W}_t$ is the standard Wiener process with respect to Q and insert the result into the solution of the SDE of S_t.

You may use the following result: Let W_t be a standard Brownian motion, then the process $\exp\left(\theta W_t - \frac{1}{2}\theta^2 t\right)$ constitutes a martingale with respect to $\mathfrak{F}_t$.

Let W_t be a standard Brownian motion and ϕ_t a bounded stochastic process with continuous sample paths, then the stochastic integral $\int_0^t \phi_u dW_u$ defines a martingale with respect to $\mathfrak{F}_t$.

Exercise 3

Assume that there exists a pair (α, β) and a positive real number x such that V_t replicates a European call option $\xi = (S_T - K)^+$, i.e. $V_T = V_0 + \int_0^T \alpha_s dS_s + \int_0^T \beta_s dP_s = \xi,\ V_0 = x$.

Prove the Black-Scholes formula, i.e. show that for $t < T$ the value at time t of the European call option on the stock S with the strike price $K > 0$ and maturity T is

$$V_t = S_t \Phi(d_1(t, S_t)) - K e^{-r(T-t)} \Phi(d_2(t, S_t))$$

where Φ is the $N(0,1)$ distribution function and

$$d_1(t,x) = \frac{\log(x/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2(t,x) = d_1(t,x) - \sigma\sqrt{T-t}.$$

Hint: Prove first: Let $Z \sim N(\gamma, \tau^2)$ then

$$E[(ae^Z - K)^+] = ae^{\gamma + \frac{1}{2}\tau^2} \Phi\left(\frac{\log(a/K) + \gamma}{\tau} + \tau\right) - K\Phi\left(\frac{\log(a/K) + \gamma}{\tau}\right)$$

and note that $V_t = e^{rt}e^{-rt}V_t = e^{rt}E_Q[e^{-rT}V_T|\mathfrak{F}_t]$ and $V_T = (S_T - K)^+$. Rewrite S_T with the help of S_t and apply the above result.

These exercises are additional and voluntary.
Hence, the exercise sheet doesn't need to be handed in.