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Stochastic II

2. Tutorial

Exercise 1

(a) Suppose that $X_n \stackrel{a.s.}{\to} X$ and $Y_n \stackrel{a.s.}{\to} Y$ and show that

$$X_n + Y_n \stackrel{a.s.}{\to} X + Y.$$

(b) Show that a corresponding result holds for the convergence in the r-th mean and in probability, but not in distribution.

Exercise 2

Let X, Y be iid rvs on $(\Omega, \mathfrak{F}, P)$ with $E[X]$,$E[Y] < \infty$ and let $\mathfrak{G}$ be a sub-σ algebra of $\mathfrak{F}$.

(a) Prove Jensen's inequality, i.e.

$$g(E(Y|\mathfrak{G})) \leq E(g(Y)|\mathfrak{G}) \ a.s.$$

for all convex functions g.

(b) Use the result derived in (a) in order to show

$$E[f(X+Y)] \geq E[f(X+E[Y])],$$

where f is a nonnegative convex function.

Exercise 3

Suppose $(X_\lambda)_{\lambda\in\Lambda}$ is a family of integrable random variables on a probability space $(\Omega, \mathfrak{F}, P)$. This family is called uniform integrable if for $c \to \infty$

$$\sup_{\lambda\in\Lambda} E[\mathbb{1}_{\{|X_\lambda|\geq c\}|X_\lambda|}] \to 0.$$

(a) Prove $(X_\lambda)_{\lambda\in\Lambda}$ is uniform integrable iff $(X_\lambda)_{\lambda\in\Lambda}$ is bounded in $L^1(\Omega)$ and for every $\varepsilon > 0$ we can find a $\delta > 0$ with

$$\sup_{\lambda\in\Lambda} E[\mathbb{1}_{\{A\}}|X_\lambda|] < \varepsilon, \quad \text{whenever} \quad P(A) < \delta.$$

(b) Suppose X, X_n, $n \in \mathbb{N}$ integrable rvs, where (X_n) is even uniform intergrable. Prove

$$X_n \xrightarrow{P} X \quad \Longleftrightarrow \quad X_n \xrightarrow{L_1} X.$$

Exercise 4

Let $(X_t)_{t\geq 0}$ be a stochastic process such that for all $n \geq 1$ and indexes $0 = t_0 < t_1 < \ldots < t_n < \infty$ the increments $X_0, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent $N(0, t_i - t_{i-1})$-normal distributed with $P(X_0 = 0) = 1$.

Compute the finite dimensional distribution of (X_t), i.e.

$$P(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n)$$

with $B_i \in \mathcal{B}(\mathbb{R})$ with the help of the finite dimensional function

$$f(y_1, \ldots, y_n) = (y_1, y_1 + y_2, \ldots, y_1 + \ldots + y_n).$$

Hint: Note, that $(X_{t_1}, \ldots, X_{t_n}) = f(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})$.

Hand in **We 03.11.10 up to 16.00** in postbox 20 at F4.