University of Konstanz 3.11.2010
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Stochastic II

3. Tutorial

Exercise 1
For two distribution functions F and G, let

$$d(F,G) := \inf\{\delta > 0 : F(x-\delta) - \delta \le G(x) \le F(x+\delta) + \delta \ \forall x \in \mathbb{R}\}.$$

(a) Show d is a metric on the space of distribution functions.

(b) Show that convergence in distribution is equivalent to convergence with respect to the metric d, defined in *(a)*.

Exercise 2
Given a sequence of i.i.d. discrete random variables $X_1, \ldots, X_n$ with $P(X_i = 1) = P(X_i = -1) = 0.5$, a simple random walk is defined by the discrete process $(S_n)_{n\in\mathbb{N}}$, with $S_n = \sum_{i=1}^n X_i$ and $S_0 = 0$.

(a) Show that both S_n and $S_n^2 - n$ are martingales.

(b) Assume that a gambler is playing a coin-flipping game. He starts with $S_0 = k$ € and earns an additional 1€ in case of a head, while he looses 1€ in case of a tail. The game ends, if either the gambler is going bankrupt or after his capital adds up to N €. Compute the probability of a ultimate ruin of the gambler and the expected duration of the game, using the results derived in *(a)*.

Exercise 3

(a) Let $X, X_1, X_2, \ldots$ be random variables and there exists $r > 0$ such that $\{|X_n|^r, n \ge 1\}$ is uniformly integrable. Prove that if $X_n \xrightarrow{d} X$ for $n \to \infty$, then

$$E[|X_n|^r] - E[|X|^r] \quad \text{for} \quad n \to \infty.$$

(b) Let $X, X_1, X_2, \ldots$ be random variables with finite moments of arbitrary order and

$$E[|X_n|^k] \to E[|X|^k] \quad \text{for} \quad n \to \infty.$$

Show that if the moments of X uniquely determines the distribution of X, which is for instance the case for the normal distribution, we have

$$X_n \xrightarrow{d} X$$

for $n \to \infty$.

Exercise 4

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with $E(X_n) = 0$ and $E(X_n^2) = 1$. Prove that

$$P\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = \infty\right) = 1$$

where $S_n = \sum_{i=1}^n X_i$.

Hand in **We 10.11.10 up to 16.00** in postbox 20 at F4.