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Stochastics II

4. Tutorial

Exercise 1

Let $(\Omega, \mathfrak{F}, P)$ be a probability space and suppose that the family $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ is adapted to $\mathfrak{F}_t$, i.e. X_t is $\mathfrak{F}_t$-measurable for $t \geq 0$. Moreover, assume that the increments of the family $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ are independent from the past, i.e. for all $s, t \in [0,\infty)$ with $s < t$ the difference $X_t - X_s$ is independent from $\mathfrak{F}_s$.

(a) Show that if all X_t are integrable then the family $Y_t := X_t - E[X_t]$ is a martingale.

(b) Show that if all X_t are square integrable then the family $Y_t := X_t^2 - E[X_t^2]$ is a martingale.
Hint: $E[X_t^2|\mathfrak{F}_s] = E[(X_s + X_t - X_s)^2|\mathfrak{F}_s]$

(c) Show that if $\alpha \in \mathbb{R}$ and $E[e^{\alpha X_t}] < \infty$ for all t, then the family $Y_t := \frac{e^{\alpha X_t}}{E[\alpha X_t]}$ is a martingale.

Hint: $e^{\alpha X_t} = e^{\alpha X_s} e^{\alpha(X_t - X_s)}$, *where the last expression is a product of independent rvs.*

Exercise 2

Let $(B_t)_{t\geq 0}$ be a one dimensional Brownian motion with $B_0 = 0$.

(a) Show the Markov property of $(B_t)_{t\geq 0}$.

(b) Show that $(-B_t)_{t\geq 0}$ and $(B_t^\lambda := \frac{1}{\lambda} B_{\lambda^2 t})_{t\geq 0}$ with $\lambda > 0$ are Brownian motions.

(c) Let $s \geq 0$ and $B_t^{(s)} := B_{t-s} - B_s$ for $t \geq 0$. Show that $(B_t^{(s)})_{t\geq 0}$ is a Brownian motion.

Exercise 3

Show that any sequence of independent random variables taking values in the countable set S is a Markov chain. Under what condition is the chain time homogen?

Exercise 4
Let $\{X_n : n \geq 1\}$ be independent identically distributed integer valued random variables. Which of the following constitute is a Markov chain

(a) $S_n = \sum_{r=1}^{n} X_r$ with $S_0 = 0$

(b) $Y_n = X_n + X_{n-1}$ with $X_0 = 0$

(c) $Z_n = \sum_{r=0}^{n} S_r$.

Hand in **We 17.11.10 up to 16.00** in postbox 20 at F4.