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Stochastic II

5. Tutorial

Exercise 1

A monkey types a text at random on a keyboard, so that each new letter is picked uniformly at random among the 26 letters of the roman alphabet. Let X_n be the n-th letter of the monkey's masterpiece, and let T be the first time when the monkey has typed the exact word ABRACADABRA, i.e.

$$T = \inf\{n \geq 0 : \ (X_{n-10}, X_{n-9}, ..., X_n)\} = (A, B, R, A, C, A, D, A, B, R, A)$$

(a) Compute $E[T]$ with the help of the optional sampling theorem.
Hint: Construct the game.

(b) Why $E[T]$ is larger than the average first time the monkey has typed ABRACADABRO?

Exercise 2

Let $(\Omega, \mathfrak{F}, P)$ be a probability space with a filtration $(\mathfrak{F}_\mathfrak{n})_{n\in\mathbb{N}}$ and let $(S_n)_{n\in\mathbb{N}}$ be a $\mathfrak{F}_\mathfrak{n}$-submartingale. Moreover, assume that for the family of rvs $(\pi_n)_{n\in\mathbb{N}}$ each π_n is $\mathfrak{F}_{\mathfrak{n}-1}$-measurable for $n \geq 1$ and let π_0 be $\mathfrak{F}_0$-measurable. Consider the process

$$V_n := \pi_0 S_0 + \sum_{j=1}^{n} \pi_j (S_j - S_{j-1}), \ n \geq 0.$$

Prove the following statements:

(a) Provided π_n is non-negative, $(V_n)_{n\in\mathbb{N}}$ is a submartingale.

(b) If $(S_n)_{n\in\mathbb{N}}$ is a martingale, $(V_n)_{n\in\mathbb{N}}$ is a martingale, even without the additional assumption of $\pi_n > 0$.

Exercise 3
Suppose $(B_t)_{t\geq 0}$ is a one dimensional Brownian motion with $B_0 = 0$. Moreover let, for $t > 0$, $0 = t_0^n < t \cdots < t_{k_n}^n = t$ be a sequence of partitions of the interval $[0, t]$ with

$$max\{t_j^n - t_{j-1}^n : 1 \leq j \leq k_n\} \to 0, \quad \text{for} \quad n \to \infty.$$

Prove

$$\sum_{j=1}^{k_n} (B_{t_j^n} - B_{t_{j-1}^n})^2 \xrightarrow{L^2} t \quad \text{for} \quad n \to \infty,$$

i.e. $E(|\sum_{j=1}^{k_n} (B_{t_j^n} - B_{t_{j-1}^n})^2 - t|^2) \to 0,\ n \to \infty$.

Hand in **We 24.11.10 up to 16.00** in postbox 20 at F4.