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Stochastic II

7. Tutorial

Exercise 1

Let $(\Omega, \mathfrak{F}, P)$ be a probability space and $(X_n)_{n\in\mathbb{N}}$ a sequence of independent rvs. Suppose the filtration $\mathfrak{F}_n$ is given by

$$\mathfrak{F}_n := \sigma(X_k : \ 0 \leq k \leq n).$$

(a) Let $S_n := \sum_{k=0}^{n} X_k$ for $n \in \mathbb{N}$ and prove that given $E[X_n] = 0$ for all $n \in \mathbb{N}$, $(S_n)_{n\in\mathbb{N}}$ is a $\mathfrak{F}_n$-martingale.

(b) Let $u > d > 0$ (up and down) and X_n be Bernoulli-distributed rvs with

$$P(X_n = u) = p, \ P(X_n = d) = 1 - p$$

and assume $S_n := \prod_{k=0}^{n} X_k$.
Prove: $(S_n)_{n\in\mathbb{N}}$ is a martingale if and only if $pu + (1 - p)d = 1$.

(c) Consider the same situation as in (b) for a finite family of rvs $(X_n)_{0\leq n\leq N}$ and choose the probability space $\Omega := \{u, d\}^{N+1}$ equipped with the measure P given by

$$P(\omega) = P((\omega_0, ..., \omega_N)) := p^{(\text{number of u's in } \omega)} \cdot (1 - p)^{(\text{number of d's in } \omega)}$$

for a $p \in (0, 1)$.
Let $X_i(\omega) = \omega_i, \ i = 0, \ldots, N$ and assume moreover that $d < 1 < u$ holds. Setting $p_* = \frac{1-d}{u-d}$, we are able to define a new probability measure P^* by

$$P^*(\omega) = P^*((\omega_0, ..., \omega_N)) := p_*^{(\text{number of u's in } \omega)} \cdot (1 - p_*)^{(\text{number of d's in } \omega)}$$

Show that P and P^* are probability measures on Ω and that X_i are a family of independent rvs under both measures. Moreover prove, that $(S_n)_{n\in\mathbb{N}}$ is martingale under P^*.

Exercise 2
Let $(B_t)_{t\geq 0}$ be a brownian motion. Consider the simple, quadratic integrable processes

$$f = f_1 \mathbb{1}_{\{t=0\}} + \sum_{j=1}^{n} f_j \mathbb{1}_{(t_{j-1},t_j]}$$

$$g = g_1 \mathbb{1}_{\{t=0\}} + \sum_{j=1}^{n} g_j \mathbb{1}_{(t_{j-1},t_j]}$$

and let f_j, g_j be $\mathfrak{F}_{t_{j-1}}$ measurable (without loss of generality, you can assume that both processes have the same sampling points t_j). Prove that

$$m_t := \left(\int_0^t f \; dB\right)\left(\int_0^t g \; dB\right) - \int_0^t f \cdot g \; ds$$

is a martingale, i.e. that

$$< \int_0^{\cdot} f \; dB, \int_0^{\cdot} g \; dB >= \int_0^{\cdot} f \cdot g \; ds$$

Exercise 3
Assume $(B_t)_{t\in[0,T]}$ is a standard brownian motion. Verify the following statements using similar arguments as for the proof of

$$\int B_t dB_t = \frac{1}{2}B_t^2 - \frac{1}{2}t,$$

which can be found in the lecture notes.

(a) $\int_0^t s dB_s = tB_t - \int_0^t B_s ds$

(b) $\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds$

(c) $E\left[\left(\int_0^t B_s dB_s\right)^2\right] = \int_0^t E(B_s^2) ds$

Hand in **We 08.12.10 up to 16.00** in postbox 20 at F4.