University of Konstanz
Department of Mathematics and Statistics
Prof. Dr. M. Kohlmann
Yevgen Shumeyko 15.12.2010

Stochastics II

9. Tutorial

Exercise 1

Let $(\Omega, \mathfrak{F}, \mathfrak{F}_{\mathfrak{n}n\in\mathbb{N}}, P)$ be a probability space with a given filtration and let $(X_n)_{n\in\mathbb{N}}$ be a martingale on this probability space. Prove the equivalence of the following statements:

(a) $(X_n)_{n\in\mathbb{N}}$ is a regular martingale.

(b) The family $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable.

(c) There exists a $\mathfrak{F}$-measurable, integrable rvs X with $X_n \xrightarrow{L^1} X,\ n \to \infty$.

(d) There exists a $\mathfrak{F}$-measurable, integrable rvs X with $X_n \xrightarrow{L^1} X,\ n \to \infty$ and $E[X|\mathfrak{F}_{\mathfrak{n}}] = X_n$.

Exercise 2

Let $(\Omega, \mathfrak{F}, P)$ be a probability space and $(X_n)_{n\in\mathbb{N}}$ be a sequence of iid rvs with $P(X_n = 1) = P(X_n = -1) = \frac{1}{2}$. We equip the probability space with the filtration generated by X_n, i.e. $\mathfrak{F}_n = \sigma(X_j : 1 \leq j \leq n)$. Consider now, the process

$$V_n := \sum_{j=1}^{n} 2^{j-1} X_j \quad \text{for} \quad n \geq 1,$$

that describes the wealth of a gambler after n rounds of fair coin tossing game. The gambler starts with a wealth of 0 and is allowed to borrow an arbitrary large amount of money. Note, V_n is a martingale (Why?). Moreover, define a stopping time $\tau := \min\{n \geq 1 : X_n = 1\}$.

(a) Describe the strategy of a gambler with the wealth process $(V_n)_{n\in\mathbb{N}}$. Describe the one of a gambler with $(V_{\tau\wedge n})_{n\in\mathbb{N}}$.

(b) Compute $E[V_{\tau\wedge n}] = E[\sum_{j=1}^{n-1} \mathbb{1}_{\{\tau=j\}} V_j + \mathbb{1}_{\{\tau\geq n\}} V_n]$ with $n \geq 1$.
Hint: Use the fact, that V_j is constant on the set $\{\tau = j\}$ and V_n is constant on $\{\tau \geq n\}$. Many thanks to Max Brixner !

(c) Compute $E(V_\tau)$ and compare with *(b)*.

Exercise 3

Let $w = (w_1, \ldots, w_d)$ be a d-dimensional Brownian motion and suppose that A and C_i, $i = 1, \ldots, d$ are adapted, essential bounded $\mathbb{R}^{n\times n}$ processes in $[0, T]$. Moreover, let $|\cdot|$ be the euclidian norm in $\mathbb{R}^n$ and x a solution of the stochastic differential equation (SDE)

$$\begin{aligned} dx(s) &= A(s)x(s)ds + \sum_{i=1}^{d} C_i(s)x(s)dw_i(s), \\ x(0) &= x_0. \end{aligned}$$

Prove

$$d|x(s)|^2 = \left(2x^T(s)A(s)x(s) + \sum_{i=1}^{d}(|C_i(s)x(s)|^2)\right) ds + 2\sum_{i=1}^{d} x^T(s)C_i(s)x(s)dw_i(s).$$

Hint: Note $d|x(s)|^2 = d\sum x_j^2(s)$.

Hand in **Mo 10.01.11 up to 15.00** in postbox 20 at F4.

The Stochastics team wishes you all a
Merry Christmas and a successful happy New Year