**Diplomarbeit - slightly improved version**

# Spectrahedra and a Relaxation of Convex, Semialgebraic Sets

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# Contents


**Introduction** v

**1. Spectrahedra and Their Projections** 1
- 1.1. Elementary Properties and the Theorem of Helton and Vinnikov . . . . . . 1
- 1.2. Convex Hulls of the Union of Spectrahedra . . . . . . . . . . . . . . . . . . . 7
- 1.3. A Local-Global Principle for Projections of Spectrahedra . . . . . . . . . . . 15

**2. Truncated Quadratic Modules and the Lasserre-Relaxation** 17
- 2.1. Quadratic Modules and Convex Cones . . . . . . . . . . . . . . . . . . . . . 17
- 2.2. The Lasserre-Relaxation . . . . . . . . . . . . . . . . . . . . . . . . . . . . 22

**3. The Exactness of the Lasserre-Relaxation** 29
- 3.1. Basic Version of the Main Results . . . . . . . . . . . . . . . . . . . . . . . 31
- 3.2. Strictly Quasiconcave Polynomials . . . . . . . . . . . . . . . . . . . . . . . 38
- 3.3. Extension of the Boundary . . . . . . . . . . . . . . . . . . . . . . . . . . . 42
- 3.4. Short Summary . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 58

**A. Concave Functions, Convex Sets and Lagrange Multipliers** 61
- A.1. Concave and Quasiconcave Functions . . . . . . . . . . . . . . . . . . . . . 61
- A.2. Convex Sets . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 64
- A.3. Optimisation and Lagrange Multipliers . . . . . . . . . . . . . . . . . . . . 67

**B. The Dimension of Semi-algebraic Sets and the Real Spectrum** 73

**C. Basics of Differential Geometry and Boundaries of Convex Sets** 77
- C.1. Hypersurfaces in $\mathbb{R}^n$ and Curvature . . . . . . . . . . . . . . . . . . . . . 77
- C.2. The Boundary of Convex Sets, Support Functions and the Minkowski Functional . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 81

**Zusammenfassung auf Deutsch** 87

**Bibliography** 88

# Introduction

A spectrahedron is a set defined by a linear matrix inequality, i.e. a set of the form

$$\{(x_1,\ldots,x_n)\in\mathbb{R}^n : A_0 + x_1A_1 + \ldots + x_nA_n \textit{ is positive semi-definite}\}$$

for real symmetric matrices $A_0,\ldots,A_n$. A spectrahedron is always a convex and basic-closed semi-algebraic set. Interest in this class of semi-algebraic sets arose in optimisation when algorithms computing arbitrarily close approximations to a solution of an optimisation problem over such sets in polynomial time were found. There are also efficient optimisation algorithms for optimisation problems over the projection of a spectrahedron (cf. [VB96] for more information and further references) which need not be a spectrahedron anymore.
Although much work has been done on these matters in optimisation and much is known about spectrahedra, not much is yet known about the question of how to decide if a given set is a spectrahedron, and even less about the question of how to decide if a given set is the projection of a spectrahedron. The last is the leading question of this work. Much work on it has been done by Helton and Nie in the papers [HN09] and [HN10]. The goal of this work is to present the results of these papers in detail. Before we come back to the contents of these papers and this work, we will briefly talk about spectrahedra:
The first study of the above question for spectrahedra has been done in [HV07] where it was shown that every spectrahedron is the slice of a hyperbolicity cone and, more important and much more difficult, the converse for dimension 2 by using the theory of determinantal representations (cf. [NPS10] for details on the translation of the original statement to the one given here). In fact, it is a reasonable and still unanswered question whether the converse holds in every dimension, i.e. if every slice of a hyperbolicity cone is a spectrahedron (it is closely related to the generalisation of the Lax-conjecture to higher dimensions, cf. [LPR05]; a strong version of the Lax-conjecture has recently been proved to be false by Brändén in [Brä10]).
The fact that a spectrahedron is always a slice of a hyperbolicity cone seems to be a very restrictive condition. But more necessary conditions are known, e.g. on the faces: The faces of spectrahedra have been studied in section 2.1 of [RG95] and the main result on this gives a parametrisation of the faces of a spectrahedron in terms of the linear matrix inequality defining it. A nice geometric consequence of this parametrisation is that all faces of a spectrahedron are exposed, i.e. cut out by a hyperplane (cf. [RG95], Corollary 1). This condition can often be easily verified in concrete examples. Unfortunately (or fortunately), it fails in general for projections of spectrahedra.
The only known general properties of projections of spectrahedra are the facts that they

are convex (which is a direct consequence of the convexity of spectrahedra) and semi-algebraic (which follows from quantifier elimination, cf. [BCR98], Theorem 2.2.1). The convexity can be used to get another necessary condition (cf. Theorem A.2.3) and although it is not special to projections of spectrahedra, it does compare to the sufficient conditions that we will establish in various theorems throughout this work (cf. for example Theorem 3.3.2), namely it boils down to non-negative versus positive curvature.
The question, whether a set is a (projection of a) spectrahedron has already been studied for various more or less concrete classes of sets : Sanyal, Sottile and Sturmfels have studied a class of sets called orbitopes (an orbitope is the convex hull of an orbit under the action of a compact and real algebraic group on a finite-dimensional real vector space) and found many examples of orbitopes that are (projections of) spectrahedra, cf. [SSS09]. Henrion has provided another class of examples, namely convex hulls of rational varieties (under severe restrictions on the dimension and further assumptions), cf. [Hen09] - his results have been generalised (in the case of rational curves) in [GN10].
In this work, we will focus on general basic-closed semi-algebraic sets and impose assumptions on the defining polynomials in order to make a constructive method, which has been proposed by Lasserre in [Las09], work. We will call this construction Lasserre-relaxation. (A similar construction called theta body has been proposed by Gouveia, Parrilo and Thomas in [GPT08] - it gives a representation of the closure of the convex hull of a real variety as the projection of a spectrahedron if and only if the Lasserre-relaxation also does.) The question whether or not the Lasserre-relaxation gives a representation as the projection of a spectrahedron for the convex hull of a given basic-closed semi-algebraic set can be related to a question of real algebra. Namely to the question whether or not all linear polynomials that are non-negative on the given semi-algebraic set have a representation in the preordering (or, more generally, the quadratic module) generated by the defining polynomials (cf. [PD01]) with degree bounds on the sums of squares involved in this representation. Some work on this method has previously been done, e.g. it has been proved in [NPS10] that it can only work in the case that all faces of the convex hull are again exposed. Therefore we cannot hope that this method produces a representation as the projection of a spectrahedron for every set which is the projection of a spectrahedron, because there are many examples of projections of spectrahedra with non-exposed faces, cf. [NPS10], Example 3.7.
Still, Helton and Nie have been the first to use this construction in [HN09] and [HN10] to prove quite general sufficient conditions for compact, convex and basic-closed semi-algebraic sets to be the projection of a spectrahedron. In fact, they have come to the conjecture that every convex and semi-algebraic set is the projection of a spectrahedron. We will now give a short overview of the contents and the organisation of this work:
We will start with a brief summary of the properties of spectrahedra and their projections and also prove them along the way except for the Theorem of Helton and Vinnikov, which is cited below (Theorem 1.1.7). After that, we will show that the convex hull of a finite union of (projections of) spectrahedra is the projection of a spectrahedron (cf. Theorem 1.2.7) - the proof is constructive in the sense that given spectrahedra that project down to the sets occurring in the union, we find a spectrahedron that projects down to

the convex hull. This result is a generalisation of [HN09], Theorem 2.2 and it can also be found in [NS09]. We will use a variant of this method to show that the interior of the projection of a spectrahedron is the projection of a spectrahedron. A direct application of the result about convex hulls (Theorem 1.2.7) will be the local-global principle for projections of spectrahedra in the last part of chapter 1 (Theorem 1.3.5).
In chapter 2, we consider a basic-closed semi-algebraic set

$$C = \mathscr{S}(g_1,\dots,g_r) = \{x \in \mathbb{R}^n : g_1(x) \geq 0,\dots,g_r(x) \geq 0\}$$

and we introduce the concepts of a (finitely generated) quadratic module and preordering. We consider their truncations to finite dimensional subspaces of the polynomial ring as cones and present some general facts about their duals as cones, i.e. the set of all linear functionals on the $\mathbb{R}$-vector space that are non-negative on the (primal) cone. In the second part of this chapter, we will specialise these abstract results in order to get the Lasserre-relaxation. Our main result is Theorem 2.2.4, which can also be found in [NPS10] (Proposition 3.1) and in a special case as Theorem 2 in [Las09]. It characterises the Lasserre-relaxation geometrically in terms of the linear polynomials that lie in the truncation of the quadratic module $\mathrm{QM}(g_1,\dots,g_r)$ generated by the defining polynomials of the basic-closed semi-algebraic set $C$.
Chapter 3 is devoted to the results of the works [HN09] and [HN10]. Again, we will consider a basic-closed semi-algebraic set $C = \mathscr{S}(g_1,\dots,g_r)$. Our approach to the results will be different from the original work of Helton and Nie. First, we will proof a basic version of our results, where we will show that the Lasserre-relaxation for a compact, convex and basic-closed semi-algebraic set gives a spectrahedron that projects down to the set if, roughly speaking, the defining polynomials of the set are concave (on the set) and have negative definite Hessian in the points on the boundary where they vanish, cf. Theorem 3.1.6. But in this Theorem, we will have the freedom to substitute the original defining polynomials of the set by others which define the set only locally. Then, in the two following sections, we will exploit this freedom in order to weaken the hypothesis to a condition only on the boundary (and we will also need technical assumptions on the hypersurfaces defined by the polynomials $g_i$). The most general result is Theorem 3.3.2.
The main ingredient of the proof of the basic Theorem 3.1.6 is an analogue of Putinar's Positivstellensatz (cf. [PD01], Theorem 5.3.8) for matrices ( [HN10], Theorem 29), which we will not prove in this work. The degree bounds on the sums of squares in the Matrix Positivstellensatz are essential for the following proofs because they relate to the truncations of the quadratic modules. Weakening the assumptions in the following two sections is based on reduction to the basic result 3.1.6 by more or less technical methods where the worst problems arise in the second. The technical difficulties in the first part are mainly elementary whereas in the second, key ingredients are properties of positively curved (analytic) hypersurfaces in $\mathbb{R}^n$ and their relation to convex sets (mainly presented in appendix C) as well as a smoothing procedure by an integral transformation.
In the appendices A and C, we have put together some facts, which will be used in this work, but are hard to find in the literature. Appendix B contains the proof of a Lemma

in chapter 2 (Lemma 2.1.17), which we will not give there because it uses very different methods and is not essential to the section.

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# 1. Spectrahedra and Their Projections

## 1.1. Elementary Properties and the Theorem of Helton and Vinnikov

**Definition 1.1.1.** (a) A subset $C \subset \mathbb{R}^n$ is a *spectrahedron* if there are real symmetric matrices $A_0, \ldots, A_n \in \mathrm{Sym}_{d\times d}(\mathbb{R})$ such that

$$C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \geq 0\}$$

(The notation $A \geq 0$ for a real symmetric matrix $A \in \mathrm{Sym}_{d\times d}(\mathbb{R})$ signifies that the matrix $A$ is positive semi-definite.)
We will use the notation $\mathscr{A}$ for the linear matrix polynomial $A_0 + X_1 A_1 + \ldots + X_n A_n \in \mathrm{Sym}_{d\times d}(\mathbb{R})[X_1, \ldots, X_n]$ and $\mathscr{A}_{(1)} = X_1 A_1 + \ldots + X_n A_n$ for the homogeneous part of degree one.
For the spectrahedron defined by the linear matrix polynomial $\mathscr{A}$, we will write $\mathscr{S}(\mathscr{A}) = \{x \in \mathbb{R}^n : \mathscr{A}(x) \geq 0\}$.
(b) A linear matrix polynomial $\mathscr{A} = A_0 + X_1 A_1 + \ldots + X_n A_n$ is said to be *monic* if $A_0 = I_d$ is the identity matrix.

**Remark 1.1.2** (alternative definition of a spectrahedron)**.** The above definition of a spectrahedron in $\mathbb{R}^n$ is equivalent to saying that a spectrahedron is the preimage of the cone $S_+ \subset \mathrm{Sym}_{d\times d}(\mathbb{R})$ of positive semi-definite real symmetric matrices for an affine linear map $\mathbb{R}^n \to \mathrm{Sym}_{d\times d}(\mathbb{R})$. Indeed, a linear matrix polynomial can be identified with an affine linear map $\mathbb{R}^n \to \mathrm{Sym}_{d\times d}(\mathbb{R})$.
Since the vector space $\mathrm{Sym}_{d\times d}(\mathbb{R})$ is isomorphic to the vector space $\mathrm{Bil}^{sym}(V)$ of symmetric bilinear forms on a $d$-dimensional $\mathbb{R}$-vector space $V$ (by choice of a basis of $V$), we see: Let $V$ be a finite dimensional $\mathbb{R}$-vector space. A spectrahedron is linearly isomorphic to the Cartesian product of a slice of the cone of symmetric, positive semi-definite bilinear forms on $V$ and $\mathbb{R}^m$ for a certain $m \in \mathbb{N}_0$.

**Remark 1.1.3.** (i) Every spectrahedron is convex: This is clear from the alternative definition. Using the original definition, we fix a representation $C = \mathscr{S}(\mathscr{A})$ with a linear matrix polynomial $\mathscr{A} \in \mathrm{Sym}_{d\times d}(\mathbb{R})[X_1, \ldots, X_n]$ and take $x, y \in C$ and $\lambda \in [0,1]$. Then we have $\mathscr{A}(x + \lambda(y-x)) = \mathscr{A}(x) - \lambda\mathscr{A}(x) + \lambda\mathscr{A}(y) \geq 0$ which means $x + \lambda(y-x) \in C$ (NB: $\mathscr{A}(\alpha x) = \alpha\mathscr{A}(x)$ is, of course, false in general).
(ii) Every spectrahedron is a basic closed semi-algebraic set, i.e. it is the solution set

of finitely many simultaneous polynomial inequalities (cf. [BCR98], Definition 2.7.1): A symmetric real matrix $A \in \mathrm{Sym}_{d\times d}(\mathbb{R})$ is positive semi-definite if and only if the signs of the coefficients of its characteristic polynomial $P_A(t) = \det(A - tI_d) = s_0 + s_1 t + \ldots + s_d t^d \in \mathbb{R}[t]$ are alternating: The characteristic polynomial $P_A(t)$ has no negative zero if and only if the polynomial $P_A(-t)$ has no positive zero. As we know that all zeros of $P_A(t)$ are real ($A$ is symmetric), this is the case if and only if all the coefficients of $P_A(-t)$ are non-negative.
So a spectrahedron is given as a semi-algebraic set by

$$C = \mathscr{S}(\mathscr{A}) = \{x \in \mathbb{R}^n : s_0(x) \geq 0, -s_1(x) \geq 0, \ldots, (-1)^d s_d(x) \geq 0\}$$

with the coefficients $s_i \in \mathbb{R}[X_1, \ldots, X_n]$ of

$$\det(\mathscr{A} - tI_d) = s_0 + s_1 t + \ldots + s_d t^d$$

(iii) A set $C \subset \mathbb{R}^n$ is the projection of a spectrahedron if and only if there is an $N \in \mathbb{N}$ and real symmetric matrices $A_0, \ldots, A_n, B_1, \ldots, B_N \in \mathrm{Sym}_{d\times d}(\mathbb{R})$ such that

$$C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \exists\, y_1, \ldots, y_N \in \mathbb{R}^N : A_0 + x_1 A_1 + \ldots + x_n A_n + y_1 B_1 + \ldots + y_N B_N \geq 0\}$$

Also in this case, we will write $\mathscr{A}$ for the linear matrix polynomial in the above equation and $\mathscr{A}_{(1)}$ (as in the preceding) for the homogeneous part of degree one. For the projection of a spectrahedron we will write $C = \pi_X(\mathscr{S}(\mathscr{A}(X,Y)))$ if $C$ is given as in the above equation.

**Examples 1.1.4.** (i) Every polyhedron is a spectrahedron. Namely, if $P = \{x \in \mathbb{R}^n : \ell_1(x) \geq 0, \ldots, \ell_r(x) \geq 0\}$, then we can also write $P$ as a spectrahedron

$$P = \left\{ x \in \mathbb{R}^n : \begin{pmatrix} \ell_1(x) & & 0 \\ & \ddots & \\ 0 & & \ell_r(x) \end{pmatrix} \geq 0 \right\}$$

(ii) The unit disc $\mathrm{D}^n \subset \mathbb{R}^n$ is a spectrahedron. A representation is given by

$$\mathrm{D}^n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \begin{pmatrix} 1 & & & x_1 \\ & \ddots & & \vdots \\ & & 1 & x_n \\ x_1 & \ldots & x_n & 1 \end{pmatrix} \geq 0 \right\}$$

This can be seen by calculating the determinant of the defining matrix polynomial which happens to be $1 - \sum_{i=1}^n X_i^2$, i.e. the equation of the disc. In the case $n = 2$, there is a representation of smaller size

$$\mathrm{D}^2 = \left\{ (x,y) \in \mathbb{R}^2 : I_2 + x \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

The deepest result on spectrahedra, known so far, is a theorem by Helton and Vinnikov. We will cite it after defining some notions necessary to state it.

**Definition 1.1.5.** (a) A closed set $C \subset \mathbb{R}^n$ is an *algebraic interior* if there is a polynomial $p \in \mathbb{R}[X_1,\dots,X_n]$ such that $C$ is the closure of a connected component of the set $\{x \in \mathbb{R}^n : p(x) > 0\}$.
(b) Let $C$ be an algebraic interior. A polynomial $p$ of minimal degree such that $C$ is a connected component of $\{x \in \mathbb{R}^n : p(x) > 0\}$ is said to be a *minimal defining polynomial* for $C$. As is shown in [HV07], Lemma 2.1, it is (up to multiplication by a real positive constant) uniquely determined. We define the degree of $C$ to be the degree of such a minimal defining polynomial.
(c) An algebraic interior of degree $d$ with minimal defining polynomial $p \in \mathbb{R}[X_1,\dots,X_n]$ is called *rigidly convex* if every line through an interior point of $C$ intersects the real, algebraic and projective hypersurface $\{x \in \mathbb{P}^n(\mathbb{R}) : p^*(x) = 0\}$ (where $p^*$ is the homogenisation of $p$) in $d$ points (when counting multiplicities). This is true if and only if the polynomial in one variable $t$ given by $p(tx + x_0)$, where $x_0$ is an interior point of $C$, has only real zeros for all $x \in \mathbb{R}^n$ (this is shown in the proof of [HV07], Theorem 3.1).

**Remark 1.1.6.** It is known that every rigidly convex set is convex (cf. [HV07], Section 5.3, Topological Property (3)). Further, every rigidly convex set is basic closed semialgebraic and the defining polynomials can be explicitly constructed from the minimal defining polynomial of the rigidly convex set as an algebraic interior by taking the so-called Renegar-derivatives (cf. [NPS10], Remark 2.6). In this paper, the authors also prove that all faces of a rigidly convex set are exposed (cf. [NPS10], Corollary 2.5).

**Theorem 1.1.7** ( [HV07], Theorem 3.1)**.** *Let $C \subset \mathbb{R}^n$ be a spectrahedron. Then $C$ is a rigidly convex, algebraic interior.*
*If $n = 2$, then the converse is also true: Every rigidly convex, algebraic interior $C \subset \mathbb{R}^2$ is a spectrahedron. More precisely, if $d$ is the degree of the algebraic interior $C \subset \mathbb{R}^2$, then there is a linear matrix polynomial $\mathscr{A} \in \mathrm{Sym}_{d\times d}(\mathbb{R})[X_1,\dots,X_n]$ such that $C = \{x \in \mathbb{R}^2 : \mathscr{A}(x) \geq 0\}$.*

**Remark 1.1.8.** (i) The minimal defining polynomial of $C$ is a factor of the determinant of the linear matrix polynomial $\mathscr{A}$ defining $C$. With the help of the multilinearity of the determinant and the fact that all eigenvalues of a symmetric matrix are real, it is not very difficult to see that all zeros of $f_x(t) := \det(\mathscr{A}(tx + x_0))$ (where $x_0$ is an interior point of $C$) in $t$ are real for all $x \in \mathbb{R}^n$. This property remains true for every factor of $f_x$. This is the idea of proof for the general direction of the above theorem. (cf. the proof of [HV07], Theorem 2.2)
(ii) The converse in the special case of dimension 2 is much more difficult to see and uses the theory of determinantal representations. It is a crucial step in the proof of the Lax-Conjecture (cf. [LPR05]). The question of the converse in higher dimensions is reasonable because all known necessary conditions on spectrahedra are met by rigidly convex sets (cf. Remark 1.1.6), but it remains open. A strong version of the generalisation of the Lax-conjecture to higher dimensions is false, as was recently shown by Brändén ( [Brä10]).

Now we will turn our attention to the connection between the topological interior of a spectrahedron and a defining linear matrix polynomial.

**Lemma 1.1.9** (cf. the proof of [HV07], Theorem 2.2)**.** *Let $C = \mathscr{S}(\mathscr{A})$ be a spectrahedron. The set of points, where the linear matrix inequality strictly holds, is a subset of the topological interior of $C$.*
*If the linear matrix polynomial describing $C$ is monic, then the converse will also be true:*

$$\operatorname{int}(C) = \{x \in \mathbb{R}^n : \mathscr{A}(x) > 0\}$$

PROOF. The inclusion $\{x \in \mathbb{R}^n : \mathscr{A}(x) > 0\} \subset \operatorname{int}(C)$ is easy to see because the eigenvalues of $\mathscr{A}(x)$ are continuous functions of $x$.
So let the matrix polynomial describing $C$ be monic. Then we have $0 \in C$. Now let $x \in \operatorname{int}(C)$ be an interior point of $C$. There is a constant $\varepsilon > 0$ such that $(1+\varepsilon)x \in C$. Let $\gamma\colon [0,1] \to \mathbb{R}^n$, $t \mapsto t(1+\varepsilon)x$, be a parametrisation of the line segment between 0 and the point $(1+\varepsilon)x$. As $C$ is convex, we have $\gamma([0,1]) \subset C$ and therefore we have $I_d + t((1+\varepsilon)x_1A_1 + \ldots + (1+\varepsilon)x_nA_n) = \mathscr{A}(\gamma(t)) > 0$ for all $t \in [0,1)$ which we see by diagonalising the matrix $((1+\varepsilon)x_1A_1 + \ldots + (1+\varepsilon)x_nA_n)$; so in particular we have $\mathscr{A}(x) > 0$. □

It is not a restriction to consider only monic linear matrix polynomials in the above lemma, because every spectrahedron with non-empty interior can be translated in such a way that it can be defined by a monic linear matrix polynomial:

**Proposition 1.1.10** ( [HV07], Lemma 2.3)**.** *Let $C = \mathscr{S}(\mathscr{A})$ be a spectrahedron and assume that* 0 *is an interior point of $C$. Then there are real symmetric matrices $A'_1, \ldots, A'_n \in \operatorname{Sym}_{d'\times d'}(\mathbb{R})$ such that*

$$C = \{x \in \mathbb{R}^n : I_{d'} + x_1A'_1 + \ldots + x_nA'_n \geq 0\}$$

PROOF. Write $\mathscr{A} = A_0 + X_1A_1 + \ldots + X_nA_n$. Since we have $0 \in \operatorname{int}(C)$, the matrix $A_0$ is positive semi-definite and there are $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that the matrices $A_0 + \varepsilon_1A_1, A_0 - \varepsilon_1A_1, \ldots, A_0 + \varepsilon_nA_n, A_0 - \varepsilon_nA_n$ are positive semi-definite. We write $f_i$ for the endomorphism on $\mathbb{R}^d$ defined by $A_i$ after choice of the canonical basis. The vector space $\mathbb{R}^d$ is the orthogonal sum of $\ker(f_0)$ and $\operatorname{im}(f_0)$ because $A_0$ is symmetric. The subspace $U := \operatorname{im}(f_0)$ is $f_0$-invariant and the endomorphism $\widetilde{f_0} := f_0|_U\colon U \to U$ is bijective. We know that all eigenvalues of $\widetilde{f_0}$ are real and positive. Now we will show $\operatorname{im}(f_i) \subset \operatorname{im}(f_0)$ for all $i = 1, \ldots, n$: Fix $i \in \{1, \ldots, n\}$ and take $x \in \ker(f_0) = \operatorname{im}(f_0)^\perp$. We have $x^t(A_0 \pm \varepsilon_iA_i)x \geq 0$ and using $x^tA_0x = 0$, we deduce that $x^tA_ix = 0$. Now we have $x^t(A_0 + \varepsilon_iA_i)x = 0$ and $A_0 + \varepsilon_iA_i \geq 0$. By factorising the matrix $A_0 + \varepsilon_iA_i$ in $D^tD$ for some matrix $D \in \mathrm{M}_{d\times d}(\mathbb{R})$ we get $(A_0 + \varepsilon_iA_i)x = 0$. With $A_0x = 0$ it follows that $A_ix = 0$, which says $x \in \ker(f_i) = \operatorname{im}(f_i)^\perp$. Thus we have $\operatorname{im}(f_i) \subset \operatorname{im}(f_0)$ as desired.
The restrictions of the endomorphisms $f_i$ to $U$ are self-adjoint endomorphisms of $U$ and thus a matrix $\widetilde{A_i} \in \mathrm{M}_{d'\times d'}(\mathbb{R})$ (where $d'$ denotes the dimension of $U$) representing $f_i$ is symmetric. We have $C = \mathscr{S}(\widetilde{\mathscr{A}})$ for the linear matrix polynomial $\widetilde{\mathscr{A}} = \widetilde{A_0} + X_1\widetilde{A_1} + \ldots + X_n\widetilde{A_n}$ because $f_0 + x_1f_1 + \ldots + x_nf_n$ is positive semi-definite if and only if

$\widetilde{f_0} + x_1\widetilde{f_1} + \ldots + x_n\widetilde{f_n}$ is positive semi-definite.
We now factorise $\widetilde{A_0}$ to $\widetilde{A_0} = B^tB$ where $B$ is an invertible matrix $B \in \mathrm{GL}(d', \mathbb{R})$ and put $A'_i := B^{-t}\widetilde{A_i}B^{-1}$. Then we have $C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : I_{d'} + x_1A'_1 + \ldots + x_nA'_n \geq 0\}$. □

We will prove an analogous result (to Lemma 1.1.9) about the topological interior of projections of spectrahedra. We will need the following lemma in the proof.

**Proposition 1.1.11.** *Let $V, W$ be finite dimensional $\mathbb{R}$-vector spaces, let $\pi : V \to W$ be a linear map and let $C \subset V$ be a convex set. Denote the relative interior of a convex set $C$, i.e. the interior in its affine hull, by* relint$(C)$*. Then the image of the relative interior is the relative interior of the image:*

$$\pi(\mathrm{relint}(C)) = \mathrm{relint}(\pi(C))$$

Proof. Let $U$ be the affine hull of $C$. After an affine transformation, we can assume that $U$ is a subspace, i.e. contains $0 \in V$ and thus we can look at the linear map $\pi|_U : U \to \pi(U)$. We see that we can assume without loss of generality that $C$ has non-empty interior in $V$ and the map $\pi$ is surjective.
The set $\pi(\mathrm{int}(C)) \subset \pi(C)$ is open because $\pi$ is an open map (after adequate change of coordinates, $\pi$ is a projection on the first factor $\mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$). This gives the inclusion $\pi(\mathrm{int}(C)) \subset \mathrm{int}(\pi(C))$.
Now let $y \in \pi(C)$ be a point of the image of $C$ and let $x \in C$ be a point such that $\pi(x) = y$. Assume that $x + v \notin \mathrm{int}(C)$ for all $v \in \ker(\pi)$. By [Bou81], Chapter II, §5, Théorème 1, there is an affine hyperplane $H$ supporting $C$ such that $x \in H \cap C$ and $\ker(\pi) \subset H$. Therefore, as $\pi$ is surjective, $\pi(H)$ is an affine hyperplane supporting $\pi(C)$ such that $y = \pi(x) \in \pi(H \cap C) \subset \pi(H) \cap \pi(C)$, i.e. $y \notin \mathrm{int}(\pi(C))$. □

**Corollary 1.1.12.** *Let $C = \pi_X(\mathscr{S}(\mathscr{A}(X,Y))) = \{x \in \mathbb{R}^n : \exists\, y \in \mathbb{R}^N : \mathscr{A}(x,y) \geq 0\}$ be the projection of a spectrahedron which is defined by a monic linear matrix polynomial. Then the topological interior of $C$ is the set of all points in which the matrix inequality is strictly satisfied, i.e.*

$$\mathrm{int}(C) = \{x \in \mathbb{R}^n : \exists\, y \in \mathbb{R}^N : \mathscr{A}(x,y) > 0\}$$

Proof. This follows from Lemma 1.1.9 by Proposition 1.1.11 because the projection $\pi : \mathbb{R}^{n+N} \to \mathbb{R}^n$, $(x_1, \ldots, x_n, y_1, \ldots, y_N) \mapsto (x_1, \ldots, x_n)$ is a linear map. □

As another elementary property of (projections of) spectrahedra, we will now study their behaviour under set operations.

**Lemma 1.1.13.** *(i) The intersection of two spectrahedra is a spectrahedron. The same is true for projections of spectrahedra.*
*(ii) The Minkowski-sum (i.e. the sum taken element-wise) of two spectrahedra is the projection of a spectrahedron. Again, we also have that the Minkowski-sum of two projections of spectrahedra is again a projection of a spectrahedron.*

PROOF. Let $C = \mathscr{S}(\mathscr{A})$ and $C' = \mathscr{S}(\mathscr{A}')$ be two spectrahedra. (i) Then we have $C \cap C' = \mathscr{S}(\mathscr{A} \boxplus \mathscr{A}')$ where we write $\mathscr{A} \boxplus \mathscr{A}'$ for the matrix polynomial $A_0 \boxplus A_0' + X_1 A_1 \boxplus A_1' + \ldots + X_n A_n \boxplus A_n'$ (we write $\mathscr{A} = A_0 + X_1 A_1 + \ldots + X_n A_n$ and analogously $\mathscr{A}' = A_0' + X_1 A_1' + \ldots + X_n A_n'$) made of block matrices

$$A \boxplus A' = \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}$$

In the case where $C = \{x \in \mathbb{R}^n : \exists\, u \in \mathbb{R}^N : \mathscr{A}(x,u) \geq 0\}$ and $C' = \{x \in \mathbb{R}^n : \exists u' \in \mathbb{R}^{N'} : \mathscr{A}'(x,u') \geq 0\}$ are projections of spectrahedra, we assume $N = N'$ (by filling in zero matrices in the shorter matrix inequality).
If we write $\mathscr{A} = A_0 + X_1 A_1 + \ldots + X_n A_n + Y_1 B_1 + \ldots + Y_N B_N$ (again, analogously for $\mathscr{A}'$), then the intersection $C \cap C'$ is represented as $\{x \in \mathbb{R}^n : \exists (u,u') \in \mathbb{R}^N \times \mathbb{R}^N : \mathscr{A}(x,u) \boxplus \mathscr{A}(x,u') \geq 0\}$ where

$$\begin{aligned} \mathscr{A} \boxplus \mathscr{A}' &= A_0 \boxplus A_0' + (X_1 A_1) \boxplus (X_1 A_1') + \ldots + (X_n A_n) \boxplus (X_n A_n') \\ &+ (Y_1 B_1) \boxplus 0 + \ldots + (Y_N B_N) \boxplus 0 + 0 \boxplus (Y_1' B_1') + 0 \boxplus (Y_N' B_N') \end{aligned}$$

Of course $(X_i A_i) \boxplus (X_i A_i') = X_i (A_i \boxplus A_i')$.
As for part (ii), we look at $C \times \mathbb{R}^n = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \mathscr{A}(x) \geq 0\}$ which is clearly a spectrahedron in $\mathbb{R}^{2n}$ and analogously $\mathbb{R}^n \times C'$. Then we know by part (i) that $C \times C' = (C \times \mathbb{R}^n) \cap (\mathbb{R}^n \times C')$ is a spectrahedron in $\mathbb{R}^{2n}$. And so $C + C' = \pi(C \times C')$ for the projection $\pi \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(x,y) \mapsto x + y$. The argument for projections of spectrahedra is analogous. □

**Examples 1.1.14.** (i) We take $C = [0,1] \times \{0\} \subset \mathbb{R}^2$ and $C' = \mathrm{D}^2 = \{(x,y) \in \mathbb{R}^2 : 1 - x^2 - y^2 \geq$

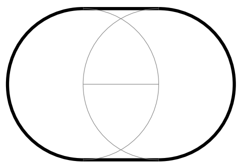

Figure 1.1.: The figure shows the set $C + C' = \mathrm{D}^2 \cup [0,1] \cup (\mathrm{D}^2 + (1,0))$ enclosed by the thick black lines and the sets $C$, $\mathrm{D}^2$ and $\mathrm{D}^2 + (1,0)$.

$0\} \subset \mathbb{R}^2$, which are spectrahedra. Then their Minkowski-sum $C + C' = [0,1] \times [-1,1] \cup \mathrm{D}^2 \cup (\mathrm{D}^2 + (1,0))$ is not a spectrahedron, because it is known not to be a basic closed semi-algebraic set. (cf. [BCR98], Example 10.4.5(a)).
(ii) Clearly, the class of spectrahedra cannot be closed under the operation of union of sets, because the class of convex sets is not. But even if the union of two spectrahedra is again convex, it need not be a spectrahedron again:
Take $C = [-1,0] \times [-1,1] \subset \mathbb{R}^2$ and again $C' = \mathrm{D}^2$. Their union $C \cup C'$ is not a spectrahedron by [RG95], Corollary 1, because the faces $\{(0,1)\}$ and $\{(0,-1)\}$ are not exposed. We will see in the following section that it is the projection of a spectrahedron.

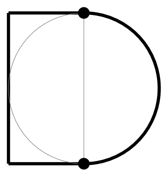

Figure 1.2.: The set $C \cup C' = [-1,0] \times [-1,1] \cup D^2$ and its non-exposed faces, marked by black dots.

## 1.2. Convex Hulls of the Union of Spectrahedra

In this section, we will prove constructively that the convex hull of a finite union of projections of spectrahedra is again the projection of a spectrahedron. We will also use this technique to show that the interior of a projection of a spectrahedron is again the projection of a spectrahedron. We start with an elementary lemma:

**Proposition 1.2.1.** *Let $C_1, \dots, C_m \subset \mathbb{R}^n$ be non-empty, convex sets. Denote by $\Delta_{m-1} = \{(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m_{\geq 0} : \lambda_1 + \ldots + \lambda_m = 1\}$ the standard simplex. Then we have*

$$\operatorname{conv}\left(\bigcup_{i=1}^m C_i\right) = \bigcup_{(\lambda_1,\dots,\lambda_m)\in\Delta_{m-1}} (\lambda_1 C_1 + \ldots + \lambda_m C_m)$$

This is a special case of [Roc70], Theorem 3.3 (there, arbitrary, not necessarily finite, families of convex sets are discussed). As the proof is very elementary and easy, we will give it here.

Proof. Take $x = \sum_{i=1}^m \sum_{j=1}^{l_i} \lambda_{ij} x_{ij}$ where $\lambda_{ij} \geq 0$, $\sum_{i=1}^m \sum_{j=1}^{l_i} \lambda_{ij} = 1$ and $x_{ij} \in C_i$ for all $j = 1, \dots, l_i$. For all $i = 1, \dots, m$ set $\lambda_i = \sum_{j=1}^{l_i} \lambda_{ij}$. Then we have because of the convexity of $C_i$ that the point $\frac{1}{\lambda_i} \sum_{j=1}^{l_i} \lambda_{ij} x_{ij} =: x_i$ is in $C_i$ if $\lambda_i \neq 0$. If $\lambda_i = 0$, then take any point $x_i \in C_i$. Thus we get $x = \lambda_1 x_1 + \ldots + \lambda_m x_m$ and $(\lambda_1, \dots, \lambda_m) \in \Delta_{m-1}$ which gives one inclusion. The other one is obvious. □

This Lemma motivates the following definition (as will become clear later, cf. Lemma 1.2.4 and Remark 1.2.5).

**Definition 1.2.2.** (a) Let $C = \pi_X(\mathscr{S}(A_0 + \mathscr{A}_{(1)}(X,Y)))$ be the projection of a spectrahedron. Define for $\lambda \in \mathbb{R}$ the linear matrix polynomial

$$\mathscr{A}_{\boxplus}(\lambda, X, Y, Z) = \left(\lambda A_0 + \mathscr{A}_{(1)}(X,Y)\right) \boxplus \begin{pmatrix} \lambda & X_1 \\ X_1 & Z \end{pmatrix} \boxplus \ldots \boxplus \begin{pmatrix} \lambda & X_n \\ X_n & Z \end{pmatrix}$$

and thereby the projection of a spectrahedron

$$\widehat{C} = \{(x,\lambda) \in \mathbb{R}^{n+1} : \exists\, (y,z) \in \mathbb{R}^{N+1}\ \mathscr{A}_{\boxplus}(\lambda, x, y, z) \geq 0\} =: \pi_{(X,\lambda)}(\mathscr{S}(\mathscr{A}_{\boxplus}(\lambda, X, Y, Z)))$$

(b) Let $C_1,\dots,C_m \subset \mathbb{R}^n$ be projections of spectrahedra. We set

$$\Delta_{m-1}^m = \{(x^1,\lambda_1,\dots,x^m,\lambda_m) \in \mathbb{R}^{nm+m} : (\lambda_1,\dots,\lambda_m) \in \Delta_{m-1}\}$$

Writing $\pi\colon (\mathbb{R}^n \times \mathbb{R}) \times \ldots \times (\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}^n$, $((x^1,\lambda_1),\dots,(x^m,\lambda_m)) \mapsto \sum_{i=1}^m x^i$, we define the set

$$\mathscr{C}(C_1,\dots,C_m) = \pi\left((\widehat{C_1} \times \ldots \times \widehat{C_m}) \cap \Delta_{m-1}^m\right)$$

**Remark 1.2.3.** (i) The set $\mathscr{C}(C_1,\dots,C_m)$ in the above definition is by construction the projection of a spectrahedron (cf. Lemma 1.1.13(i)).
(ii) Write

$$\widetilde{C} = \{(x,\lambda) \in \mathbb{R}^{n+1} : \exists\, y \in \mathbb{R}^N\ \lambda A_0 + \mathscr{A}_{(1)}(x,y) \geq 0\}$$

If we set $\mathscr{C}(\mathscr{A}^1,\dots,\mathscr{A}^m) = \pi((\widetilde{C_1} \times \ldots \times \widetilde{C_m}) \cap \Delta_{m-1}^m)$, then this definition will agree with the definition of $\mathscr{C}$ in [HN09], Theorem 2.2. The definition given there is the following

$$\begin{aligned}\mathscr{C} \quad = \quad & \{x \in \mathbb{R}^n : \forall\, k \in \{1,\dots,m\}\ \exists\, x^k \in \mathbb{R}^n\ \exists y^k \in \mathbb{R}^{N_k}\ \exists\, \lambda \in \Delta_{m-1} \text{ such that} \\ & \lambda_k A_0^k + \sum_{i=1}^n x_i^k A_i^k + \sum_{j=1}^{N_k} y_j^k B_j^k \geq 0 \text{ and } x = \sum_{k=1}^m x^k\}\end{aligned}$$

The above construction of $\mathscr{C}(C_1,\dots,C_m)$ is a generalisation of this and allows us to prove a stronger version of [HN09], Theorem 2.2 (cf. Theorem 1.2.7).

As already promised, the following lemma will cast some more light on Definition 1.2.2.

**Lemma 1.2.4.** *Let $\emptyset \neq C = \pi_X(\mathscr{S}(\mathscr{A}(X,Y)))$ be the projection of a spectrahedron. Put $C^0 = \pi_X(\mathscr{S}(\mathscr{A}_{(1)}(X,Y)))$.*
*(i) For all $\lambda > 0$ the equation $\lambda C = \pi_X(\mathscr{S}(\lambda A_0 + \mathscr{A}_{(1)}(X,Y)))$ holds.*
*(ii) We have $C^0 \subset \operatorname{cl}(\bigcup_{0<\lambda\leq 1} \lambda C)$.*
*(iii) If $C$ is bounded, then we have $C^0 = \{0\}$.*
*(iv) For all $\lambda \geq 0$ we have*

$$\lambda C = \{x \in \mathbb{R}^n : \exists\,(y,z) \in \mathbb{R}^{N+1}\ \mathscr{A}_{\boxplus}(\lambda,x,y,z) \geq 0\} = \pi_X(\mathscr{S}(\mathscr{A}_{\boxplus}(\lambda,X,Y,Z)))$$

*In the case $\lambda < 0$, the set $\mathscr{S}(\mathscr{A}_{\boxplus}(\lambda,X,Y,Z)) \subset \mathbb{R}^{n+N+2}$ is empty.*

Proof. (i) Let $\lambda > 0$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^N$ be such that $\lambda A_0 + \mathscr{A}_{(1)}(x,y) \geq 0$. We have $\lambda\left(A_0 + \mathscr{A}_{(1)}(\frac{1}{\lambda}x, \frac{1}{\lambda}y)\right) \geq 0$, which implies $\frac{1}{\lambda}x \in C$, i.e. $x \in \lambda C$. Conversely, let $x \in \lambda C$. Then there is a $x' \in C$ such that $x = \lambda x'$. Therefore there exists a $y' \in \mathbb{R}^N$ such that $A_0 + \mathscr{A}_{(1)}(x',y') \geq 0$. We deduce

$$0 \leq \lambda\left(A_0 + \mathscr{A}_{(1)}(x',y')\right) = \lambda A_0 + \mathscr{A}_{(1)}(x,\lambda y')$$

(ii) Let $x \in C^0$ and let $y$ be in $\mathbb{R}^N$ such that $\mathscr{A}_{(1)}(x,y) \geq 0$. Since $C \neq \emptyset$, there is a $x' \in C$ and a $y' \in \mathbb{R}^N$ such that $A_0 + \mathscr{A}_{(1)}(x',y') \geq 0$. For all $\lambda \in (0,1]$ we have

$$0 \leq \lambda\left(A_0 + \mathscr{A}_{(1)}(x',y')\right) + \mathscr{A}_{(1)}(x,y) = \lambda A_0 + \mathscr{A}_{(1)}(x + \lambda x', y + \lambda y')$$

Using part (i) we get $x+\lambda x' \in \lambda C$. The claim now follows from the fact that $x+\lambda x'$ converges to $x$ for $\lambda \to 0$.
(iii) If $C$ is bounded, then so is $\mathrm{cl}(\bigcup_{0<\lambda\le 1} \lambda C)$ and therefore $C^0$ is bounded (using part (ii)). But obviously $C^0$ is a closed, convex cone and so it must be equal to $\{0\}$.
(iv) Recall

$$\mathscr{A}_{\boxplus}(\lambda,X,Y,Z) = (\lambda A_0 + \mathscr{A}_{(1)}(X,Y)) \boxplus \begin{pmatrix} \lambda & X_1 \\ X_1 & Z \end{pmatrix} \boxplus \ldots \boxplus \begin{pmatrix} \lambda & X_n \\ X_n & Z \end{pmatrix}$$

In the case of $\lambda > 0$, the proof of this part is analogous to the proof of part (i), because the additional condition

$$\exists\, z \in \mathbb{R} \text{ such that } \begin{pmatrix} \lambda & x_i \\ x_i & z \end{pmatrix} \succeq 0$$

is true for all $i \in \{1,\ldots,n\}$ if one chooses a sufficiently large $z \in \mathbb{R}_{\ge 0}$ (we need $\lambda z - x_i^2 \ge 0$). In the case of $\lambda = 0$, this condition holds if and only if $x_i = 0$, which implies the first part of the claim. In the case $\lambda < 0$, the above condition cannot hold for any $z \in \mathbb{R}$. □

**Remark 1.2.5.** (i) We see from the above Lemma that the equality $\mathscr{C}(C_1,\ldots,C_m) = \mathscr{C}(\mathscr{A}^1,\ldots,\mathscr{A}^m)$ holds if all the sets $C_1,\ldots,C_m$ are bounded. Namely, it follows directly from claims (i), (iii) and (iv) that even the equality of spectrahedra

$$(\widehat{C_1} \times \ldots \times \widehat{C_m}) \cap \Delta_{m-1}^m = \bigcup_{\lambda \in \Delta_{m-1}} (\lambda_1 C_1 \times \{\lambda_1\}) \times \ldots \times (\lambda_m C_m \times \{\lambda_m\}) = (\widetilde{C_1} \times \ldots \times \widetilde{C_m}) \cap \Delta_{m-1}^m$$

holds. This equality is stronger than the stated equality of sets (both sets are by definition the image of this spectrahedron under the projection $\pi\colon (\mathbb{R}^n \times \mathbb{R}) \times \ldots \times (\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}^n$, $((x^1,\lambda_1),\ldots,(x^m,\lambda_m)) \mapsto \sum_{i=1}^m x^i$).
(ii) Since the equality

$$(\widehat{C_1} \times \ldots \times \widehat{C_m}) \cap \Delta_{m-1}^m = \bigcup_{\lambda \in \Delta_{m-1}} (\lambda_1 C_1 \times \{\lambda_1\}) \times \ldots \times (\lambda_m C_m \times \{\lambda_m\})$$

holds without any further assumptions on the projections of spectrahedra $C_1,\ldots,C_m$, we see that this set does not depend on the chosen representations for the sets $C_1,\ldots,C_m$. In particular, the set $\mathscr{C}(C_1,\ldots,C_m)$ does not depend on the chosen representations.
(iii) If one of the sets $C_1,\ldots,C_m$ is unbounded, the set $\mathscr{C}(\mathscr{A}^1,\ldots,\mathscr{A}^m)$ may depend on the chosen representations $\mathscr{A}^1,\ldots,\mathscr{A}^m$ for the sets $C_1,\ldots,C_m$, as the following example shows.

**Example 1.2.6.** We set $C_1 = \{(0,1)\}$ and $C_2 = \{(x_1,0)\colon x_1 \in \mathbb{R}\}$ in $\mathbb{R}^2$. For the set $C_1$ we choose the following representation as the projection of a spectrahedron

$$C_1 = \{(x_1,x_2) \in \mathbb{R}^2 \colon \mathscr{A}^1(x_1,x_2) = \mathrm{diag}(x_1,-x_1,x_2-1,-x_2+1) \succeq 0\}$$

and for $C_2$ we choose two different representations

$$\begin{aligned} C_2 &= \{(x_1,x_2) \in \mathbb{R}^2 \colon \mathscr{A}^2(x_1,x_2) = \mathrm{diag}(x_2,-x_2) \succeq 0\} \\ C_2 &= \left\{ (x_1,x_2) \in \mathbb{R}^2 \colon \exists\, y \in \mathbb{R}\ \mathscr{B}^2(x_1,x_2,y) = \begin{pmatrix} 1 & x_1 & 0 & 0 \\ x_1 & y & 0 & 0 \\ 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & -x_2 \end{pmatrix} \succeq 0 \right\} \end{aligned}$$

By Lemma 1.2.4(i) we see that the sets $\mathscr{C}(\mathscr{A}^1,\mathscr{A}^2)$ and $\mathscr{C}(\mathscr{A}^1,\mathscr{B}^2)$ depend only on what the sets $\pi_X(\mathscr{S}(\mathscr{A}^2_{(1)}(X,Y)))$ and $\pi_X(\mathscr{S}(\mathscr{B}^2_{(1)}(X,Y))$ are (cf. Remark 1.2.5(i) and/or further below in this remark). And indeed we get different results for these two representations:
Namely, the set $\mathscr{C}(\mathscr{A}^1,\mathscr{A}^2) = \{(x_1,x_2) \in \mathbb{R}^2 \colon x_1 \in \mathbb{R}, x_2 \in [0,1]\}$ is a closed strip, and

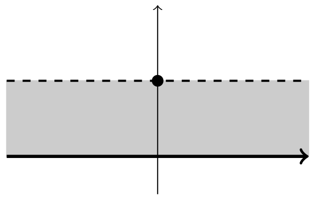

Figure 1.3.: The set $C^1$ is the thick point and the set $C^2$ the thick line. The dashed black line is contained in $\mathscr{C}(\mathscr{A}^1,\mathscr{A}^2)$ and not contained in $\mathscr{C}(\mathscr{A}^1,\mathscr{B}^2)$.

$\mathscr{C}(\mathscr{A}^1,\mathscr{B}^2) = \{(x_1,x_2) \in \mathbb{R}^2 \colon x_1 \in \mathbb{R}, x_2 \in [0,1)\} \cup \{(0,1)\}$ is the convex hull of the union of the two sets.
The reason for this difference lies in the fact that $\pi_X(\mathscr{S}(\mathscr{A}^2_{(1)}(X,Y))) = \mathbb{R}\times\{0\}$, because the matrix polynomial $\mathscr{A}^2 = \mathscr{A}^2_{(1)}$ does not have a constant term. On the other hand, $\pi_X(\mathscr{S}(\mathscr{B}^2_{(1)}(X,Y)) = \{(0,0)\}$ because the upper left $2\times 2$ matrix gives the condition $-x_1^2 \geq 0$ (i.e. $x_1 = 0$) if we omit the constant term in the matrix polynomial $\mathscr{B}^2$.
The set $\mathscr{C}(\mathscr{A}^1,\ldots,\mathscr{A}^m)$ always depends on the set $C_i^0$, i.e. on the behaviour of $\lambda\mathscr{A}^i$ as $\lambda$ approaches zero. This is where the definition of $\mathscr{C}(C_1,\ldots,C_m)$ (in part (b) of definition 1.2.2) improves the definition $\mathscr{C}(\mathscr{A}^1,\ldots,\mathscr{A}^m)$.

We have finished with the technical preliminaries and now come to the main result of this section. We will now constructively prove that the convex hull of a union of (projections of) spectrahedra is the projection of a spectrahedron:

**Theorem 1.2.7** (cf. [HN09], Theorem 2.2)**.** *Let $C_1,\ldots,C_m \subset \mathbb{R}^n$ be non-empty projections of spectrahedra. Then we have*

$$\operatorname{conv}\left(\bigcup_{i=1}^m C_i\right) = \mathscr{C}(C_1,\ldots,C_m)$$

*In particular, the convex hull of the union of the sets $C_i$ is again the projection of a spectrahedron.*

Proof. Proposition 1.2.1 states

$$\operatorname{conv}\left(\bigcup_{i=1}^m C_i\right) = \bigcup_{(\lambda_1,\ldots,\lambda_m)\in\Delta_{m-1}} (\lambda_1 C_1 + \ldots + \lambda_m C_m)$$

By Lemma 1.2.4(iv), we know that

$$\lambda C_i = \pi_X(\mathscr{S}(\mathscr{A}_{\boxplus}(\lambda,X,Y,Z)))$$

for all $\lambda \geq 0$. Therefore the claim follows directly from the definition of $\mathscr{C}(C_1,\ldots,C_m)$ (cf. Definition 1.2.2). □

**Remark 1.2.8.** (i) The set $\mathscr{C}(C_1,\dots,C_m)$ has an explicit representation as the projection of a spectrahedron, which can be easily computed (given representations for $C_1,\dots,C_m$). Namely, given two projections of spectrahedra $C_i = \pi_X(\mathscr{S}(\mathscr{A}^i(X,Y)))$ for $i = 1,2$, we write $\mathscr{A}^i_{\boxplus}$ for the matrix polynomials in $\mathrm{Sym}_{(d+2n)\times(d+2n)}(\mathbb{R})[\lambda,X_1,\dots,Y_N,Z]$, which describe the sets $\widehat{C_i}$, as in Definition 1.2.2(a). Then the Cartesian product $\widehat{C_1}\times\widehat{C_2}$ is $\pi_{(X,X')}(\mathscr{S}(\mathscr{B}(\lambda,\lambda',X,X',Y,Y',Z,Z')))$, where

$$\mathscr{B} = \mathscr{A}^1(\lambda,X,Y,Z)\boxplus 0 + 0\boxplus\mathscr{A}^2(\lambda',X',Y',Z')$$

which is a matrix polynomial $\mathscr{B}\in\mathrm{Sym}_{2(d+2n)\times 2(d+2n)}(\mathbb{R})[\lambda,\lambda',X_1,\dots,X'_n,Y_1,\dots,Y'_N,Z,Z']$. Next, we need a representation as the projection of a spectrahedron for $\Delta^2_1\subset\mathbb{R}^{2n+2}$, i.e.

$$\Delta^2_1 = \{(x^1,\lambda_1,x^2,\lambda_2)\in\mathbb{R}^{2n+2} : \mathscr{D}(\lambda_1,\lambda_2) := \mathrm{diag}(\lambda_1,\lambda_2,\lambda_1+\lambda_2-1,1-\lambda_1-\lambda_2)\geq 0\}$$

Then, we know from Lemma 1.1.13(a) that the intersection of $\widehat{C_1}\times\widehat{C_2}$ and $\Delta^2_1$ is represented in the following way

$$(\widehat{C_1}\times\widehat{C_2})\cap\Delta^2_1 = \pi_{(X,X')}(\mathscr{S}(\mathscr{B}\boxplus\mathscr{D}(\lambda,\lambda',X,X',Y,Y',Z,Z',\Lambda)))$$

and $\mathscr{B}\boxplus\mathscr{D}\in\mathrm{Sym}_{2(d+2n+2)\times 2(d+2n+2)}(\mathbb{R})[\lambda,\lambda',X_1,\dots,Y'_N,Z,Z',\Lambda_1,\Lambda_2]$ (here, we have $2(n+N+3)$ indeterminates). Finally, the convex hull of $C_1\cup C_2$ is the projection of $(C_1\times C_2)\cap\Delta^2_1$ by the projection $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n$, $(x^1,\lambda_1,x^2,\lambda_2)\to x^1+x^2$ (cf. Theorem 1.2.7).
(ii) By using the construction of $\widehat{C}$, we can give a representation as the projection of a spectrahedron of the convex cone over the convex set $C$ if $C$ is itself the projection of a spectrahedron. We simply have to note that the set

$$\pi_X(\mathscr{S}(\mathscr{A}_{\boxplus}(\lambda,X,Y,Z))) = \{x\in\mathbb{R}^n : \exists\,(\lambda,y,z)\in\mathbb{R}^{N+2}\ \mathscr{A}_{\boxplus}(\lambda,x,y,z)\geq 0\}$$

is the convex cone over $C$ by part (iv) of Lemma 1.2.4.
Using this result, we can show that the convex hull of the union of two sets $C_1$ and $C_2$ which are projections of spectrahedra is the projection of a spectrahedron: To do so, we embed these two sets in $\mathbb{R}^{n+1}$ via the map $\mathbb{R}^n\to\mathbb{R}^{n+1}$, $x\mapsto(x,1)$. The images of $C_1$ and $C_2$ under this map are clearly projections of spectrahedra again. Now we take the convex cones $K_1$ over $C_1$ and $K_2$ over $C_2$. By Lemma 1.1.13 we know that $K_1+K_2$ is the projection of a spectrahedron. Now we have $(K_1+K_2)\cap\{(x,1)\in\mathbb{R}^{n+1} : x\in\mathbb{R}^n\} = \mathrm{conv}(C_1\cup C_2)\times\{1\}$ which shows that the convex hull of the union of $C_1$ and $C_2$ is the projection of a spectrahedron.

We now turn to a technical proposition which we will need to prove that the interior of (the projection of) a spectrahedron is the projection of a spectrahedron.

**Proposition 1.2.9.** *Let $C_1,\dots,C_m\subset\mathbb{R}$ be non-empty, convex sets.*
*We have*

$$\mathrm{conv}\left(\bigcup_{i=1}^m C_i\right) = \bigcup_{\lambda\in\Delta_{m-1}}(\lambda_1C_1+\ldots+\lambda_mC_m)\subset\mathrm{cl}\left(\bigcup_{\lambda\in\mathring{\Delta}_{m-1}}(\lambda_1C_1+\ldots+\lambda_mC_m)\right)$$

*In particular, the closure of the set on the left-hand side of the above equation is equal to the right-hand side.*

PROOF. The first equality is Proposition 1.2.1. To the second: Take $x = \sum_{i=1}^m \lambda_i x^i$ where $(\lambda_1,\dots,\lambda_m) \in \Delta_{m-1} \setminus \mathring{\Delta}_{m-1}$ and $x^i \in C_i$. Put $I = \{i \in \{1,\dots,m\} : \lambda_i = 0\} \neq \{1,\dots,m\}$ and take $j \notin I$. We define for $m \in \mathbb{N}$

$$y^m = \left(\lambda_j - \frac{\#I}{m}\right)x^j + \sum_{i\in I} \frac{1}{m} x^i + \sum_{i\notin I, i\neq j} \lambda_i x^i$$

For sufficiently large $m \in \mathbb{N}$ we have $\lambda_j - \frac{\#I}{m} > 0$, i.e. $y^m \in \bigcup_{\lambda\in\mathring{\Delta}_{m-1}} (\lambda_1 C_1 + \ldots + \lambda_m C_m)$. By construction we have $y^m \to x$ for $m \to \infty$. The second part of the claim is trivial. □

As a byproduct, we easily get more information, which can also be found in [HN09], Theorem 2.2, on the projection of a spectrahedron $\mathscr{C}(\mathscr{A}^1,\dots,\mathscr{A}^m)$ as defined in Remark 1.2.3. Although it depends on the chosen representations of $C_1,\dots,C_m$ as projections of spectrahedra, it always contains the convex hull of the union of the $C_i$ and cannot be bigger than its closure:

**Corollary 1.2.10** (cf. [HN09], Theorem 2.2)**.** *Let $C_1,\dots,C_m \subset \mathbb{R}^n$ be non-empty projections of spectrahedra. Then the set $\mathscr{C}(\mathscr{A}^1,\dots,\mathscr{A}^m)$ is contained in the closure of the convex hull of the union of the $C_i$ for all choices of representations $C_i = \pi_X(\mathscr{S}(A^i(X,Y)))$ of the $C_i$:*

$$\operatorname{conv}(\bigcup_{i=1}^m C_i) \subset \mathscr{C}(\mathscr{A}^1,\dots,\mathscr{A}^m) \subset \operatorname{cl}\left(\operatorname{conv}\left(\bigcup_{i=1}^m C_i\right)\right)$$

*In particular, the closures are the same.*

PROOF. Using Lemma 1.2.4(i) and (ii) we get the inclusion

$$\widetilde{C} := (\widetilde{C_1} \times \ldots \times \widetilde{C_m}) \cap \widetilde{\Delta}_{m-1} \subset \operatorname{cl}\left(\bigcup_{\lambda\in\mathring{\Delta}_{m-1}} (\lambda_1 C_1 \times \{\lambda_1\}) \times \ldots \times (\lambda_m C_m \times \{\lambda_m\})\right) =: K$$

From continuity of the projection $\pi\colon \mathbb{R}^{nm+m} \to \mathbb{R}$, $(x^1,\lambda_1,\dots,x^m\lambda_m) \mapsto \sum_{i=1}^m x^i$, we deduce

$$\mathscr{C}(\mathscr{A}^1,\dots,\mathscr{A}^m) = \pi(\widetilde{C}) \subset \pi(K) \subset \operatorname{cl}\left(\bigcup_{\lambda\in\mathring{\Delta}_{m-1}} \pi(\lambda_1 C_1 \times \{\lambda_1\} \times \ldots \times \lambda_m C_m \times \{\lambda_m\})\right)$$

Now Proposition 1.2.9 tells us that the right-hand side of the above equation is nothing but the closure of the convex hull of the union of the sets $C_i$. Now to the first inclusion: From Lemma 1.2.4(i), we get

$$\bigcup_{\lambda\in\Delta_{m-1}} (\lambda_1 C_1 \times \{\lambda_1\}) \times \ldots \times (\lambda_m C_m \times \{\lambda_m\}) \subset (\widetilde{C_1} \times \ldots \times \widetilde{C_m}) \cap \Delta_{m-1}^m$$

(cf. Remark 1.2.5(i)), which implies the first inclusion. □

**Example 1.2.11.** The set $\mathscr{C}(\mathscr{A}^1,\dots,\mathscr{A}^m)$ need not be either the convex hull or its closure, as the following example shows: Let $C^1 = \{(0,0)\}$ with any representation as a spectrahedron $C^1 = \mathscr{S}(\mathscr{A}^1)$ and let $C^2 = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_1x_2 - 1 \geq 0\} = \pi_X(\mathscr{S}(\mathscr{A}^2(X,Y)))$ where

$$\mathscr{A}^2 = \begin{pmatrix} X_1 & & & & \\ & X_1 & 1 & & \\ & 1 & X_2 & & \\ & & & 1 & X_1 \\ & & & X_1 & Y \end{pmatrix}$$

In this example, $\mathscr{C}(\mathscr{A}^1,\mathscr{A}^2)$ is the union of the open quadrant $\{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ and the non-negative $x_2$-axis $\{(0,x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$ because $C_2^0 = \pi_X(\mathscr{S}(\mathscr{A}^2_{(1)}(X,Y))) = \{0\} \times \mathbb{R}_{\geq 0}$ (cf. Example 1.2.6), which is neither open nor closed.

Of course, this can also be seen by taking the convex hull of projections of spectrahedra,

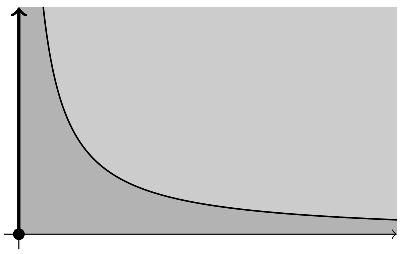

Figure 1.4.: The set $C^2$ is the area shaded in light grey and the set $\mathscr{C}(\mathscr{A}^1,\mathscr{A}^2)$ consists of the grey shaded areas as well as the thick black lines.

because $C^2$ is the projection of a spectrahedron, the non-negative $x_2$-axis, too, and the convex hull of these sets is exactly the one described above.

We continue on our way to prove that the topological interior of the projection of a spectrahedron is the projection of a spectrahedron (again in a constructive way). We need some more technical preparations which will allow us to reduce to the case of a polyhedron where we can give an explicit linear matrix polynomial defining its interior.

**Proposition 1.2.12.** *Let $C_1,\dots,C_m$ be non-empty, convex sets and assume that the interior of the set $C_j$ is non-empty for some $j \in \{1,\dots,m\}$.*
*(i) The set*

$$\bigcup_{\lambda \in \mathring{\Delta}_{m-1}} (\lambda_1 C_1 + \ldots + \lambda_m C_m)$$

*is open.*
*(ii) The interior of the convex hull of the union of the sets $C_i$ equals the set of part (i):*

$$\bigcup_{\lambda \in \mathring{\Delta}_{m-1}} (\lambda_1 C_1 + \ldots + \lambda_m C_m) = \operatorname{int}\left(\operatorname{conv}\left(\bigcup_{i=1}^m C_i\right)\right)$$

PROOF. (i) If $\operatorname{int}(C_j)$ is non-empty, then the set $\lambda_j C_j + y$ is open for all $\lambda_j > 0$ and $y \in \mathbb{R}^n$. Thus the set $\lambda_1 C_1 + \ldots + \lambda_m C_m$ is open for all $\lambda \in \mathring{\Delta}_{m-1}$ which implies that the union of these sets is open. (ii) This follows from Proposition and 1.2.9 by claim (i). □

**Definition 1.2.13** (cf. Definition 1.2.2)**.** Let $C_i = \pi_X(\mathscr{S}(\mathscr{A}^i(X,Y)))$, $i = 1,\ldots,m$, be projections of spectrahedra. Write

$$\widetilde{C} = \{(x,\lambda) \in \mathbb{R}^{n+1} : \exists\, y \in \mathbb{R}^N\ \lambda A_0 + \mathscr{A}_{(1)}(x,y) \geq 0\}$$

We set

$$\mathring{\Delta}^m_{m-1} = \{(x^1,\lambda_1,\ldots,x^m,\lambda_m) \in \mathbb{R}^{nm+m} : (\lambda_1,\ldots,\lambda_m) \in \mathring{\Delta}_{m-1}\}$$

where $\mathring{\Delta}_{m-1}$ is the interior of $\Delta_{m-1}$ in its affine hull, i.e. $\mathring{\Delta}_{m-1} = \{(\lambda_1,\ldots,\lambda_m) \in \Delta_{m-1} : \lambda_1 > 0,\ldots,\lambda_m > 0\}$.
Writing $\pi : (\mathbb{R}^n \times \mathbb{R}) \times \ldots \times (\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}^n$, $((x^1,\lambda_1),\ldots,(x^m,\lambda_m)) \mapsto \sum_{i=1}^m x^i$, we define the set

$$\mathscr{C}_0(C_1,\ldots,C_m) = \pi((\widetilde{C_1} \times \ldots \times \widetilde{C_m}) \cap \mathring{\Delta}^m_{m-1})$$

**Remark 1.2.14.** The set $\mathscr{C}_0(C_1,\ldots,C_m)$ does not depend on the chosen linear matrix polynomials defining the projections of spectrahedra $C_1,\ldots,C_m$, because we have (again by Lemma 1.2.4)

$$(\widetilde{C_1} \times \ldots \times \widetilde{C_m}) \cap \mathring{\Delta}^m_{m-1} = \bigcup_{\lambda \in \mathring{\Delta}_{m-1}} (\lambda_1 C_1 \times \{\lambda_1\}) \times \ldots \times (\lambda_m C_m \times \{\lambda_m\})$$

**Lemma 1.2.15.** *Let $C$ be the projection of a spectrahedron and let $F \subset C$ be an exposed face (i.e. by definition the intersection of a supporting hyperplane of $C$ with $C$ itself). Then the set $C \setminus F$ is again the projection of a spectrahedron.*

PROOF. By the definition of an exposed face, there is a linear polynomial $\ell \in \mathbb{R}[X_1,\ldots,X_n]$ such that $\ell|_C \geq 0$ and $\{x \in \mathbb{R}^n : \ell(x) = 0\} \cap C = F$. Now we define

$$\left\{x \in \mathbb{R}^n : \exists\,(y,z) \in \mathbb{R}^{N+1}\ \mathscr{A}(x,y) \boxplus \begin{pmatrix} \ell(x) & 1 \\ 1 & z \end{pmatrix} \geq 0\right\}$$

This set is equal to $C \setminus F$ because we find a $z \in \mathbb{R}$ such that the condition

$$\begin{pmatrix} \ell(x) & 1 \\ 1 & z \end{pmatrix} \geq 0$$

holds if and only if $\ell(x) > 0$. □

**Remark 1.2.16.** (i) Let $P = \{x \in \mathbb{R}^n : \ell_1(x) \geq 0,\ldots,\ell_r(x) \geq 0\}$ be a polyhedron defined by linear polynomials $\ell_1,\ldots,\ell_r \in \mathbb{R}[X_1,\ldots,X_n]$. Then the interior of $P$ in its affine hull is $\{x \in \mathbb{R}^n : \ell_1(x) > 0,\ldots,\ell_r(x) > 0\}$ which can be written as

$$\left\{x \in \mathbb{R}^n : \exists\,(z_1,\ldots,z_r) \in \mathbb{R}^r \begin{pmatrix} \ell_1(x) & 1 \\ 1 & z_1 \end{pmatrix} \boxplus \ldots \boxplus \begin{pmatrix} \ell_r(x) & 1 \\ 1 & z_r \end{pmatrix} \geq 0\right\}$$

The linear matrix inequality describing the spectrahedron given in the above equation is of lower matrix dimensions than the one given in the proof of the above Lemma.
(ii) Since we get by part (i) a representation of $\mathring{\Delta}_{m-1}$ as the projection of a spectrahedron, we now know that $\mathscr{C}_0(C_1,\dots,C_m)$ as defined in 1.2.2 for projections of spectrahedra $C_1,\dots,C_m$ is again the projection of a spectrahedron.

Now, the result is easy to prove:

**Theorem 1.2.17.** *The interior of a set which is the projection of a spectrahedron is again the projection of a spectrahedron.*

PROOF. Let $C$ be the projection of a spectrahedron. If the interior of $C$ is empty, then there is nothing to prove. If the interior of $C$ is non-empty, then take $x \in \operatorname{int}(C)$. Remark 1.2.16(ii) tells us that $\mathscr{C}_0(C,\{x\})$ is the projection of a spectrahedron. By Proposition 1.2.12 (cf. Remark 1.2.14) this is the interior of the convex hull of the union of $C$ and $\{x\}$, which is obviously the interior of $C$. □

# 1.3. A Local-Global Principle for Projections of Spectrahedra

We will prove a result which will imply that a compact convex set is the projection of a spectrahedron if and only if it is locally on its boundary. To state it precisely, we need some basics of convex geometry:

**Definition 1.3.1.** Let $C$ be a convex set. A point $a \in C$ is called an *extreme point* of $C$ if for all $b$, $c \in C$ such that $\frac{1}{2}(b+c) = a$ we have $b = a = c$. We will write $\operatorname{Ex}(C)$ for the set of all extreme points of $C$.

**Lemma 1.3.2.** *Let $D \subset \mathbb{R}^n$ be a set. A point $a \in \mathbb{R}^n$ is an extreme point of the convex hull of $D$ if and only if $a$ is in $D$ and not in the convex hull of $D \setminus \{a\}$.*

PROOF. Take $x \in \operatorname{conv}(D)\setminus D$. Then we have $x = \sum_{i=1}^m \lambda_i d_i$ where $\lambda_i > 0$, $\sum_{i=1}^m \lambda_i = 1$, $d_i \in D$, $d_i \neq d_j$ for all $i \neq j$ and $m \geq 2$. Without loss of generality we can assume that $\lambda_2 \geq \lambda_1$. We set $b = \frac{1}{2}\lambda_1 d_1 + (\lambda_2 + \frac{1}{2}\lambda_1)d_2 + \sum_{i=3}^m \lambda_i d_i$ and $c = \frac{3}{2}\lambda_1 d_1 + (\lambda_2 - \frac{1}{2}\lambda_1)d_2 + \sum_{i=3}^m \lambda_i d_i$. These two points lie in the convex hull of $D$ and satisfy $\frac{1}{2}(b+c) = x$. As we also have $b - c = \lambda_1(d_2 - d_1) \neq 0$, the point $x$ cannot be an extreme point of the convex hull of $D$. Thus every extreme point of $\operatorname{conv}(D)$ is an element of $D$.
An extreme point of $\operatorname{conv}(D)$ cannot be a convex combination of other elements of $\operatorname{conv}(D)$: Let $a$ be an extreme point of $\operatorname{conv}(D)$ and $a = \sum_{i=1}^m \lambda_i d_i$ where $\lambda_i > 0$, $m \geq 2$. Then we have $a = \frac{1}{2}d_1 + \frac{1}{2}\frac{1}{1-\lambda_1}\left(\sum_{i=2}^m \lambda_i d_i\right)$, whence we deduce $d_1 = a$ and $\frac{1}{1-\lambda_1}\sum_{i=2}^m \lambda_i d_i = a$. By iteration of this argument, we finally arrive at $d_i = a$ for all $i = 1,\dots,m$ and thus $a \notin \operatorname{conv}(D \setminus \{a\})$.
Conversely, take $a \in D$ such that $a \notin \operatorname{conv}(D\setminus\{a\})$. Take $b, c \in \operatorname{conv}(D)$ such that $\frac{1}{2}(b+c) = a$. Then we have $a \in \operatorname{conv}(\{b,c\})$. Now it follows from $a \notin \operatorname{conv}(D)\setminus\{a\}$ that $b = a = c$ which means that $a$ is an extreme point of $\operatorname{conv}(D)$. □

**Theorem 1.3.3** (Theorem of Minkowski, cf. [Bar02], Theorem II.3.3)**.** *Let $C \subset \mathbb{R}^n$ be a non-empty, compact and convex set. Then $C$ is the convex hull of the set of its extreme points:*

$$C = \operatorname{conv}(\operatorname{Ex}(C))$$

As the proof is a short and simple proof by induction on the dimension of $C$, we include it here:

PROOF. If the dimension of $C$ is 0, then $C$ is a point and therefore there is nothing to show. We proceed by induction: Let the dimension of $C \subset \mathbb{R}^n$ be $n$ (if the dimension of $C$ is $d < n$, then it is contained in an affine subspace of $\mathbb{R}^n$ (cf. [Bar02], Theorem II.2.4) and this case is included in the induction hypothesis). If $x \in \partial C$, then $x$ is contained in a face $F$ of $C$ (cf. [Bar02], Corollary II.2.8) and therefore a convex combination of extreme points of this face by induction hypothesis. This implies the result in this case because $\operatorname{Ex}(F) \subset \operatorname{Ex}(C)$.
Now take $x \in \operatorname{int}(C)$ and take any line $L$ through $x$. Then $L \cap C = [y,z] = \{\lambda y + (1-\lambda)z : \lambda \in [0,1]\}$ for some $y, z \in \partial C$. Since $y$ and $z$ are convex combinations of extreme points of $C$, the same is true for $x$, which finishes the proof. □

**Remark 1.3.4.** The set of all extreme points of a convex set $C$ is the smallest set (with respect to inclusion) such that $C$ is the convex hull of this set. Note that it need not be closed if $n \geq 3$.

**Theorem 1.3.5.** *Let $C \subset \mathbb{R}^n$ be a compact and convex set. Then $C$ is the projection of a spectrahedron if and only if every $x \in \operatorname{cl}(\operatorname{Ex}(C))$ possesses a neighbourhood $U_x$ such that the set $U_x \cap C$ is the projection of a spectrahedron.*

PROOF. If there is a neighbourhood $U_x$ for all $x \in \operatorname{cl}(\operatorname{Ex}(C))$ such that $U_x \cap C$ is the projection of a spectrahedron, then we can find (as $\operatorname{cl}(\operatorname{Ex}(C))$ is compact) a finite number of points $x_1, \ldots, x_r \in \operatorname{cl}(\operatorname{Ex}(C))$ such that $\operatorname{cl}(\operatorname{Ex}(C)) \subset (U_{x_1} \cup \ldots \cup U_{x_n}) \cap C$. Now we know by Theorem 1.2.7 and Theorem 1.3.3 that $C$ is the projection of a spectrahedron.
The converse is obvious because $\operatorname{cl}(B(x,r))$ is a spectrahedron for all $x \in \mathbb{R}$ and all $r \geq 0$. □

**Remark 1.3.6.** Theorem 1.3.5 is a generalisation of [HN09], Proposition 4.3.

# 2. Truncated Quadratic Modules and the Lasserre-Relaxation

## 2.1. Quadratic Modules and Convex Cones

Throughout this section, we fix polynomials $g_1,\dots,g_r \in \mathbb{R}[X_1,\dots,X_n]$ and denote by $C = \{x \in \mathbb{R}^n : g_1(x) \geq 0,\dots,g_r(x) \geq 0\}$ the basic closed semi-algebraic set defined by those polynomials. Furthermore we set $g_0 = 1 = g^{(0,\dots,0)}$.

Firstly, we will introduce and talk about preorderings and quadratic modules.

**Definition 2.1.1.** (a) The set $\mathrm{PO}(g_1,\dots,g_r) = \{\sum_{\alpha\in\{0,1\}^r} \sigma_\alpha g^\alpha : \sigma_\alpha \in \sum \mathbb{R}[X_1,\dots,X_n]^2\}$ is called the *preordering* generated by $g_1,\dots,g_r$. For $k \in \mathbb{N}$, we call the set

$$\mathrm{PO}(g_1,\dots,g_r)_k = \left\{ \sum_{\alpha\in\{0,1\}^r} \sigma_\alpha g^\alpha : \sigma_\alpha \in \sum \mathbb{R}[X_1,\dots,X_n]^2,\ \deg(\sigma_\alpha g^\alpha) \leq k \right\} \subset \mathbb{R}[X_1,\dots,X_n]_k$$

the *truncated preordering* of degree $k$.
(b) The set $\mathrm{QM}(g_1,\dots,g_r) = \{\sum \sigma_i g_i : \sigma_i \in \sum \mathbb{R}[X_1,\dots,X_n]^2\}$ is called the *quadratic module* generated by $g_1,\dots,g_r$. Analogously, we write

$$\mathrm{QM}(g_1,\dots,g_r)_k = \left\{ \sum_{i=0}^{r} \sigma_i g_i : \sigma_i \in \sum \mathbb{R}[X_1,\dots,X_n]^2,\ \deg(\sigma_i g_i) \leq k \right\} \subset \mathbb{R}[X_1,\dots,X_n]_k$$

for $k \in \mathbb{N}$ and call this set the *truncated quadratic module* of degree $k$.

**Remark 2.1.2.** (i) Every preordering is a quadratic module.
(ii) If we choose different defining polynomials $p_1,\dots,p_s$ for the set $C$ and we have $\mathrm{QM}(g_1,\dots,g_r) = \mathrm{QM}(p_1,\dots,p_s)$, then by writing $p_j = \sum f_{ij}^2 g_i$ with polynomials $f_{ij} \in \mathbb{R}[X_1,\dots,X_n]$ we get $\mathrm{QM}(p_1,\dots,p_s)_k \subset \mathrm{QM}(g_1,\dots,g_r)_{k'}$, where $k' = k + 2\max\{\deg(f_{ij})\} + \max\{\deg(g_i) - \deg(p_j) : i = 1,\dots,r \text{ and } j = 1,\dots,s\}$.

In fact, every truncated quadratic module is a convex cone.

**Definition 2.1.3.** (a) A non-empty subset $K$ of an $\mathbb{R}$-vector space is a *cone* if for all $x \in K$ and $\alpha \in \mathbb{R}_{\geq 0}$ we have $\alpha x \in K$.
(b) A cone $K$ is called *pointed* if it does not contain a line, i.e. for all $x \in K \setminus \{0\}$ we have $-x \notin K$.

**Remark 2.1.4.** (i) A cone $K$ is convex if and only if it is closed under addition, i.e. if for all $x, y \in K$ we have $x + y \in K$: If the cone is closed under addition, then convexity is obvious. Conversely, if $K$ is convex and we take $x, y \in K$, then it is closed under addition by the equality $x + y = \frac{1}{2}(2x) + \frac{1}{2}(2y)$.
(ii) If we have a cone in $\mathbb{R}^n$, then its topological closure will again be a cone.

**Example 2.1.5.** (i) Obviously, every union of lines is a cone and so is every union of sets of the form $\{ax \colon a \in \mathbb{R}_{\geq 0}\}$ for $x \in V$.
(ii) Every quadratic module, and in particular also every preordering, in $\mathbb{R}[X_1, \ldots, X_n]$ is a convex cone.

We will now investigate how topological and algebraic properties of truncated quadratic modules relate. It is convenient to have the following notion at hand.

**Definition 2.1.6.** Let $V$ be an $\mathbb{R}$-vector space and $A \subset V$ a convex subset. A point $x \in A$ is called an *algebraic interior point* if for all $v \in V$ there is a scalar $a > 0$ such that $x + av \in A$.

**Remark 2.1.7.** (i) If $V$ is a topological $\mathbb{R}$-vector space, then every interior point of $A$ is an algebraic interior point. (cf. [Köt69], page 177).
(ii) If $V = \mathbb{R}^n$, then $x \in A$ is an algebraic interior point of $A$ if and only if $x$ is in the topological interior of $A$.

As a first, basic and abstract result, we have the following:

**Lemma 2.1.8.** *Let $V$ be an $\mathbb{R}$-vector space, let $K \subset V$ be a cone.*
*(i) If $K$ has an algebraic interior point, then $V = K - K = \{x - y \colon x, y \in K\}$.*
*(ii) If $V$ is finite dimensional, $K$ is convex and $V = K - K$, then $K$ has an algebraic interior point.*

PROOF. (i) Let $x_0 \in K$ be an algebraic interior point of $K$ and let $\{v_i\}_{i \in I}$ be an $\mathbb{R}$-basis of $V$. Take $y \in V$ and write $y = \sum_{i \in I} a_i v_i$ where $a_i = 0$ for almost all $i \in I$. Put $J = \{i \in I \colon a_i \neq 0\}$. Choose for every $j \in J$ a real number $\varepsilon_j$ such that $a_j \varepsilon_j > 0$ and $x_0 + \varepsilon_j v_j \in K$. Thus we have $-\frac{a_j}{\varepsilon_j} x_0 \in -K$ and therefore $y = \sum_{j \in J} \left( \frac{a_j}{\varepsilon_j}(x_0 + \varepsilon_j v_j) - \frac{a_j}{\varepsilon_j} x_0 \right) \in K - K$.
(ii) If $K$ has no algebraic interior point, then the interior of $K$ is empty (by Remark 2.1.7(i)) and therefore $K$ is contained in a hyperplane $H \subset V$ by Theorem II.2.4 in [Bar02]. Hence we have $K - K \subset H \subsetneq V$. □

**Remark 2.1.9.** We cannot drop any of the additional assumptions in part (ii) of the preceding Lemma as the following examples show:
(i) Take $V = \oplus_{i \in \mathbb{N}} \mathbb{R} = \{(x_1, x_2, \ldots) \colon x_i \in \mathbb{R}, x_i = 0 \text{ for almost all } i \in \mathbb{N}\}$ and define the set $A = \{(a_1, a_2, \ldots) \colon a_j > 0 \text{ for } j = \max\{i \in \mathbb{N} \colon a_i \neq 0\}\}$. It is a convex subset of the infinite dimensional $\mathbb{R}$-vector space $V$ and we have $A - A = V$. But $A$ does not contain any algebraic interior point: Let $a = (a_1, a_2, \ldots) \in A$ and $j = \max\{i \in \mathbb{N} \colon a_i \neq 0\}$. Put $x = (x_1, x_2, \ldots) \in V$, $x_i = 0$ for all $i \neq j + 1$ and $x_{j+1} = -1$. Then the point $a + tx$ does not lie in $A$ for any choice of $t > 0$. The set $A$ is thus also an example for a convex set which does

not contain any algebraic interior point and is not contained in any hyperplane in $V$.
(ii) Take $V = \mathbb{R}^n$ and let $\{e_i\}_{i=1,\dots,n}$ be the standard basis. Put $K = \{(x_1, 0, \dots, 0) \in \mathbb{R}^n : x_1 \geq 0\} \cup \{(0, x_2, \dots, x_n) : (x_2, \dots, x_n) \in \mathbb{R}^{n-1}\}$. The set $K$ is a cone (as it is the union of sets of the form $\{\alpha x : \alpha \in \mathbb{R}_{\geq 0}\}$ for some $x \in V$) and we have $K - K = V$, but the interior of $K$ is empty.

This abstract setup specialises to the following, useful result on truncated quadratic modules:

**Lemma 2.1.10** (cf. [KMS05], p. 4287)**.** *Let $k \in \mathbb{N}$ be even. Then the interior of the cone $\mathrm{QM}(1)_k \subset \mathbb{R}[X_1, \dots, X_n]$ (the truncation of the cone of sums of squares) is non-empty. In particular, the interior of any truncation of a quadratic module $\mathrm{QM}(g_1, \dots, g_r)_k$ (or $\mathrm{PO}(g_1, \dots, g_r)_k$) is non-empty.*

PROOF. By Lemma 2.1.8 it suffices to show that $\mathrm{QM}(1)_k - \mathrm{QM}(1)_k = \mathbb{R}[X_1, \dots, X_n]_k$. From the equation

$$pq = \frac{1}{2}\left((p+q)^2 - p^2 - q^2\right)$$

we get that $X^\alpha \in \mathrm{QM}(1)_k - \mathrm{QM}(1)_k$ for all $\alpha \in \mathbb{N}^r$, $|\alpha| \leq k$, which completes the proof. □

**Remark 2.1.11.** The above lemma is in general false for odd $k \in \mathbb{N}$, because it might occur that we have $\mathrm{QM}(g_1, \dots, g_r)_{k-1} = \mathrm{QM}(g_1, \dots, g_r)_k$. This is for example the case for the cone of sums of squares $\sum \mathbb{R}[X_1, \dots, X_n]^2 = \mathrm{QM}(1)$ and all odd $k \in \mathbb{N}$.

We will get more information by making the dual space part of our considerations in the following way:

**Definition 2.1.12.** Let $V$ be an $\mathbb{R}$-vector space and let $K \subset V$ be a cone. We define the *dual cone* $K^\vee$ of $K$ to be the set of all linear functionals on $V$ that only take non-negative values on $K$

$$K^\vee = \{L \in V^\vee : L|_K \geq 0\} \subset V^\vee$$

**Remark 2.1.13.** Let $D \subset \mathbb{R}^n$ be a set. Then the closure of the convex hull of $D$ is the intersection of all closed half-spaces containing $D$, i.e.

$$\mathrm{cl}(\mathrm{conv}(D)) = \bigcap \{\ell^{-1}([0,\infty)) : \ell \in \mathbb{R}[X_1, \dots, X_n]_1, \ell|_D \geq 0\}$$

This statement is an easy corollary to the separation theorem for convex sets (cf. for example [Bar02], Theorem III.1.3): Take $x \in \mathbb{R}^n \setminus \mathrm{cl}(\mathrm{conv}(D))$. Then there exists a real number $c \in \mathbb{R}$ and a linear functional $\ell : \mathbb{R}^n \to \mathbb{R}$ such that $\ell(x) < c$ and $\ell|_{\mathrm{cl}(\mathrm{conv}(D))} \geq c$.
In the case of $D$ being a cone we get

$$\mathrm{cl}(\mathrm{conv}(D)) = \bigcap \{\ell^{-1}([0,\infty)) : \ell : \mathbb{R}^n \to \mathbb{R} \text{ linear}, \ell|_D \geq 0\}$$

We gather the relationships of the properties of a cone and its dual:

**Proposition 2.1.14.** *Let $V$ be a finite-dimensional $\mathbb{R}$-vector space and let $K \subset V$ be a cone.*
*(i) The dual cone $K^\vee$ of $K$ is a closed, convex cone.*
*(ii) If the interior of $K$ is non-empty, then the dual cone $K^\vee$ is pointed.*
*(iii) The interior of the dual cone $K^\vee$ consists of all functionals which are positive on* $\mathrm{cl}(K) \setminus \{0\}$*, i.e.* $\mathrm{int}(K^\vee) = \{L \in V^\vee : \forall\, x \in \mathrm{cl}(K) \setminus \{0\} : L(x) > 0\}$.
*(iv) If the closure of $K$ is pointed, then the interior of the dual cone $K^\vee$ is non-empty.*
*(v) The closure of the convex hull of $K$ is canonically homeomorphic to the bidual cone of $K$. In particular, the closure of a convex cone is pointed if and only if the interior of its dual cone is non-empty.*

Proof. (i) is clear. (ii) If $K^\vee$ is not pointed, then there is a functional $L \in K^\vee \setminus \{0\}$ such that $-L \in K^\vee$, i.e. $L|_K \geq 0$ and $-L|_K \geq 0$. This means $K \subset \{x \in V : L(x) = 0\}$ and therefore the interior of $K$ must be empty.
(iii) Take $L_0 \in \mathrm{int}(K^\vee)$. Then there is for all $L \in V^\vee$ a scalar $a > 0$ such that $L_0 + aL \in K^\vee$ (cf. Remark 2.1.7). Let $x \in \mathrm{cl}(K) \setminus \{0\}$ and take $L \in V^\vee$ such that $L(x) < 0$. Then we get $L_0 + aL(x) \geq 0$ for some $a > 0$ and therefore we have $L(x) > 0$. Conversely, if we take $L_0 \in V^\vee$ such that $L_0(x) > 0$ for all $x \in \mathrm{cl}(K) \setminus \{0\}$, then we obviously have $L_0 \in K^\vee$ and $L_0 + L \in K^\vee$ for all $L \in K^\vee$. So take $L \in V^\vee \setminus K^\vee$ and set $a_0 = \min\{L_0(x) \colon x \in \mathrm{cl}(K), \|x\| = 1\} > 0$ and $a = \min\{L(x) \colon x \in \mathrm{cl}(K), \|x\| = 1\} < 0$. Choose $\varepsilon > 0$ such that $a_0 + \varepsilon a > 0$. Then we have $(L_0 + \varepsilon L)|_K \geq 0$, because of the following inequality that is valid for all $x \in K$

$$(L_0 + \varepsilon L)\left(\frac{x}{\|x\|}\right) = L_0\left(\frac{x}{\|x\|}\right) + \varepsilon L\left(\frac{x}{\|x\|}\right) \geq a_0 + \varepsilon a > 0$$

This shows that the chosen $L_0$ is an algebraic interior point of $K^\vee$.
(iv) Put $K_1 = \{x \in \mathrm{cl}(K) \colon \|x\| = 1\}$. Then $K_1$ is compact and therefore $A := \mathrm{conv}(K_1)$ is convex and compact (cf. [Bar02], Theorem I.2.4). If we have $0 \in A$, then $0 = \sum_{i=1}^r \lambda_i x_i$ where $x_i \in K_1$, $\lambda_i > 0$, $\sum_{i=1}^r \lambda_i = 1$. We deduce $0 \neq -\lambda_1 x_1 = \sum_{i=2}^r \lambda_i x_i \in \mathrm{cl}(K)$, which means that the closure of $K$ is not pointed. As this is an assumption, we know $0 \notin A$. By applying [Bar02], Theorem III.1.3, we find a scalar $a \in \mathbb{R}$ and a linear polynomial $\ell \in \mathbb{R}[X_1,\ldots,X_n]_1$ such that $\ell|_A > a$ and $\ell(0) < a$. By defining $L \colon \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \ell(x) - \ell(0)$ we get a linear functional on $\mathbb{R}^n$ satisfying $L|_A > 0$ which implies $L|_{\mathrm{cl}(K)\setminus\{0\}} > 0$. By part (iii) we know that this functional $L$ lies in the interior of $K^\vee$.
(v) The canonical homeomorphism will, of course, be the canonical isomorphism which identifies $V$ and $V^{\vee\vee}$, i.e. the map that associates to $x \in V$ the evaluation $\mathrm{ev}_x \in V^{\vee\vee}$ of a functional at $x$. Now let $x \notin \mathrm{cl}(\mathrm{conv}(K))$. Then there is a linear functional $\ell \in V^\vee$ such that $\ell|_{\mathrm{cl}(\mathrm{conv}(K))} \geq 0$ and $\ell(x) < 0$ (again by [Bar02], Theorem III.1.3 and the fact that $K$ is a cone), i.e. $\ell \in K^\vee$. But we have $\mathrm{ev}_x(\ell) = \ell(x) < c \leq 0$ which means $\mathrm{ev}_x \notin K^{\vee\vee}$. Conversely, take $x \in \mathrm{cl}(\mathrm{conv}(K))$. As we know that $\mathrm{cl}(\mathrm{conv}(K)) = \bigcap\{\ell^{-1}([0,\infty)) \colon \ell \in V^\vee, \ell|_K \geq 0\}$ (cf. Remark 2.1.13), we have $\ell(x) \geq 0$ for all $\ell \in K^\vee$, i.e. $\mathrm{ev}_x \in K^{\vee\vee}$. The second part of the claim follows by using parts (ii) and (iv). □

After the discussion of this general setup, we focus again on truncated quadratic modules and their duals. They will be essential for the definition of the Lasserre-relaxation in the next section. For the remainder of this section, let $T_k$ denote $\mathrm{QM}(g_1,\dots,g_r)_k$ or $\mathrm{PO}(g_1,\dots,g_r)_k$.

**Definition 2.1.15.** For all $k \in \mathbb{N}$ we define

$$\begin{aligned}\widehat{\mathfrak{L}_k} = T_k^\vee &= \{L : \mathbb{R}[X_1,\dots,X_n]_k \to \mathbb{R} : L \text{ linear}, L(T_k) \subset [0,\infty)\} \\ \mathfrak{L}_k &= \{L : \mathbb{R}[X_1,\dots,X_n]_k \to \mathbb{R} : L \text{ linear}, L(T_k) \subset [0,\infty), L(1) = 1\}\end{aligned}$$

**Remark 2.1.16.** (i) The set $\widehat{\mathfrak{L}_k}$ is the dual cone of $T_k$; the set $\mathfrak{L}_k$ is convex and closed.
(ii) The interior of the cone $\widehat{\mathfrak{L}_k} \subset \mathbb{R}[X_1,\dots,X_n]_k^\vee$ is non-empty if and only if the cone $T_k$ is pointed (cf. Proposition 2.1.14). This condition is more closely reviewed in the following lemma.

**Lemma 2.1.17.** *(i) If the interior of $C$ is non-empty, then $T_k$ is pointed for all $k \in \mathbb{N}$.*
*(ii) If we have $T_k = \mathrm{PO}(g_1,\dots,g_r)_k$, then the converse is also true: If $\mathrm{PO}(g_1,\dots,g_r)_k$ is pointed for all $k \in \mathbb{N}$, then the interior of $C$ is non-empty.*

PROOF. (i) If $T_k$ is not pointed for some $k \in \mathbb{N}$, then there is a polynomial $p \in T_k \setminus \{0\}$ such that $-p \in T_k$. We deduce $C \subset \mathrm{Z}(p)$ and therefore $\mathrm{int}(C) = \emptyset$.
(ii) We give a proof in appendix B. □

So far, we have only considered $\mathbb{R}[X_1,\dots,X_n]$ as an $\mathbb{R}$-vector space. Now we will take a short look at the multiplicative structure and its consequences for elements of the dual space.

**Proposition 2.1.18.** *Let $L \in \mathbb{R}[X_1,\dots,X_n]_k^\vee$ be a functional such that $L(p^2) \geq 0$ for all $p \in \mathbb{R}[X_1,\dots,X_n]$ of degree $\deg(p) \leq \frac{k}{2}$. Then we have for all $p,q \in \mathbb{R}[X_1,\dots,X_n]$ with $\deg(p),\deg(q) \leq \frac{k}{2}$ the following inequality*

$$L(pq)^2 \leq L(p^2)L(q^2)$$

*In particular, if $L(1) = 0$, we get $L(p) = 0$ for all $p \in \mathbb{R}[X_1,\dots,X_n]$ of degree $\deg(p) \leq \frac{k}{2}$.*

PROOF. Take $p,q \in \mathbb{R}[X_1,\dots,X_n]$, $\deg(p),\deg(q) \leq \frac{k}{2}$. For all $r \in \mathbb{R}$ we have

$$L(p^2) + 2rL(pq) + r^2L(q^2) = L((p+rq)^2) \geq 0$$

The left-hand side has no real zero or a real zero of multiplicity two, considered as a polynomial in $r$. Therefore we get for the discriminant of the polynomial

$$4L(pq)^2 - 4L(p^2)L(q^2) \leq 0$$

which implies the claim.
The second part of the claim is a direct consequence of the following inequality

$$L(p)^2 = L(p1)^2 \leq L(1^2)L(p^2) = 0$$

□

**Remark 2.1.19.** (i) Despite this Lemma, the conic hull of $\mathfrak{L}_k$ in $\mathbb{R}[X_1,\dots,X_n]_k^\vee$ might in general be a proper subset of $\widehat{\mathfrak{L}_k}$, as the following example shows:
Take $\mathrm{PO}(X) \subset \mathbb{R}[X]$. The functional $L\colon \mathbb{R}[X]_1 \to \mathbb{R}$ defined by $L(1) = 0$ and $L(X) = c > 0$ is non-negative on $\mathrm{PO}(X)_1$ and therefore we have $L \in \widehat{\mathfrak{L}_1}$ but obviously not in the conic hull of $\mathfrak{L}_1$.
This functional cannot be extended to $\mathbb{R}[X]_2$ in such a way that the restriction of the extension to $\mathrm{PO}(X)_2$ remains non-negative. This follows from Proposition 2.1.18 and can be seen directly as follows: For all $a > 0$ we have $f_a := X^2 - 2aX + a^2 = (X-a)^2 \in PO(X)_2$. But for every extension of $L$ to $\mathbb{R}[X]_2$ there is a choice of $a > 0$ such that the extension is negative on $f_a$.
(ii) In general, it holds true that the closure of the conic hull of $\mathfrak{L}_k$ in $\mathbb{R}[X_1,\dots,X_n]_k^\vee$ is $\widehat{\mathfrak{L}_k}$: Take $L \in \widehat{\mathfrak{L}_k}$ and $L_0 \in \mathfrak{L}_k$. Then the functional $L + \alpha L_0$ is in the conic hull of $\mathfrak{L}_k$ for all $\alpha > 0$ and we have $(L + \alpha L_0) \to L$ for $\alpha \to 0$.

## 2.2. The Lasserre-Relaxation

Recall that we have fixed polynomials $g_1,\dots,g_r \in \mathbb{R}[X_1,\dots,X_n]$ and denote by $C$ the basic closed semi-algebraic set defined by these polynomials. We have also put $T_k = \mathrm{QM}(g_1,\dots,g_r)_k$ or $T_k = \mathrm{PO}(g_1,\dots,g_r)_k$ and $\mathfrak{L}_k = \{L \in T_k^\vee : L(1) = 1\}$; the following constructions work for both cases.

**Definition 2.2.1.** Let $k$ be in $\mathbb{N}$. The *Lasserre-relaxation* of $C$ of degree $k$ is defined to be

$$C_k = \{(L(X_1),\dots,L(X_n)) : L \in \mathfrak{L}_k\} \subset \mathbb{R}^n$$

A more abstract definition has been developed in [GN10].

**Remark 2.2.2.** (i) The relaxation $C_k$ is convex because it is the image of the convex set $\mathfrak{L}_k$ under the linear map $\pi\colon \mathbb{R}[X_1,\dots,X_n]_k^\vee \to \mathbb{R}^n$, $L \mapsto (L(X_1),\dots,L(X_n))$.
(ii) We have $C \subset C_k$ for all $k \in \mathbb{N}$: Fix $k \in \mathbb{N}$. For all $x \in \mathbb{R}^n$ we know that the point evaluation at $x$ (denoted by $\mathrm{ev}_x$) is a linear functional on $\mathbb{R}[X_1,\dots,X_n]_k$ and we have $\mathrm{ev}_x \in \mathfrak{L}_k$ if $x \in C$. Hence we get $C \subset C_k$.

Next, we will give a description of the Lasserre-relaxation in terms of the truncated quadratic module and not its dual (cf. Theorem 2.2.4). The following, easy lemma will prove to be very useful in the following.

**Proposition 2.2.3.** *Let $p \in \mathbb{R}[X_1,\dots,X_n]_k$. If for all $L \in \mathfrak{L}_k$ the inequality $L(p) \geq 0$ holds, then $p$ is in the closure of $T_k$.*

PROOF. By Remark 2.1.19(ii) we know that we also have the inequality $L(p) \geq 0$ for all $L \in \widehat{\mathfrak{L}_k}$. This implies the claim by application of the separation theorem for convex sets (cf. Remark 2.1.13). □

**Theorem 2.2.4.** *The closure of the Lasserre-relaxation of degree $k \in \mathbb{N}$ is the intersection of all closed half-spaces whose defining, linear polynomials lie in the closure of $T_k$, i.e. we have for all $k \in \mathbb{N}$*

$$\operatorname{cl}(C_k) = \{x \in \mathbb{R}^n : \forall\, \ell \in \operatorname{cl}(T_k), \deg(\ell) = 1 : \ell(x) \geq 0\}$$

PROOF. An element $x \in C_k$ is contained in the set on the right-hand side because we have for all linear polynomials $\ell \in \mathbb{R}[X_1,\dots,X_n]_1 \cap \operatorname{cl}(T_k)$ and all $L \in \mathfrak{L}_k$

$$\ell(L(X_1),\dots,L(X_n)) = L(\ell) \geq 0$$

Since the set on the right-hand side is closed, we get one inclusion.
Now take $x \notin \operatorname{cl}(C_k)$. Since $\{x\}$ is compact, $\operatorname{cl}(C_k)$ is closed and both sets are convex, there is a linear polynomial $\ell \in \mathbb{R}[X_1,\dots,X_n]_1$ such that $\ell(x) < 0$ and $\ell|_{\operatorname{cl}(C_k)} > 0$ (cf. [Bar02], Theorem III.1.3). Again by the identity $\ell(L(X_1),\dots,L(X_n)) = L(\ell) \geq 0$ we get $L(\ell) \geq 0$ for all $L \in \mathfrak{L}_k$ and by Proposition 2.2.3 we get $\ell \in \operatorname{cl}(T_k)$. We now have the other inclusion by $\ell(x) < 0$. □

**Remark 2.2.5.** If the interior of $C$ is non-empty, then the cone $T_k$ is closed for all $k \in \mathbb{N}$ (cf. [Mar08], 4.1.4, or [PS01], 2.6(b)). In this case, we can drop the closure of $T_k$ in the claim of the above theorem:

$$\operatorname{cl}(C_k) = \{x \in \mathbb{R}^n : \forall\, \ell \in T_k, \deg(\ell) = 1 : \ell(x) \geq 0\}$$

As a special case of Theorem 2.2.4, we get the following corollary which will be used extensively in section 3.

**Corollary 2.2.6.** *Let $k \in \mathbb{N}$. The closure of the Lasserre-relaxation of degree $k$ is the closure of the convex hull of $C$ if and only if every linear polynomial which is non-negative on $C$ is contained in the closure of $T_k$.*

PROOF. If we know that every linear polynomial which is non-negative on $C$ is contained in the closure of $T_k$, then we get the desired claim by Theorem 2.2.4 and and the separation theorem for convex sets (cf. Remark 2.1.13). Conversely, if the closure of the Lasserre-relaxation of degree $k$ is the closure of the convex hull of $C$ and we take a linear polynomial $p \in \mathbb{R}[X_1,\dots,X_n]$ which is non-negative on $C$, then we get $L(p) = p(L(X_1),\dots,L(X_n)) \geq 0$ ($L \in \mathfrak{L}_k$) by $\operatorname{cl}(C_k) = \operatorname{cl}(\operatorname{conv}(C))$. Now we apply Proposition 2.2.3 and get the desired claim. □

In order to state another corollary to the theorem, we need the following notion:

**Definition 2.2.7.** A quadratic module $\mathrm{QM} \subset \mathbb{R}[X_1,\dots,X_n]$ is called archimedean if there is an $N \in \mathbb{N}$ such that the polynomial $N - \sum_{i=1}^n X_i^2$ lies in the quadratic module (cf. [PD01] for more information on archimedean quadratic modules).

We can now state this new corollary:

**Corollary 2.2.8.** *Let $C$ be compact and $\mathrm{QM}(g_1,\dots,g_r)$ be archimedean. Then the closure of the convex hull of $C$ is the intersection of all its Lasserre-relaxations:*

$$\mathrm{cl}(\mathrm{conv}(C)) = \bigcap_{k\in\mathbb{N}} C_k$$

PROOF. This follows from Theorem 2.2.4 by using the separation theorem for convex sets (cf. Remark 2.1.13) and Putinar's Positivstellensatz (cf. [PD01], Theorem 5.3.8). □

In order to prove that all linear polynomials which are non-negative on $C$ lie in the closure of $T_k$ it is sometimes useful to apply Farkas's Lemma in the following form:

**Corollary 2.2.9** (to Farkas's Lemma)**.** *Let $\emptyset \neq P \subset \mathbb{R}^n$ be a non-empty polyhedron, say*

$$P = \{x\in\mathbb{R}^n : \ell_1(x)\ge 0,\dots,\ell_r(x)\ge 0\}$$

*where $\ell_1,\dots,\ell_r \in \mathbb{R}[X_1,\dots,X_n]_1$ are linear polynomials. Let $\ell \in \mathbb{R}[X_1,\dots,X_n]_1$ be a linear polynomial which is non-negative on $P$ and $\min\{\ell(x): x\in P\} = 0$.*
*Then there are $a_1,\dots,a_r \ge 0$ such that $\ell = a_1\ell_1+\ldots+a_r\ell_r$.*

PROOF. After an affine transformation we may assume that $\ell(0) = 0$ and $\ell_i(0)\ge 0$ ($i = 1,\dots,r$), i.e. we move an intersection point of $\mathrm{Z}(\ell)$ and $P$ to the origin. We identify the polynomials with their sequences of coefficients, i.e. we write $\ell = (\ell_0,\dots,\ell_n)$ and $\ell_j = (\ell_0^j,\dots,\ell_n^j)$ ($j = 1,\dots,r$). We put $A = (\ell_i^j)_{0\le i\le n, 1\le j\le r} \in \mathrm{M}_{(n+1)\times r}(\mathbb{R})$ and $b = (\ell_0,\dots,\ell_n)$. For all $y\in\mathbb{R}^n$ we have that $y\in P$ if and only if the vector $(1,y)^t A = (\ell_1(y),\dots,\ell_r(y))$ has only non-negative entries.
If there is a vector $y'\in\mathbb{R}^{n+1}$ such that $y'^t b < 0$ and $y'^t A$ has only non-negative entries, then there also is a vector $y\in\mathbb{R}^n$ such that $(1,y)\in\mathbb{R}^{n+1}$ has the same properties because the first entry of the vector $b$ is $\ell_0 = 0$ and there are only non-negative coefficients in the first row of the matrix $A$: If the first entry $y_1'$ of $y'$ is non-positive, then we can put $y = (1,0,\dots,0)+y'$. If $y_1'$ is positive, we put $y' = \frac{1}{y_1'}y'$.
This point $y\in\mathbb{R}^n$ lies in the polyhedron $P$, but we have $\ell(y) = (1,y)^t b < 0$, which is a contradiction. So by Farkas's Lemma A.3.12 there is a vector $x\in\mathbb{R}^r$ which only has non-negative entries such that $Ax = b$, i.e. there are non-negative real numbers $x_1,\dots,x_r$ such that $\ell = x_1\ell_1+\ldots+x_r\ell_r$. □

**Remark 2.2.10.** (i) In the situation of Corollary 2.2.9, let $u\in P$ be such that $\ell(u) = 0$. Then we have $0 = \ell(u) = a_1\ell_1(u)+\ldots+a_r\ell_r(u)$ and because of $\ell_i(u)\ge 0$ and $a_i\ge 0$ for all $i = 1,\dots,r$ we get $a_i\ell_i(u) = 0$ from this equation. So those defining polynomials of $P$ which vanish in $u$ suffice to write $\ell$ as a linear combination with non-negative coefficients.
(ii) This is the reason for the failure of Corollary 2.2.9 if the polynomial $\ell$ is positive on $P$:
Let $P = \{(x,y)\in\mathbb{R}^2 : x\ge 0\}$ and $\ell = X+1$. Then we have $\ell|_P > 0$ and $\ell$ is no multiple of the defining polynomial $X$ of $P$.
This is of course no problem in our case because we are only interested in whether or

not the given polynomial lies in some quadratic module. And if it is strictly positive on $C$, we subtract the minimum, show then, that it is in the quadratic module and can afterwards add it again, as it is a positive constant, i.e. a square.

We will now define a short notion for the situation about the Lasserre-relaxation which we will be most interested in for the remainder of this work.

**Definition 2.2.11.** We say that the Lasserre-relaxation of $C$ by the quadratic module (resp. the preordering) is *exact* if there is a $k \in \mathbb{N}$ such that the closure of the Lasserre-relaxation of degree $k$ constructed by using $T_k = \mathrm{QM}(g_1,\dots,g_r)_k$ (resp. $T_k = \mathrm{PO}(g_1,\dots,g_r)_k$) equals the closure of the convex hull of $C$, i.e. if $\mathrm{cl}(C_k) = \mathrm{cl}(\mathrm{conv}(C))$.
The Lasserre-relaxation of $C$ of degree $k$ is called exact if we have $\mathrm{cl}(C_k) = \mathrm{cl}(\mathrm{conv}(C))$.

**Remark 2.2.12.** The Lasserre-relaxation of a fixed degree depends, in general, on the chosen defining polynomials for $C$. But as far as exactness is concerned, the statement whether or not there is a degree such that the Lasserre-relaxation of this degree is exact, depends only on the quadratic module which is generated by the chosen set of defining polynomials for $C$ (cf. Remark 2.1.2(ii)). Consider the following examples:
(i) Let $g = g_1 = 1 - X^2 \in \mathbb{R}[X]$. Then we have $C = [-1,1]$ and the Lasserre-relaxation of degree two will be exact by the identity

$$1 + X = \frac{1}{2}(1+X)^2 + \frac{1}{2}(1-X^2) = \frac{1}{2} + X + \frac{1}{2}X^2 + \frac{1}{2} - \frac{1}{2}X^2$$

because this identity implies that all linear polynomials which are non-negative on $C$ lie in $\mathrm{QM}(g)_2$ (we get $1-X$ by applying $X \mapsto -X$ to the above equation).
(ii) Consider again $g = g_1 = 1 - X^2 \in \mathbb{R}[X]$ and the Lasserre-relaxation constructed by using $\mathrm{QM}(g^3)$. Then it will not be exact, because $f := 1 + X \notin \mathrm{QM}(g^3)$. For if $f = Pg^3 + Q$ with sums of squares $P, Q \in \sum \mathbb{R}[X]^2$, then $f$ would have a zero of order at least two at the point $1 \in C$.
But the Lasserre-relaxations approximate $C$ as the degree grows: Let $d > 0$, then $1 - X^2 + d \in \mathrm{QM}(g^3)_N$ for all $N \geq C_1\sqrt{\frac{4(1+d)}{d}}\log\left(\frac{4(1+d)}{d}\right)$ (with a constant $C_1 > 0$) by [Ste96], Theorem 5. Therefore the polynomials $1 + X + \frac{d}{2}$ and $1 - X + \frac{d}{2}$ are also in $\mathrm{QM}(g^3)_N$ by

$$1 + X + d = \frac{1}{2}(1+X)^2 + \frac{1}{2}(1 - X^2 + d)$$

In the same paper, we also find lower bounds for the degree of the sums of squares needed to represent linear polynomials: If we have $1 + X + d \in \mathrm{QM}(g^3)_k$, then we also have by the invariance of $g$ under the coordinate transformation $X \mapsto -X$ (reflexion on the origin) $1 - X + d \in \mathrm{QM}(g^3)_k$. Hence we get $1 - X^2 + 2d + d^2 = (1+X+d)(1-X+d) \in \mathrm{QM}(g^3)_{2k}$. By [Ste96], Theorem 4, we get $2k \geq C_2\frac{1}{\sqrt{2d+d^2}}$ for a constant $C_2 > 0$.
(iii) Consider the set $C = [-1,1] \times \mathbb{R} \subset \mathbb{R}^2$ defined by $g = (1 - X_1^2)X_2^2$, then the Lasserre-relaxation of $C$ of arbitrary degree is $\mathbb{R}^2$ because no non-constant, linear polynomial lies in $\mathrm{QM}(g)$. For if $\ell = aX_1 + bX_2 + c \in \mathrm{QM}(g)$, then we must have $b = 0$ (or else $\ell$ would not be non-negative on $C$). Now write $\ell = \sigma_0 + \sigma_1 g$ with sums of squares $\sigma_0, \sigma_1 \in \mathbb{R}[X_1, X_2]$.

By substituting $X_2 = 0$ we get that $aX_1 + c$ is a sum of squares in $\mathbb{R}[X_1]$ and therefore we have $a = 0$, which means $\ell$ is constant.
If we take as defining polynomials $g_1 = 1 - X_1$ and $g_2 = 1 + X_1$ both in $\mathbb{R}[X_1, X_2]$, then obviously the Lasserre-relaxation of degree one will be exact. Yet, if we take as defining polynomial $f = 1 - X_1^2 \in \mathbb{R}[X_1, X_2]$, then we have $\mathrm{QM}(g_1, g_2) = \mathrm{QM}(f)$ (by $1 - X_1 = \frac{1}{2}(1 - X_1)^2 + \frac{1}{2}(1 - X_1^2)$ and analogously $1 + X_1 = \frac{1}{2}(1 + X_1)^2 + \frac{1}{2}(1 - X_1^2)$). But in order to get an exact Lasserre-relaxation in the case $\mathrm{QM}(f)$, we have to take the relaxation of degree two.

We will now show that the set $\mathfrak{L}_k$ is in fact a spectrahedron. So the Lasserre-relaxation is a constructive method which allows us to find a representation of the convex hull of a basic closed, semi-algebraic set as the projection of a spectrahedron, provided that it is exact.

**Lemma 2.2.13.** *Let $k \in \mathbb{N}$, put $g_0 = 1$ and $d_i = \max\{l \in \mathbb{N} \colon 2l + \deg(g_i) \le k\}$ (we assume $\deg(g_i) \le k$ for all $i = 1, \dots, r$). Put*

$$V = \bigoplus_{i=0}^{r} \mathbb{R}[X_1, \dots, X_n]_{d_i}$$

*and $U_k = \{L \in \mathbb{R}[X_1, \dots, X_n]_k^\vee : L(1) = 1\}$. Then the map*

$$\Phi \colon \mathbb{R}[X_1, \dots, X_n]_k^\vee \to \mathrm{Bil}^{sym}(V) \quad , \quad L \mapsto B_L$$
$$B_L((p_0, \dots, p_r), (q_0, \dots, q_r)) \quad = \quad \sum_{i=0}^{r} L(p_i q_i g_i)$$

*(recall that we have put $g_0 = 1$) is linear. Furthermore we have*
*(i) Let $L \in U_k$. Then $L$ is in $\mathfrak{L}_k$ if and only if $B_L$ is positive semi-definite.*
*(ii) If $k$ is even, then $\Phi$ is injective.*
*(iii) For all $L \in \ker(\Phi)$, we have $L(1) = 0$.*

PROOF. The linearity of the map $\Phi$ is obvious.
(i) If $L \in \mathfrak{L}_k$, then we have $B_L((p_0, \dots, p_r), (p_0, \dots, p_r)) = \sum_{i=0}^{r} L(p_i^2 g_i) \ge 0$ for all vectors $(p_0, \dots, p_r) \in V$. Conversely, if we have $L \in U_k$ and we know that $B_L$ is positive semi-definite, then we get $L \in \mathfrak{L}_k$ because of the identity

$$0 \le B_L((0, \dots, p, \dots, 0), (0, \dots, p, \dots, 0)) = L(p^2 g_i)$$

for all $p \in \mathbb{R}[X_1, \dots, X_n]_{d_i}$.
(ii) Take $L \in \mathbb{R}[X_1, \dots, X_n]_k$ such that $B_L = 0$, which implies that for all $i = 0, \dots, r$ and $p_i, q_i \in \mathbb{R}[X_1, \dots, X_n]_{d_i}$ we have $L(p_i q_i g_i) = 0$. By taking $i = 0$ we get $L(pq) = 0$ for all $p, q \in \mathbb{R}[X_1, \dots, X_n]_{\frac{k}{2}}$. As every monomial in $\mathbb{R}[X_1, \dots, X_n]_k$ of degree less than or equal to $k$ is the product of two monomials of degree at most $\frac{k}{2}$, this implies the claim.
(iii) This follows directly from $L(1) = B_L((1, 0, \dots, 0), (1, 0, \dots, 0)) = 0$. □

**Theorem 2.2.14.** *The set $\mathfrak{L}_k$ is a spectrahedron for all $k \in \mathbb{N}$.*

PROOF. As $\mathrm{QM}(g_1,\dots,g_r)_k = \mathrm{QM}(g_i : \deg(g_i) \le k)_k$, we can assume without loss of generality that $k \ge \max\{\deg(g_1),\dots,\deg(g_r)\}$. Now $\mathfrak{L}_k$ is the preimage of the cone of positive semi-definite bilinear forms on the real vector space $V$ by the map $\Phi$ both introduced in the above Lemma 2.2.13. By the alternative definition given in Remark 1.1.2, we see that $\mathfrak{L}_k$ is a spectrahedron for all $k \in \mathbb{N}$.
The case $T_k = \mathrm{PO}(g_1,\dots,g_r)_k$ is also covered because a truncated preordering is still a truncated quadratic module $\mathrm{PO}(g_1,\dots,g_r)_k = \mathrm{QM}(g^\alpha : \alpha \in \{0,1\}^r, \deg(g^\alpha) \le k)_k$. □

**Corollary 2.2.15.** *Let $C$ be a basic closed semi-algebraic set. The Lasserre-relaxation $C_k$ is the projection of a spectrahedron for every $k \in \mathbb{N}$.*
*In particular, if $C$ has an exact Lasserre-relaxation, which implies that $C$ is convex, then it is the projection of a spectrahedron.* □

**Remark 2.2.16.** This result is well known and is presented for example in [Sch05b] (cf. Lemma 24) or in [Mar08] (cf. 10.5.4(a)). There, the same construction is done by choosing the canonical basis of monomials on the ring of polynomials $\mathbb{R}[X_1,\dots,X_n]_k$ and by looking at $\mathrm{Sym}_{d\times d}(\mathbb{R})$ instead of $\mathrm{Bil}^{sym}(V)$. The linear matrix polynomial which is obtained by this choices of bases is the following (again we assume for simplicity $\deg(g_i) \le k$ for all $i = 1,\dots,r$): Identify a linear functional $L \in \mathbb{R}[X_1,\dots,X_n]_k^\vee$ with the sequence of its moments $(L(X^\alpha))_{|\alpha|\le k} = (a_\alpha)_{|\alpha|\le k}$. Every functional gives rise to a bilinear form as above

$$B_L : V \times V \to \mathbb{R},\ ((p_0,\dots,p_r),(q_0,\dots,q_r)) \mapsto \sum_{i=0}^{r} L(p_i q_i g_i)$$

This bilinear form is positive semi-definite if and only if the matrix

$$A_i := \left( \sum_{|\alpha|\le k} \gamma_\alpha^i a_{\alpha+\beta+\delta} \right)_{|\beta|,|\delta|\le d_i} \in \mathrm{M}_{d_i\times d_i}(\mathbb{R})$$

where we write $g_i = \sum_{|\alpha|\le k} \gamma_\alpha^i X^\alpha$, is positive semi-definite for all $i = 1,\dots,r$, which is nothing but Lemma 2.2.13(i). Thus we get the desired linear matrix inequality

$$\mathfrak{L}_k = \bigcap_{i=1}^{r} \left\{ (a_\alpha)_{|\alpha|\le k} : A_i((a_\alpha)) \ge 0 \right\} = \left\{ (a_\alpha)_{|\alpha|\le k} : A_1((a_\alpha)) \boxplus \dots \boxplus A_r((a_\alpha)) \ge 0 \right\}$$

and the matrices occurring in this inequality have dimension $d \times d$ for $d = \sum_{i=1}^{r} \binom{d_i+n}{d_i}$.

**Remark 2.2.17.** We can also give the construction of the Lasserre-relaxation completely free of coordinates:
Let $V$ be a finite-dimensional $\mathbb{R}$-vector space, $\dim(V) = n$. Then we have an isomorphism of the ring of polynomials $\mathbb{R}[X_1,\dots,X_n]$ (graded by degree) and the symmetric algebra $\mathrm{Sym}(V^\vee)$ over the dual space of $V$ as graded rings (cf. [Eis95], Corollary A2.3(c)). Denote by $\mathrm{ev}_x$ the evaluation at $x \in V$, which is a linear functional on $A := \mathrm{Sym}(V^\vee)$ defined on homogeneous elements as $\mathrm{ev}_x(t_1 \otimes \dots \otimes t_m) = t_1(x) \cdot \ldots \cdot t_m(x)$.
Fix elements $g_1,\dots,g_r \in A$ and set $C = \{x \in V : \mathrm{ev}_x(g_1) \ge 0,\dots,\mathrm{ev}_x(g_r) \ge 0\}$. For all $k \in \mathbb{N}$

we define $\mathfrak{L}_k$ to be the intersection of the dual cone of $T_k$ (where $T_k = \mathrm{QM}(g_1,\ldots,g_r)_k \subset \mathrm{Sym}_{\le k}(V^\vee)$ or $T_k = \mathrm{PO}(g_1,\ldots,g_r)_k$) and the affine space $U_k = \{L \in \left(\mathrm{Sym}_{\le k}(V^\vee)\right)^\vee : L(1) = 1\}$ (where $A_k := \mathrm{Sym}_{\le k}(V^\vee)$ denotes the subspace of elements of degree less than or equal to $k$ in $A$). Now we fix a surjective, linear map $\pi\colon A_k^\vee \to V$, $L \mapsto \pi(L)$ such that $\ell(\pi(L)) = L(\ell)$ for all elements $\ell$ of $A_1$ and all functionals $L \in A_k^\vee$. We will see below that such a map exists and that the equality $\pi(\mathrm{ev}_x) = x$ holds.
We define the Lasserre-relaxation of $C$ of degree $k$ to be $C_k := \pi(\mathfrak{L}_k)$ for all $k \in \mathbb{N}$.
By the property $\pi(\mathrm{ev}_x) = x$ we get $C \subset C_k$ for all $k \in \mathbb{N}$ and in fact, we also get Theorem 2.2.4:
For all $k \in \mathbb{N}$ we have

$$\mathrm{cl}(C_k) = \{x \in V : \forall\, \ell \in \mathrm{cl}(T_k), \deg(\ell) = 1 : \mathrm{ev}_x(\ell) \ge 0\}$$

The proof is completely analogous because we only used Proposition 2.2.3 and the two properties of $L \mapsto (L(X_1),\ldots,L(X_n))$ which we assumed for $\pi$.
In fact, there is only one projection $\pi\colon A_k^\vee \to V$ such that $\ell(\pi(L)) = L(\ell)$ holds for all $\ell \in A_1$ and all $L \in A_k^\vee$.
For take a basis $\{X_i\}_{i=1,\ldots,n}$ of $V^\vee$ and denote by $\{e_i\}_{i=1,\ldots,n}$ the dual basis on $V^{\vee\vee} \cong V$. Now take $L \in A_k^\vee$. Then we have $X_i(\pi(L)) = L(X_i)$. Now write $\pi(L) = \sum_{j=1}^n a_j e_j$. Then we get $a_i = X_i(\sum_{j=1}^n a_j e_j) = X_i(\pi(L)) = L(X_i)$ which means that the coordinate vector of $\pi(L)$ in the chosen basis is exactly $(L(X_1),\ldots,L(X_n))$ which is the projection used in Definition 2.2.1 to define the Lasserre-relaxation.
From this argument we also see that the property $\pi(\mathrm{ev}_x) = x$ is implied by the property $\ell(\pi(L)) = L(\ell)$.

# 3. The Exactness of the Lasserre-Relaxation

In this chapter, we will prove the exactness of the Lasserre-relaxation for a convex, compact, basic closed semi-algebraic set by imposing assumptions on the defining polynomials. As in the preceding chapter, we will fix throughout this chapter polynomials $g_1,\dots,g_r \in \mathbb{R}[X_1,\dots,X_n]$ and $g_0 = 1 \in \mathbb{R}[X_1,\dots,X_n]$ and mostly write $C$ for the basic closed semi-algebraic set defined by these polynomials. As we will always assume that the set $C$ is compact, we will restrict our attention to archimedean quadratic modules (and talk briefly about it before starting with the presentation of our results). This covers also the case of preorderings, since every preordering is a quadratic module and every preordering defining a compact set is archimedean by Schmüdgen's Positivstellensatz (cf. [PD01], Theorem 5.2.9). In the work ( [HN10]) of Helton and Nie, both cases are covered. As we will not present any result for preorderings which would be superior in any way to the analogous result for archimedean quadratic modules, we will not follow this course. The analoguous statements for preorderings are always obtained by substituting the quadratic module in the statement by the corresponding preordering without any other change to the assumptions, i.e. even though $\mathrm{PO}(g_1,\dots,g_r) = \mathrm{QM}(g_i g_j : i,j = 0,\dots,r)$ it is not necessary to assume the hypothesis for the pairwise products $g_i g_j$ of the defining polynomials $g_i$. The reason is that this is the case for the Matrix Positivstellensatz cited below (cf. [HN10], Theorems 27 and 29).
We begin this chapter by presenting that analogue for matrices of Putinar's Positivstellensatz for polynomials (cf. [PD01], Theorem 5.3.8), which will be an essential tool in the following proofs of theorems ensuring the exactness of the Lasserre-relaxation.

We will talk briefly about archimedean quadratic modules:

**Definition 3.0.1.** A quadratic module $\mathrm{QM} \subset \mathbb{R}[X_1,\dots,X_n]$ is called archimedean if there is an $N \in \mathbb{N}$ such that the polynomial

$$N - \sum_{i=1}^{n} X_i^2$$

lies in the quadratic module QM.

**Remark 3.0.2.** (i) Note that an archimedean quadratic module always defines a compact semi-algebraic set. The converse is not true in general. Yet it is true, if the quadratic module happens to be closed under multiplication because it is then a preordering and therefore archimedean by Schmüdgen's Positivstellensatz ( [PD01], Theorem 5.2.9).

(ii) To assume that the quadratic module $\mathrm{QM}(g_1,\dots,g_r)$ is archimedean is not a grave restriction if we assume anyway that $C = \{x \in \mathbb{R}^n : \forall\, f \in \mathrm{QM}(g_1,\dots,g_r) \colon f(x) \geq 0\}$ is compact: We can then consider the quadratic module $\mathrm{QM}(g_1,\dots,g_r,g_{r+1})$ with $g_{r+1} = N - \sum_{i=1}^{n} X_i^2$ and $N \in \mathbb{N}$ so large that $C \subset B(0,N)$. Obviously, this quadratic module is archimedean and it also defines the same set $C$.
(iii) The importance of this property lies in Putinar's Positivstellensatz (cf. [PD01], Theorem 5.3.8): It is the essential hypothesis which distinguishes Schmüdgen's Positivstellensatz, which is formulated for preorderings, from this more special case of quadratic modules.

Now, we will briefly summarise the facts we need to understand the analogue of Putinar's Positivstellensatz for matrices and then cite it:

**Definition 3.0.3.** The *degree* $\deg(F)$ of a matrix $F = (f_{ij})_{i,j} \in \mathrm{M}_{m\times n}(\mathbb{R}[X_1,\dots,X_n])$ is the maximum of the degrees of its entries, i.e. $\deg(F) = \max\{\deg(f_{ij}) \colon 1 \leq i \leq m, 1 \leq j \leq n\}$.

**Definition 3.0.4.** A symmetric matrix $F \in \mathrm{Sym}_{d\times d}(\mathbb{R}[X_1,\dots,X_n])$ is said to be a *sum of squares* if there is a $k \in \mathbb{N}$ and a matrix $G \in \mathrm{M}_{k\times d}(\mathbb{R}[X_1,\dots,X_n])$ such that $F = G^t G$.

**Remark 3.0.5.** If a matrix $F = (f_{ij}) \in \mathrm{Sym}_{d\times d}(\mathbb{R}[X_1,\dots,X_n])$ is a sum of squares, then the diagonal entries of $F$ are sums of squares $f_{ii} \in \sum \mathbb{R}[X_1,\dots,X_n]^2$ $(i = 1,\dots,d)$ of polynomials.

For the following definition, observe that $\mathrm{M}_{d\times d}(\mathbb{R}[X_1,\dots,X_n]) = \mathrm{M}_{d\times d}(\mathbb{R})[X_1,\dots,X_n]$, i.e. a square matrix polynomial is the same as a polynomial with square matrices as coefficients.

**Definition 3.0.6.** (a) Let $F \in \mathrm{M}_{d\times d}(\mathbb{R})$ be a square matrix. We write $\lambda_{max}(F)$ for the greatest absolute value of an eigenvalue of $F$, i.e. $\max\{|\lambda| \colon \lambda \in \mathbb{C}, \lambda \textit{ eigenvalue of } F\}$.
(b) For a square matrix $F = \sum_{\alpha\in\mathbb{N}_0^n} F_\alpha X^\alpha \in \mathrm{M}_{d\times d}(\mathbb{R}[X_1,\dots,X_n])$ with real coefficient matrices $F_\alpha \in \mathrm{M}_{d\times d}(\mathbb{R})$ we define the *norm* of $F$ to be

$$\|F\| = \max\left\{\lambda_{max}(F_\alpha)\frac{\alpha_1!\cdot\ldots\cdot\alpha_n!}{|\alpha|!} \colon \alpha \in \mathbb{N}_0^n\right\}$$

Next we cite the Positivstellensatz, we will use. A proof can be found in the given reference.

**Theorem 3.0.7** (cf. [HN10], Theorem 29)**.** *Let* $\mathrm{QM}(g_1,\dots,g_r)$ *be archimedean and let* $F \in \mathrm{Sym}_{d\times d}(\mathbb{R}[X_1,\dots,X_n])$ *be a symmetric matrix. Assume that there is a* $\delta > 0$ *such that for all* $x \in C$ *we have* $F(x) \geq \delta I_d > 0$*. Then there are a constant* $c > 0$ *and sums of squares* $G_i \in \mathrm{Sym}_{d\times d}(\mathbb{R}[X_1,\dots,X_n])$ *such that*

$$F = G_0 + g_1 G_1 + \ldots + g_r G_r$$

*and* $\deg(g_i G_i) \leq c\left(\deg(F)^2 n^{\deg(F)} \frac{\|F\|}{\delta}\right)^c$.

**Remark 3.0.8.** If $F(x) > 0$ is positive definite for all $x \in C$, then there exists such a $\delta > 0$ as in the hypothesis of the preceding theorem due to the compactness of $C$ and the fact that the eigenvalues of $F(x)$ are continuous functions of $x \in C$.

## 3.1. Basic Version of the Main Results

In this section, we prove a basic theorem on which all results coming up in this chapter will be reduced to. We begin with some technical preparations.

**Remark 3.1.1.** If we have two matrices $A=(a_{ij}), B=(b_{ij})\in \mathrm{Sym}_{d\times d}(\mathbb{R}[X_1,\dots,X_n])$ and the polynomials $F:=(Y_1,\dots,Y_d)A(Y_1,\dots,Y_d)^t=(Y_1,\dots,Y_d)B(Y_1,\dots,Y_d)^t$ are equal, where $Y_1,\dots,Y_d$ are indeterminates, then we already have $A=B$.
This is simply the well-known fact that there is a unique, symmetric matrix representing a quadratic form: The quadratic form in question is

$$F=\sum_{i,j=1}^{d}\frac{1}{2}(a_{ij}+a_{ji})Y_iY_j=\sum_{i,j=1}^{d}a_{ij}Y_iY_j=\sum_{i,j=1}^{d}b_{ij}Y_iY_j$$

The following lemma is very convenient if one wants to check whether a matrix is a sum of squares or not.

**Lemma 3.1.2.** *A symmetric matrix $A\in\mathrm{Sym}_{d\times d}(\mathbb{R}[X_1,\dots,X_n])$ is a sum of squares if and only if the polynomial $(Y_1,\dots,Y_d)A(Y_1,\dots,Y_d)^t\in\mathbb{R}[X_1,\dots,X_n,Y_1,\dots,Y_d]$ is a sum of squares.*

Proof. Write $\underline{Y}=(Y_1,\dots,Y_d)$. If $A$ is a sum of squares, then there is a matrix $G\in \mathrm{M}_{k\times d}(\mathbb{R}[X_1,\dots,X_n])$ such that $A=G^tG$. So the polynomial $f:=\underline{Y}A\underline{Y}^t=\underline{Y}G^tG\underline{Y}^t$ is a sum of squares in $\mathbb{R}[X_1,\dots,X_n,Y_1,\dots,Y_d]$.
Conversely, let $f$ be a sum of squares in $\mathbb{R}[X_1,\dots,X_n,Y_1,\dots,Y_d]$, say $f=f_1^2+\dots+f_k^2$. Since $f$ is homogeneous of degree 2 in the variables $Y_1,\dots,Y_d$, all polynomials $f_i$ in the above representation of $f$ are homogeneous of degree 1 with respect to the variables $Y_1,\dots,Y_d$. Let $B\in\mathrm{M}_{k\times d}(\mathbb{R}[X_1,\dots,X_n])$ be a matrix such that $B\underline{Y}^t=(f_1,\dots,f_k)^t$ (the $j^{th}$ row of $B$ is the coefficient vector of $f_j$ as a polynomial in the variables $Y_i$). Then we have $\underline{Y}A\underline{Y}^t=\underline{Y}B^tB\underline{Y}^t$ and therefore $A=B^tB$ is a sum of squares (cf. Remark 3.1.1). □

With the help of the above lemma we can prove:

**Lemma 3.1.3** (cf. [HN10], Lemma 7)**.** *Let $P\in\mathrm{Sym}_{d\times d}(\mathbb{R}[X_1,\dots,X_n])$ be a sum of squares and write $\underline{X}=(X_1,\dots,X_n)$. Then for all $u\in\mathbb{R}^n$ the matrix $F\in\mathrm{Sym}_{d\times d}(\mathbb{R}[X_1,\dots,X_n])$ defined by*

$$F=\int_0^1\int_0^t P(u+s(\underline{X}-u))\mathrm{d}s\mathrm{d}t$$

*is a sum of squares (the integration is to be understood entrywise).*

Proof. The matrix $F\in\mathrm{Sym}_{d\times d}(\mathbb{R}[X_1,\dots,X_n])$ is a sum of squares if and only if the polynomial $(Y_1,\dots,Y_d)F(Y_1,\dots,Y_d)^t$ is a sum of squares in $\mathbb{R}[X_1,\dots,X_n,Y_1,\dots,Y_d]$ (by Lemma 3.1.2). Write $\underline{Y}=(Y_1,\dots,Y_d)$ and $\underline{X}=(X_1,\dots,X_n)$.
By linearity of integration we have

$$\underline{Y}F\underline{Y}^t=\int_0^1\int_0^t \underline{Y}P(u+s(\underline{X}-u))\underline{Y}^t\mathrm{d}s\mathrm{d}t$$

Since $P$ is a sum of squares, so is the polynomial $f_u := \underline{Y}^t P(u+s(\underline{X}-u))\underline{Y} \in \mathbb{R}[s,\underline{X},\underline{Y}]$. Now fix a sum of squares representation of $f_u$, say $f_u = f_1^2 + \ldots + f_k^2$ with polynomials $f_1,\ldots,f_k \in \mathbb{R}[s,\underline{X},\underline{Y}]$. Let $A_u(s) \in \mathrm{Sym}_{N\times N}(\mathbb{R})$ (where $N = \binom{n+r}{r} + d$ is the sum of the number of monomials in $X_1,\ldots,X_n$ of degree $\leq r$ and the number of variables $Y_i$) be a positive semi-definite Gram matrix corresponding to the representation $f_u(s,\underline{X},\underline{Y}) = f_1(s,\underline{X},\underline{Y})^2 + \ldots + f_k(s,\underline{X},\underline{Y})^2$ (cf. [Sch08, Kapitel V], Satz 1.11) as a polynomial in the variables $\underline{X}$ and $\underline{Y}$, i.e. the entries in the $i$-th row of $A_u(s)$ are the coefficients of $f_i$ as a polynomial in $\underline{X}$ and $\underline{Y}$ and are therefore polynomial functions of the parameter $s \in \mathbb{R}$. We have

$$\begin{aligned} \underline{Y}F\underline{Y}^t &= \int_0^1 \int_0^t (1,X_1,\ldots,X_n^r,Y_1,\ldots,Y_d)A_u(s)(1,X_1,\ldots,X_n^r,Y_1,\ldots,Y_d)^t \mathrm{d}s\mathrm{d}t \\ &= (1,X_1,\ldots,X_n^r,Y_1,\ldots,Y_d)\int_0^1 \int_0^t A_u(s)\mathrm{d}s\mathrm{d}t(1,X_1,\ldots,X_n^r,Y_1,\ldots,Y_d)^t \end{aligned}$$

Note that the integral $\int_0^1 \int_0^t A_u(s)\mathrm{d}s\mathrm{d}t$ exists because we chose $A_u(s)$ in such a way that the coefficients are continuous (in fact polynomial) functions of $s$.
Since the chosen Gram matrix $A_u(s)$ is positive semi-definite for all values of $s \in \mathbb{R}$, the matrix $\int_0^1 \int_0^t A_u(s)\mathrm{d}s\mathrm{d}t$ is also positive semi-definite. Now it is a well known fact (it follows for example from principal axis transformation) that there is a matrix $D \in \mathrm{M}_{N\times N}(\mathbb{R})$ such that $D^tD = \int_0^1 \int_0^t A_u(s)\mathrm{d}s\mathrm{d}t$. Hence, the polynomial $\underline{Y}F\underline{Y}^t$ is a sum of squares in $\mathbb{R}[\underline{X},\underline{Y}]$ which implies the claim. □

Now to the last technical step we will need for now.

**Proposition 3.1.4.** *Let $A \in \mathrm{Sym}_{d\times d}(\mathbb{R}[X_1,\ldots,X_n])$, let $x,u \in \mathbb{R}^n$. Assume $A(u+s(x-u)) \geq 0$ for all $s \in [0,1]$ and $A(u) > 0$. Then the real matrix*

$$\int_0^1 \int_0^t A(u+s(x-u))\mathrm{d}s\mathrm{d}t \in \mathrm{Sym}_{d\times d}(\mathbb{R})$$

*is positive definite.*

PROOF. Linearity of integration implies that the matrix

$$F := \int_0^1 \int_0^t A(u+s(x-u))\mathrm{d}s\mathrm{d}t \in \mathrm{Sym}_{d\times d}(\mathbb{R})$$

is positive semi-definite.
Now let $z \in \mathbb{R}^n$ such that $z^tFz = 0$, i.e. $\int_0^1 \int_0^t z^tA(u+s(x-u))z\mathrm{d}s\mathrm{d}t = 0$. From the assumption $A(u+s(x-u)) \geq 0$, we get $z^tA(u+s(x-u))z = 0$ for all $s \in [0,1]$. From $A(u) > 0$ we therefore deduce $z = 0$, which means that $F$ is positive definite. □

**Definition 3.1.5.** A function $f : D \to \mathbb{R}$ on a convex set $D$ is called *concave* if the following inequality holds for all $x,y \in D$ and $t \in [0,1]$

$$f(x+t(y-x)) \geq f(x) + t(f(y)-f(x)) = (1-t)f(x) + tf(y)$$

(cf. sections A.1 and A.2 for more details)

Now we are ready to state and prove the main result of this section:

**Theorem 3.1.6.** *Let $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0,\dots,g_r(x) \geq 0\}$ be a compact set and suppose that its convex hull $C := \operatorname{conv}(S)$ has non-empty interior. Let $p_1,\dots,p_s \in \operatorname{QM}(g_1,\dots,g_r)$ and assume that there is an open set $U \subset \mathbb{R}^n$ with the property $\{x \in U : p_1(x) \geq 0,\dots,p_s(x) \geq 0\} = C$. Further assume that for all $i \in \{1,\dots,s\}$ one of the following two conditions holds:*

1. *The matrix $-p_i'' \in \operatorname{Sym}_{n\times n}(\mathbb{R}[X_1,\dots,X_n])$ is a sum of squares.*
2. *For all $x \in Z(p_i) \cap \operatorname{cl}(\operatorname{Ex}(C))$ the matrix $p_i''(x) \in \operatorname{Sym}_{n\times n}(\mathbb{R})$ is negative definite and $p_i$ is concave as a function on $C$.*

*If $\operatorname{QM}(g_1,\dots,g_r)$ is archimedean, then there is a number $N \in \mathbb{N}$ such that every linear polynomial $\ell \in \mathbb{R}[X_1,\dots,X_n]_1$ which is non-negative on $S$ (and therefore also non-negative on $C$) lies in the truncated quadratic module $\operatorname{QM}(g_1,\dots,g_r)_N$. In particular, the Lasserre-relaxation for $C$ is exact.*

**Remark 3.1.7.** (i) In the following proof, we will need the existence of Lagrange multipliers for a linear polynomial as an objective function of an optimisation problem over $C$ in a point, which lies in $\operatorname{cl}(\operatorname{Ex}(C))$. This hypothesis means for a semi-algebraic set $C = \{x \in \mathbb{R}^n : p_1(x) \geq 0,\dots,p_s(x) \geq 0\}$, which we will consider, that there are a minimiser $u \in \operatorname{cl}(\operatorname{Ex}(C))$ of the linear polynomial $\ell$ (note that by [Bar02], Corollary II.3.4, every linear polynomial has a minimiser on $C$ which is an extreme point of $C$) and non-negative scalars $\lambda_1,\dots,\lambda_r \in \mathbb{R}$, which we call Lagrange multipliers, such that the two following conditions hold

$$\begin{aligned} \lambda_i g_i(u) &= 0 \text{ for all } i = 1,\dots,r \\ \ell'(u) - \sum_{i=1}^{r} \lambda_i g_i'(u) &= 0 \end{aligned}$$

The first of these two equations is often referred to as the complementary slackness of Lagrange multipliers because it states simply the fact, that the Lagrange multiplier for every non-active polynomial at $u$ (i.e. $g_i(u) \neq 0$) is zero. In Appendix A.3, we will introduce so-called constraint qualifications (which is a name for sufficient conditions ensuring the existence of Lagrange multipliers). If for example the Slater constraint qualification or the Mangasarian-Fromowitz constraint qualification hold, this implies the existence of Lagrange multipliers (cf. A.3.6 or A.3.11). We will also give examples of sets which do not admit Lagrange multipliers for some objective function.
In our case, the existence of Lagrange multipliers is ensured by the so-called Mangasarian-Fromowitz constraint qualification:
Assume that $C$ is convex and has non-empty interior as in the hypothesis of the preceding theorem and let $u \in \partial C$. If the gradients at $u$ of all polynomials $p_i$ which vanish there are non-zero, then the Mangasarian-Fromowitz constraint qualification is met at $u$:
In every neighbourhood of $u$, there is a point $x$ such that $p_i(x) > 0$ for all $i = 1,\dots,s$, because the interior of $C$ is non-empty. This means that there is an $x \in C$ such that

$p_i'(u)(x-u) > 0$ if $p_i(u) = 0$ (this follows from Taylor expansion at $u$) which is exactly the Mangasarian-Fromowitz constraint qualification at $u$ (cf. Definition A.3.8(b)).
So, if for all $i = 1,\dots,s$ the gradient $p_i'(u)$ is non-zero for all $u \in \partial C$ where $p_i$ vanishes, then the Mangasarian-Fromowitz constraint qualification holds for all linear polynomials as objective functions.
But, since $p_i$ is concave on $C$ for all $i = 1,\dots,s$, this is the case by Remark A.1.3 and we therefore have Lagrange multipliers for all linear polynomials as objective functions.
(ii) We can substitute the assumption of $C$ having non-empty interior in the hypothesis of Theorem 3.1.6 by the assumption that Lagrange multipliers exist for all linear polynomials as objective functions in a minimiser in $\mathrm{Ex}(C)$. The interior of $C$ is of no other importance in the proof than assuring the existence of Lagrange multipliers by the Mangasarian-Fromowitz constraint qualification as just explained.

PROOF OF THEOREM 3.1.6. We want to start the proof with a technical step, which we will need later and which would only cloud the view to the core of the proof by its sheer length if we were to give it in the place where we need it: Let $K \subset \{1,\dots,s\}$ be the set of indices such that $p_i$ does not satisfy condition 1. We will show that there is a $\delta > 0$ such that the inequality

$$\delta I_n \le \int_0^1 \int_0^t -p_i''(u+s(x-u))\mathrm{d}s\mathrm{d}t =: A_i(x,u)$$

holds for all $u \in \mathrm{Z}(p_i) \cap \mathrm{cl}(\mathrm{Ex}(C))$, all $x \in S$ and all $i \in K$.
This claim holds because the matrices $A_i(x,u)$ are positive definite by Proposition 3.1.4 (applied to $A = -p_i''$) for all $i \in K$, all $x \in S$ and all $u \in \mathrm{Z}(p_i) \cap \mathrm{cl}(\mathrm{Ex}(C))$. Since the set $S \times (\mathrm{Z}(p_i) \cap \mathrm{cl}(\mathrm{Ex}(C))) \subset \mathbb{R}^n \times \mathbb{R}^n$ is compact for all $i \in K$, the existence of a $\delta_i > 0$ with the desired property follows because the lowest eigenvalue of $A_i(x,u)$ is a continuous function of $x$ and $u$. Now take $\delta = \min\{\delta_i : i \in K\}$ to be the smallest of these $\delta_i$.
By the Matrix-Positivstellensatz 3.0.7, there are for all $i \in K$ and all $u \in \mathrm{Z}(p_i) \cap \mathrm{cl}(\mathrm{Ex}(C))$ matrices $G_j^{(i)} \in \mathrm{Sym}_{n\times n}(\mathbb{R}[X_1,\dots,X_n])$ $(j = 0,\dots,r)$ which are sums of squares with degree bounds $\deg(G_j^{(i)} g_j) \le 2N_i - 2$ where

$$N_i = \max\left\{ l \in \mathbb{N} : \max\{\deg(g_1),\dots,\deg(g_r)\} \le 2l-2 \le c\left(\deg(A_i)^2 n^{\deg(A_i)} \frac{\|A_i\|}{\delta}\right)^c \right\}$$

such that

$$A_i(\underline{X},u) = G_0^{(i)} + \sum_{j=1}^r G_j^{(i)} g_j$$

Note that the degree of $A_i$ is bounded (independently of $u$) by the degree of $A_i(X,0)$. The norm $\|A_i(X,u)\|$ also depends on $u$, but it is a continuous function of $u$ (because it only involves the greatest eigenvalues of the coefficient matrices of the matrix polynomial $A_i(X,u)$) and we define $\|A_i\|$ to be the maximum of $\|A_i(X,u)\|$ over $u \in \mathrm{Z}(p_i) \cap \mathrm{cl}(\mathrm{Ex}(C))$.
By taking $N := \max\{N_i : i = 1,\dots,s\}$, we get a uniform degree bound for the matrices $G_j^{(i)}$ (this bound only depends on the dimension $n$, the maximal degree of $p_1,\dots,p_s$, the

norm of the matrices $A_i$ and on $\delta$).
Now we start with the essential argument:
Let $\ell \in \mathbb{R}[X_1,\dots,X_n]_1$ be a linear polynomial which is non-negative on $C$. By Remark 3.1.7, there is a minimiser $u \in \mathrm{Ex}(C)$ and there are Lagrange multipliers $\lambda_1,\dots,\lambda_s \geq 0$ such that

$$\begin{aligned} \ell'(u) &= \lambda_1 p_1'(u) + \ldots + \lambda_s p_s'(u) \\ \lambda_i p_i(u) &= 0 \text{ for all } i = 1,\dots,s \end{aligned}$$

We define the polynomial $f_\ell := \ell - \ell_* - \sum_{i=1}^s \lambda_i p_i \in \mathbb{R}[X_1,\dots,X_n]$ where $\ell_*$ denotes the minimum of $\ell$ on $C$. Note that we have $\ell \in \mathrm{QM}(g_1,\dots,g_r)_{N+N'}$ if we have $p_1,\dots,p_s \in \mathrm{QM}(g_1,\dots,g_r)_{N'}$ (such an $N' \in \mathbb{N}$ exists by hypothesis) and $f_\ell \in \mathrm{QM}(g_1,\dots,g_r)_N$ which we will now show:
By the properties of the Lagrange multipliers, the polynomial $f_\ell$ and its derivative vanish at $u$, i.e. $f_\ell(u) = 0$ and $f_\ell'(u) = 0$. So the following equality holds for all $x \in \mathbb{R}^n$ due to the fundamental theorem of calculus

$$f_\ell(x) = \int_0^1 \int_0^t f_\ell''(u + s(x-u))\mathrm{d}s\mathrm{d}t = \sum_{i=1}^s \lambda_i (x-u)^t \left( \int_0^1 \int_0^t -p_i''(u + s(x-u))\mathrm{d}s\mathrm{d}t \right)(x-u)$$

Put $I = \{i \in \{1,\dots,s\} \colon p_i(u) = 0\}$, the set of indices corresponding to active inequalities at $u$. Put $F_i = \int_0^1 \int_0^t -p_i''(u + s(\underline{X} - u))\mathrm{d}s\mathrm{d}t = A_i(\underline{X}, u) \in \mathrm{Sym}_{n\times n}(\mathbb{R}[X_1,\dots,X_n])$ (which is symmetric because the Hessian matrix of a polynomial is) for all $i \in I$ and for all $i \notin I$ put $F_i = 0 \in \mathrm{Sym}_{n\times n}(\mathbb{R}[X_1,\dots,X_n])$. By the complementary slackness of the Lagrange multipliers we have

$$f_\ell(x) = \sum_{i=1}^s \lambda_i (x-u)^t F_i(x)(x-u) = \sum_{i\in I} \lambda_i (x-u)^t F_i(x)(x-u)$$

Let $i \in I$. If $-p_i''$ is a sum of squares, then the matrix $F_i \in \mathrm{Sym}_{n\times n}(\mathbb{R}[X_1,\dots,X_n])$ is a sum of squares by Lemma 3.1.3.
Recall that we defined the set of indices $K := \{i \in I \colon -p_i'' \textit{ is not a sum of squares}\}$. By the preliminary step, which we have taken at the beginning of the proof, there are matrices $G_j^{(i)}$ $(j = 1,\dots,r)$ with a degree bound $N \in \mathbb{N}$ such that

$$F_i = G_0^{(i)} + \sum_{j=1}^r G_j^{(i)} g_j$$

and $\deg(G_j^{(i)} g_j) \leq N$ for all $j = 1,\dots,r$ and $i \in K$. Write

$$\begin{aligned} \sigma_0 &:= \sum_{i\in I\setminus K} \lambda_i (\underline{X} - u)^t F_i (\underline{X} - u) + \sum_{i\in K} \lambda_i (\underline{X} - u)^t G_0^{(i)} (\underline{X} - u) \\ \sigma_j &:= \sum_{i\in K} \lambda_i (\underline{X} - u)^t G_j^{(i)} (\underline{X} - u) \text{ for } j = 1,\dots,r \end{aligned}$$

Thus we get sums of squares $\sigma_j \in \sum \mathbb{R}[X_1,\dots,X_n]^2$ and end up with a representation $f_\ell = \sigma_0 + \sigma_1 g_1 + \ldots + \sigma_r g_r$ of $f_\ell$ with uniform (in $\ell$) degree bounds $\deg(\sigma_j g_j) \leq 2N$. So we

get $f_\ell \in \mathrm{QM}(g_1,\ldots,g_r)_{2N}$ as desired.
The second part of the claim about the exactness of the Lasserre-relaxation is simply an application of Corollary 2.2.6. □

**Remark 3.1.8.** This Theorem in particular states the following:
Let $C = \{x \in U : p_1(x) \geq 0,\ldots,p_s(x) \geq 0\}$ (for an open set $U \subset \mathbb{R}^n$) be a locally basic-closed semi-algebraic and compact set with non-empty interior such that the defining polynomials $p_1,\ldots,p_s$ satisfy the hypothesis of Theorem 3.1.6, i.e. condition 1 or 2. Then the Lasserre-relaxation for $C$ is exact for all archimedean quadratic modules $\mathrm{QM}(g_1,\ldots,g_r)$ such that the convex hull of the set $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0,\ldots,g_r(x) \geq 0\}$ is $C$.

As an immediate corollary, we get a first criterion for the exactness of the Lasserre-relaxation.

**Corollary 3.1.9** (cf. [HN10], Theorem 5 and 6)**.** *Let $C = \{x \in \mathbb{R}^n : g_1(x) \geq 0,\ldots,g_r(x) \geq 0\}$ be convex and compact with non-empty interior. Assume that one of the two following conditions holds for all $i \in \{1,\ldots,r\}$:*

1. *The matrix $-g_i''$ is a sum of squares in $\mathrm{Sym}_{n\times n}(\mathbb{R}[X_1,\ldots,X_n])$.*
2. *For all $x \in \mathrm{Z}(g_i) \cap \mathrm{cl}(\mathrm{Ex}(C))$ the matrix $g_i''(x) \in \mathrm{Sym}_{n\times n}(\mathbb{R})$ is negative definite and $g_i$ is concave as a function on $C$.*

*If $\mathrm{QM}(g_1,\ldots,g_r)$ is archimedean, then the Lasserre-relaxation (by the quadratic module) is exact.* □

**Remark 3.1.10.** This corollary implies [HN10], Theorem 1. There, the authors assume that the matrix $\sum_{i=1}^r \lambda_i g_i''(u_\ell)$ is negative definite for all linear polynomials $\ell$ and every minimiser $u_\ell \in C$ of $\ell$ on $C$ as well as every possible tuple of Lagrange multipliers for $\ell$ at $u_\ell$, which they need in order to make a compacity argument in the proof work. Since the defining polynomials are supposed to be concave and $C$ to have non-empty interior in the hypothesis of this theorem, we conclude from Remark A.1.3 that the gradient of $g_i$ does not vanish in any zero of $g_i$ on $C$.
In particular, if we take a zero $x \in C$ of $g_i$, then $(0,\ldots,1,\ldots,0)$ is a valid tuple of Lagrange multipliers for the linear functional $v \mapsto g_i'(x)v$, which minimises in $x$ because $\{v \in \mathbb{R}^n : g_i'(x)(v-x) \geq 0\}$ is a supporting hyperplane to $C$ at $x$. Therefore that hypothesis says that $g_i''(x)$ is negative definite. This is essentially condition 2 in the hypothesis of the above corollary 3.1.9.

**Example 3.1.11.** Consider $g = 1 - X_1^4 - X_2^4 - X_1^2X_2^2 \in \mathbb{R}[X_1,X_2]$ and the set $C = \{(x_1,x_2) \in \mathbb{R}^2 : g(x_1,x_2) \geq 0\}$. We have $g' = \begin{pmatrix} -4X_1^3 - 2X_1X_2^2 & -4X_2^3 - 2X_1^2X_2 \end{pmatrix}$ and the Hessian matrix of $g$ is

$$g'' = \begin{pmatrix} -12X_1^2 - 2X_2^2 & -4X_1X_2 \\ -4X_1X_2 & -12X_2^2 - 2X_1^2 \end{pmatrix}$$

Indeed, $-g_i''$ is a sum of squares: By Lemma 3.1.2 we need to show that the polynomial

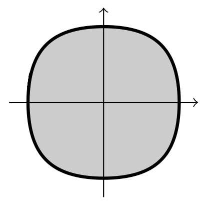

Figure 3.1.: The grey shaded area is the set $\{(x_1,x_2) \in \mathbb{R}^n : 1 - x_1^4 - x_2^4 - x_1^2 x_2^2 \geq 0\}$.

$$f := Y_1^2(12X_1^2 + 2X_2^2) + Y_2^2(12X_2^2 + 2X_1^2) + 8Y_1Y_2X_1X_2 = \begin{pmatrix} Y_1 & Y_2 \end{pmatrix}(-g'')\begin{pmatrix} Y_1 & Y_2 \end{pmatrix}^t$$

is a sum of squares in $\mathbb{R}[X_1,X_2,Y_1,Y_2]$. We show that this polynomial is non-negative on $\mathbb{R}^4$:
The polynomial $f$ is globally non-negative if the polynomial

$$\tilde{f} := \frac{f}{Y_1^2} = \left(\frac{Y_2}{Y_1}\right)^2 (12X_2^2 + 2X_1^2) + \left(\frac{Y_2}{Y_1}\right) 8X_1X_2 + 12X_1^2 + 2X_2^2 \in \mathbb{R}[\frac{Y_2}{Y_1}, X_1, X_2]$$

is non-negative on the whole of $\mathbb{R}^3$. We look at this polynomial as a polynomial in the variable $\frac{Y_2}{Y_1}$ and calculate its discriminant

$$\delta_{\tilde{f}} = 64X_1^2X_2^2 - 4(12X_2^2 + 2X_1^2)(12X_1^2 + 2X_2^2) = -512X_1^2X_2^2 - 16(6X_1^4 + 6X_2^4 + X_1^2X_2^2) \leq 0$$

As the leading coefficient of $\tilde{f}$ as a polynomial in $\frac{Y_2}{Y_1}$ is positive, we get that $\tilde{f}$ is non-negative on $\mathbb{R}^3$. From [CLR80], Theorem 7.1, which states roughly that every globally non-negative biform of degree (m,2) is a sum of squares, we know that $f$ is a sum of squares in $\mathbb{R}[X_1,X_2,Y_1,Y_2]$, so the Lasserre-relaxation for $C$ is exact by Corollary 3.1.9 above.
(The quadratic module $\mathrm{QM}(g)$ is archimedean because $C$ is compact and $\mathrm{QM}(g) = \mathrm{PO}(g)$ and therefore we have $N - X_1^2 - X_2^2 \in \mathrm{QM}(g)$ for all $N \in \mathbb{N}$ such that $N - x_1^2 - x_2^2 > 0$ for all $(x_1,x_2) \in C$ by Schmüdgen's Positivstellensatz (cf. [PD01], Theorem 5.2.9).)

We can now easily deduce the following result:

**Corollary 3.1.12.** *Let $C = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$ be compact, and assume that $\mathrm{QM}(g_1,\ldots,g_r)$ is archimedean. Then there is an $N \in \mathbb{N}$ such that for all $k \geq N$ the closure $\mathrm{cl}(C_k)$ of the Lasserre-relaxation of degree $k$ is compact.*

In [Las09], it is shown using Putinar's Positivstellensatz with results on degree bounds (proved in [NS07]) that under these hypothesis, the Lasserre-relaxations approximate $C$ in the Hausdorff-metric (cf. [Las09], Theorem 6). This is mainly a combination of this corollary and Corollary 2.2.8.

Proof. By the archimedean condition there is an $R > 0$ such that $f := R - \sum_{i=1}^n X_i^2 \in \mathrm{QM}(g_1,\ldots,g_r)$, i.e. $\mathrm{QM}(f) \subset \mathrm{QM}(g_1,\ldots,g_r)$. By Corollary 3.1.9 the Lasserre-relaxation for $\mathrm{cl}(B(0,R))$ is exact (because the negative Hessian matrix of $f$ is a sum of squares). Thus there is a $N \in \mathbb{N}$ such that for all $k \geq N$ we have $\mathrm{cl}(C_k) \subset \mathrm{cl}(B(0,R))$. □

## 3.2. Strictly Quasiconcave Polynomials

In this section, we will begin to exploit the fact that Theorem 3.1.6 allows us to substitute the defining polynomials. In order to do that in a helpful way, we need again some preparations.

**Proposition 3.2.1** (cf. [HN10], Lemma 13)**.** *For every constant $M > 0$ there exists a polynomial $q \in \mathbb{R}[X]$ in one variable such that for all $t \in [0,1]$ we have*

$$\begin{array}{rcl} q(t) & > & 0 \\ q(t) + q'(t)t & > & 0 \\ \dfrac{2q'(t) + q''(t)t}{q(t) + q'(t)t} & \leq & -M \end{array}$$

PROOF. We consider the function

$$h : \mathbb{R} \to \mathbb{R}, t \mapsto h(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}(M+1)^k t^k = \begin{cases} 1 & \text{, if } t = 0 \\ \frac{1-\exp(-(M+1)t)}{(M+1)t} & \text{, if } t \in (0,1] \end{cases}$$

This function is in $h \in \mathscr{C}^\infty([0,1]) = \mathscr{C}^\infty((0,1)) \cap \mathscr{C}([0,1])$ because

$$\lim_{t\to 0} h(t) = \lim_{t\to 0} \frac{1-\exp(-(M+1)t)}{(M+1)t} = \lim_{t\to 0} \frac{(M+1)\exp(-(M+1)t)}{M+1} = 1$$

by l'Hôpital's rule. Next, we calculate for $t \in (0,1)$

$$\begin{aligned} h(t) + th'(t) = (th(t))' &= \exp(-(M+1)t) \\ 2h'(t) + th''(t) = (h(t) + th'(t))' &= -(M+1)\exp(-(M+1)t) \end{aligned}$$

and put $q_N := \sum_{k=0}^{N} \frac{(-1)^k}{(k+1)!}(M+1)^k t^k$, i.e. the initial part of the series defining $h$ up to degree $N$. Obviously, the sequence of polynomials $(q_N)_{N\in\mathbb{N}}$ converges (and all its derivatives) pointwise to the function $h$ (and its derivatives) and, since $[0,1]$ is compact and the limit is differentiable of any order, it converges also in $\mathscr{C}^2([0,1])$ equipped with the norm

$$\|f\|_{\mathscr{C}^2} = \sum_{i=0}^{2} \frac{1}{i!} \max\{f^{(i)}(x) \colon x \in [0,1]\}$$

(i.e. the sequence $\{q_N^{(i)}\}_{N\in\mathbb{N}}$ converges uniformly in $\mathscr{C}([0,1])$ for $i = 0,1,2$). So there is for all $\varepsilon > 0$ an $N \in \mathbb{N}$ such that for all $t \in [0,1]$

$$\begin{array}{rcl} |q_N(t) - h(t)| & < & \varepsilon \\ |q_N'(t) - h'(t)| & < & \varepsilon \\ |q_N''(t) - h''(t)| & < & \varepsilon \end{array}$$

As $h$ satisfies the claimed properties and the third of them with $-M-1$ on the right hand side, so does $q_N$ for sufficiently large $N \in \mathbb{N}$. □

The assumption on the polynomial that we will need in order to substitute it in a reasonable way will be the following:

**Definition 3.2.2.** Let $D \subset \mathbb{R}^n$ be open. A twice differentiable function $f : D \to \mathbb{R}$ is called *strictly quasiconcave* in $x \in D$ if for all $y \in \mathbb{R}^n$, $y \neq 0$ such that $f'(x)y = 0$ the following inequality holds

$$y^t f''(x) y < 0$$

It is said to be *strictly quasiconcave* on $D$ if it is strictly quasiconcave in all points $x \in D$ of $D$.

**Remark 3.2.3.** Note that, if the gradient of the function $f$ in the above definition vanishes in the point $x \in D$, then the definition means simply that the Hessian of the polynomial in that point is negative definite.

**Lemma 3.2.4** (cf. [HN10], Proposition 10)**.** *Let $D \subset \mathbb{R}^n$ be a convex and compact set and let $g \in \mathbb{R}[X_1,\dots,X_n]$ be a polynomial which is strictly quasiconcave and non-negative on $D$. Then there is a polynomial $h \in \mathbb{R}[X_1,\dots,X_n]$ satisfying the following conditions:*
*(i) $h$ is positive on $D$, i.e. $h(x) > 0$ for all $x \in D$.*
*(ii) The Hessian matrix of the polynomial $p = gh$ is negative definite for all $x \in D$, i.e. $p''(x) < 0$ for all $x \in D$. In particular, $p$ is concave as a function on $D$.*

PROOF. Without loss of generality, we can assume that $g(D) \subset [0,1]$ (by scaling the coefficients of the polynomial). For any polynomial $q \in \mathbb{R}[X]$, we calculate the second derivative of $p = gq(g)$ by applying the product rule

$$\begin{aligned} p'' &= \big(q(g) + q'(g)g\big)\, g'' + \big(2q'(g) + q''(g)g\big)\, g'^t g' \\ &= \big(q(g) + q'(g)g\big) \left(g'' + \frac{2q'(g) + q''(g)g}{q(g) + q'(g)g} g'^t g'\right) \end{aligned}$$

We now see that if we choose $q$ to be a polynomial as in the above Proposition 3.2.1 with the constant $M$ given by Remark A.2.2 and making the right expression in brackets in the above equation a negative definite matrix, i.e. we choose $M$ greater than the maximum over all non-negative eigenvalues of $g''(x)$ for $x \in D$ (or any $M > 0$ if there are no non-negative eigenvalues of $g''(x)$), then we get the desired claims by putting $h := q(g)$. □

We can now prove the following

**Theorem 3.2.5** (cf. [HN10], Theorem 2)**.** *Let $C = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_r(x) \geq 0\}$ be convex and compact. Assume that one of the three following conditions is satisfied for all $i = 1,\dots,r$:*

1. *The matrix $-g_i'' \in \operatorname{Sym}_{n\times n}(\mathbb{R}[X_1,\dots,X_n])$ is a sum of squares.*

2. *For all $x \in \mathrm{Z}(g_i) \cap \mathrm{cl}(\mathrm{Ex}(C))$ the matrix $g_i''(x) \in \operatorname{Sym}_{n\times n}(\mathbb{R})$ is negative definite and $g_i$ is concave as a function on $C$.*

3. *The polynomial $g_i$ is strictly quasiconcave as a function on $C$.*

*If* $\mathrm{QM}(g_1,\dots,g_r)$ *is archimedean, then the Lasserre-relaxation for $C$ by the quadratic module* $\mathrm{QM}(g^\alpha : |\alpha| \le 2)$ *(i.e. we take the quadratic module generated by $g_1,\dots,g_r$ and all the products of two of them) is exact.*

PROOF. We substitute all defining polynomials which do not satisfy condition 1 or 2 by polynomials $p_i$ as guaranteed by Lemma 3.2.4. These polynomials lie in the quadratic module $\mathrm{QM}(g^\alpha : |\alpha| \le 2)$ because the polynomial $h_i$ in Lemma 3.2.4 is in $\mathrm{QM}(g_1,\dots,g_r)$, which is archimedean by hypothesis, by Putinar's Positivstellensatz (cf. [PD01], Theorem 5.3.8). We put $p_i = g_i$ if $g_i$ satisfies condition 1 or 2. Thus we get polynomials $p_1,\dots,p_r \in \mathrm{QM}(g^\alpha : |\alpha| \le 2)$ as in the hypothesis of Theorem 3.1.6 and so get the claim. □

**Remark 3.2.6.** (i) Condition 3 in the above Corollary 3.2.5 compares to condition 2 in points $x \in \mathrm{Z}(g_i) \cap \mathrm{cl}(\mathrm{Ex}(C))$, namely condition 2 implies condition 3 in these points. But in other points and foremost in interior points, they do not compare: As we show in Remark A.1.6(iii) and (iv), the assumption on a function to be concave or to be strictly quasiconcave on a convex set are logically independent of each other. We will take a closer look at the comparability of the three conditions in the above Corollary 3.2.5 in the following examples.
(ii) If we take the preordering generated by $g_1,\dots,g_r$ instead of the quadratic module $\mathrm{QM}(g_1,\dots,g_r)$ in the above corollary, then the Lasserre-relaxation by the same preordering $\mathrm{PO}(g_1,\dots,g_r)$ is exact because a preordering is stable under multiplication and archimedean if and only if the described set is compact (by Schmüdgen's Positivstellensatz, cf. [PD01], Theorem 5.2.9).

**Examples 3.2.7.** (i) We consider the polynomials $g_1 = 1 - X_1^2 \in \mathbb{R}[X_1,X_2]$ and $g_2 = 1 - X_2^2 \in \mathbb{R}[X_1,X_2]$ and the convex and compact set

$$C = \{(x_1,x_2) \in \mathbb{R}^2 : g_1(x_1,x_2) \ge 0, g_2(x_1,x_2) \ge 0\} = [-1,1] \times [-1,1]$$

We calculate the Hessian matrix of the two polynomials

$$\begin{aligned} g_1'' &= \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \\ g_2'' &= \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \end{aligned}$$

which immediately shows that they are both a sum of squares in $\mathrm{Sym}_{2\times 2}(\mathbb{R}[X_1,X_2])$. So we know by Corollary 3.1.9 as well as Theorem 3.2.5 that the Lasserre-relaxation by the quadratic module $\mathrm{QM}(g_1,g_2)$ (which is archimedean because of the identity $g_1 + g_2 = 2 - X_1^2 - X_2^2$) is exact.
But neither the polynomial $g_1$ nor the polynomial $g_2$ is strictly quasiconcave and none of them has a negative definite Hessian matrix if evaluated in points $x \in \mathrm{Z}(g_i) \cap \partial C$. So condition 1 in Corollary 3.2.5 is independent of conditions 2 or 3.

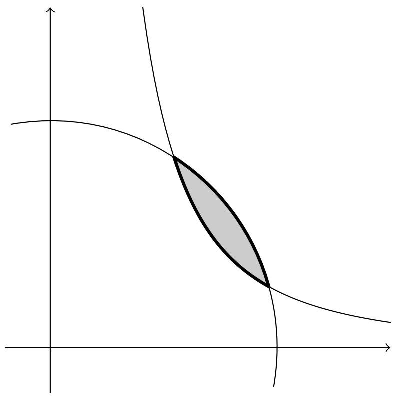

Figure 3.2.: The grey area defined in terms of polynomials by $X_1$, $1-X_1^2-X^2$ and $4X_1X_2-1$ is the projection of a spectrahedron. This example shows that Theorem 3.2.5 is more general than Corollary 3.1.9.

(ii) Here, consider $g_1 = X_1$, $g_2 = 1-X_1^2-X_2^2$, $g_3 = 4X_1^2X_2-1 \in \mathbb{R}[X_1,X_2]$, all as polynomials in two variables (cf. Figure 3.2). The Hessian matrices of the polynomials $g_1$ and $g_2$ are both sums of squares in $\mathrm{Sym}_{2\times 2}(\mathbb{R}[X_1,X_2])$ and the polynomial $g_3$ is strictly quasiconcave on the set $C := \{(x_1,x_2) \in \mathbb{R}^2 : g_1(x_1,x_2) \geq 0, g_2(x_1,x_2) \geq 0, g_3(x_1,x_2) \geq 0\}$. For the derivative of $g_3$ is $g_3'(x_1,x_2) = 4\begin{pmatrix} 2x_1x_2 & x_1^2 \end{pmatrix}$ and the Hessian matrix is

$$g_3''(x_1,x_2) = \begin{pmatrix} 8x_2 & 8x_1 \\ 8x_1 & 0 \end{pmatrix}$$

For all $x_1 > 0$ and $x_2 > 0$ we have: $g_3'(x_1,x_2)\begin{pmatrix} x_1^2 & -2x_1x_2 \end{pmatrix}^t = 0$ and

$$\begin{pmatrix} x_1^2 & -2x_1x_2 \end{pmatrix} g_3''(x_1,x_2) \begin{pmatrix} x_1^2 \\ -2x_1x_2 \end{pmatrix} = -24x_1^4x_2 < 0$$

Observing that $C$ is convex and compact and $\mathrm{QM}(g_1,g_2,g_3)$ archimedean (because of $g_2$), we get from Theorem 3.2.5 the exactness of the Lasserre-relaxation by the quadratic module. Corollary 3.1.9 does not give this result in this case because the Hessian matrix of $g_3$ is indefinite (the determinant is $\det(g_3''(x_1,x_2)) = -64x_1^2$).

(iii) Corollary 3.1.9 is also not strong enough to cover the following example of a spectrahedron: Take $g_1 = 1-X_1^2-X_2^2, g_2 = 4X_1X_2-1, g_3 = X_1 \in \mathbb{R}[X_1,X_2]$ (cf. Figure 3.3). Since the Hessian matrix of $g_2$ is indefinite, this polynomial is not concave. This set is a spectrahedron because it is the intersection of two spectrahedra, namely the disc $\mathrm{D}^2$ and $\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0, 4x_1x_2 - 1 \geq 0\}$. The latter is rigidly convex and therefore a spectrahedron by the Theorem of Helton and Vinnikov 1.1.7; a representation can also be given explicitly

$$\begin{pmatrix} 2x_1 & 1 \\ 1 & 2x_2 \end{pmatrix}$$

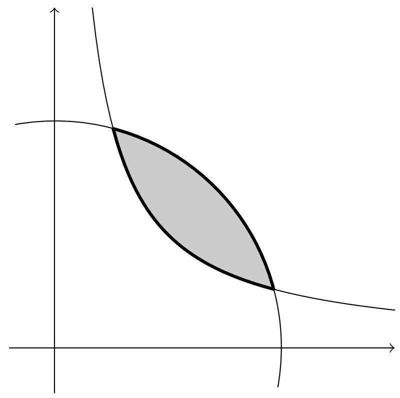

Figure 3.3.: The grey area is a spectrahedron defined in terms of polynomials by $X_1$, $1-X_1^2-X_2^2$ and $4X_1X_2-1$.

Since the polynomial $g_2$ is strictly quasiconcave in the first quadrant, we know at least by Theorem 3.2.5 that it is the projection of a spectrahedron.

## 3.3. Extension of the Boundary

We still fix a basic closed, semi-algebraic set $C \subset \mathbb{R}^n$ defined by $g_1, \dots, g_r \in \mathbb{R}[X_1, \dots, X_n]$. With the help of Theorem 3.2.5 and the local-global principle for projections of spectrahedra 1.3.5, we can easily prove the following

**Corollary 3.3.1.** *Let $C = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_r(x) \geq 0\}$ be compact and convex with non-empty interior. Assume that for all $i = 1, \dots, r$ one of the following conditions holds:*

1. *The matrix $-g_i''$ is a sum of squares in $\mathrm{Sym}_{d\times d}(\mathbb{R})[X_1, \dots, X_n]$.*

2. *The polynomial $g_i$ is strictly quasiconcave in every point $x \in \mathrm{Z}(g_i) \cap \partial C$.*

*Then the set $C$ is the projection of a spectrahedron.*

Proof. Condition 1 is a global condition and it suffices to observe that condition 2 is open which is done in A.2.1(iii): Since $C$ is compact, we can now cover the boundary of $C$ with a finite number of sets of the form $C \cap \mathrm{B}(x_i, r_i)$ for points $x_i \in \partial C$, $r_i > 0$, such that the hypothesis of Theorem 3.2.5 is met (note that the canonical defining polynomial of the disk $\mathrm{cl}(\mathrm{B}(x_i, r_i))$ satisfies condition 1). The archimedean condition on the quadratic module can be forced by adding the polynomial $N - \sum_{i=1}^n X_i^2$ for sufficiently large $N \in \mathbb{N}$ to the list of defining polynomials (again, note that this polynomial satisfies condition 1). So, by the local-global principle (cf. Theorem 1.3.5), we get the claim. □

We will now further exploit the possibility of replacing the defining polynomials by others which still lie in the quadratic module $\mathrm{QM}(g_1, \dots, g_r)$ generated by the defining polynomials of $C$ in order to improve this result. We would like to show that the

Lasserre-relaxation by the quadratic module $\mathrm{QM}(g_1,\dots,g_r)$ is exact, which has the advantage of knowing, in theory, how to construct a representation explicitly. In order to do this in practise, we might come to too high dimensions to calculate the representation. On the other hand, if we need to apply the local-global principle, then we first have to find a covering of the boundary by projections of spectrahedra, which is not an easier problem at all. Unfortunately, we will only get that the Lasserre-relaxation by the quadratic module $\mathrm{QM}(g^\alpha : |\alpha| \le 2)$ is exact (as in Theorem 3.2.5) and, which is worse, we will have to strengthen the hypothesis. We will prove the following

**Theorem 3.3.2.** *Let $C = \{x \in \mathbb{R}^n : g_1(x) \ge 0,\dots,g_r(x) \ge 0\}$ be compact and convex with non-empty interior. Assume that for all $i = 1,\dots r$ one of the following two conditions holds:*

1. *The matrix $-g_i''$ is a sum of squares in $\mathrm{Sym}_{d\times d}(\mathbb{R})[X_1,\dots,X_n]$.*
2. *The polynomial $g_i$ does not vanish in any interior point of $C$ and is strictly quasi-concave in every point $x \in \mathrm{Z}(g_i)\cap\partial C$. Further, there is an open set $U_i \subset \mathbb{R}^n$ such that $\mathrm{Z}(g_i)\cap\partial C \subset U_i \cap \mathrm{Z}(g_i)$ and $U_i \cap \mathrm{Z}(g_i)$ is a smooth and strictly convex hypersurface in $\mathbb{R}^n$ (in particular, $g_i'(x) \neq 0$ for all $x \in U_i \cap \mathrm{Z}(g_i)$).*

*If the quadratic module $\mathrm{QM}(g_1,\dots,g_r)$ is archimedean, then the Lasserre-relaxation for $C$ by the quadratic module $\mathrm{QM}(g^\alpha : |\alpha| \le 2)$ is exact.*

The condition that $U \cap \mathrm{Z}(g_i)$ is a strictly convex hypersurface is a technical condition that is necessary for the constructions in the proof to work. We will define it later (cf. Definition 3.3.3). Before we start with the rather complicated technical preparations, we will first give a short sketch of the idea of the proof:
We want to substitute the polynomials satisfying condition 2 as we did for the proof of Theorem 3.2.5 in Proposition 3.2.4, i.e. we want to find for every such polynomial a new polynomial that is strictly positive on $C$ such that the product of the two of them has negative definite Hessian in every $x \in C$. In order to substitute $g_i$, we will in a first step construct a convex and compact set $K_i$ with non-empty interior such that the intersection of the boundary of this set and the boundary of $C$ is $\mathrm{Z}(g_i)\cap\partial C$, the zero set of $g_i$ on $C$. We will make sure that the boundary of $K_i$ is a smooth and positively curved hypersurface (cf. appendix C.1 for definitions and some details). For this construction, we need the assumption of $U \cap \mathrm{Z}(g_i)$ being a strictly convex hypersurface.
In a second step, we will then construct a function $G_i \in \mathscr{C}^2(\mathbb{R}^n;\mathbb{R})$ which defines $K_i$ locally (i.e. $K_i = \{x \in U : G_i(x) \ge 0\}$ for a neighbourhood $U$ of $K_1$) and has negative definite Hessian in all points of $K_i$. By some technical conditions on $G_i$, we will make sure that the quotient $\frac{G_i}{g_i}$ is a well defined and twice differentiable function on an open set containing $K_i$. By using a Weierstraß approximation of this quotient, we will get the desired substitute for $g_i$.
Now, we will start with the construction of this compact and convex set $K_i$. We will do this in an abstract setting and need the following definition for that:

**Definition 3.3.3.** (a) A set $M \subset \mathbb{R}^n$ is called a *smooth hypersurface* in $\mathbb{R}^n$, if for every $p \in M$ there is an open neighbourhood $U \subset \mathbb{R}^n$ of $p$ and a function $f \in \mathscr{C}^\infty(U;\mathbb{R})$ such that $U \cap M = \{x \in U : f(x) = 0\}$ and $f'(x) \neq 0$ for all $x \in U$.
(b) A hypersurface $M$ is called *orientable* if there is a smooth unit normal vector field $N \to S^{n-1}$ on $M$, i.e. a map such that $\langle v, N(p)\rangle = 0$ for all tangent vectors $v \in T_pM$ and $p \in M$. An oriented hypersurface is a pair of an orientable hypersurface $M$ and a fixed unit normal vector field $N$ on $M$.
(c) An oriented hypersurface $(M,N)$ is called *strictly convex*, if for all $p \in M$ the following conditions are satisfied:

1. $M$ has positive curvature at $p$ (i.e. the second fundamental form of $M$ at $p$ is positive definite, cf. appendix C.1).
2. The intersection $M \cap \widehat{T}_pM$ of $M$ and the affine tangent space to $M$ at $p$ is $\{p\}$.
3. $M$ lies on one side of the affine tangent hyperplane $\widehat{T}_pM$ for every $p \in M$, i.e. $\langle q - p, n\rangle \geq 0$ for all $q \in M$ and a normal vector of $T_pM$ (with suitable sign).

**Remark 3.3.4.** (i) If we require in the following that a subset $M \subset \mathbb{R}^n$ is a strictly convex hypersurface, we implicitly assume that it is orientable and that we fixed a smooth unit normal vector field $N$ on $M$, i.e. we assume that we deal with an oriented hypersurface and often drop the fixed unit normal vector field in notation.
(ii) We will say that an oriented smooth hypersurface $(M,N)$ lies strictly on one side of its affine tangent hyperplanes, if conditions 2 and 3 of the above definition are satisfied.
(iii) From the fact that a strictly convex hypersurface is positively curved we know that it lies for every $p \in M$ to the side of its affine tangent hyperplane at $p$ defined by $N(p)$, i.e. $\langle q - p, N(p)\rangle \geq 0$ ($q \in M$): It is true locally because of the positive curvature of $M$ (cf. Remark C.1.17) and therefore it follows globally by the definition of a strictly convex hypersurface.
(iv) If $M$ is connected, then conditions 1 and 2 imply condition 3: Take $p \in M$ and consider the function

$$h_p : M \setminus \{p\} : \mathbb{R}, q \to \langle q - p, N(p)\rangle$$

which is continuous. Therefore, since $M \setminus \{p\}$ is connected, so is $h_p(M_p)$. But $h_p(q) \neq 0$ for all $q \in M \setminus \{p\}$ and $h_p(q) > 0$ for all $q$ in a neighbourhood of $p$ because $M$ is positively curved (cf. Remark C.1.17). That means that $h_p(M \setminus \{p\}) \subset (0,\infty)$, i.e. condition 3.
(v) We will often say that a set $S$ lies on the inner side (resp. outer side) of an affine tangent hyperplane $\widehat{T}_pM$ for an oriented smooth hypersurface $M$ with unit normal vector field $N : M \to \mathbb{R}^n$. By this we mean that $\langle s - p, N(p)\rangle \geq 0$ (resp. $\langle s - p, N(p)\rangle \leq 0$) for all $s \in S$. So in this manner of speaking, a strictly convex smooth hypersurface always lies on the inner side of its affine tangent hyperplanes.

**Lemma 3.3.5.** *Let $(M,N)$ be a strictly convex hypersurface in $\mathbb{R}^n$. For all $r \in \mathbb{R}$ we define the set*

$$M_r = \{p + rN(p) : p \in M\}$$

*and the corresponding map*

$$f_r \colon M \to M_r, p \mapsto p + rN(p)$$

*There is an* $r_0 > 0$ *such that for all* $r \in (-\infty, r_0]$ *the following assertions (i)-(iii) hold (we write* $p_r$ *for* $f_r(p)$*):*
*(i)* $M_r$ *is a smooth hypersurface, called a parallel hypersurface to* $M$*.*
*(ii)* $M_r$ *is positively curved.*
*(iii) The map* $N_r \colon M_r \to \mathbb{R}^n$*,* $N_r(p_r) = N(p)$ *defines a smooth unit normal vector field on* $M_r$*.*
*(iv) For all* $r < 0$ *the hypersurface* $M_r$ *lies strictly on one side of the hyperplanes* $\widehat{T}_x M_r$ *(*$x \in M_r$*) and*

$$\operatorname{dist}(p, M_r) = \operatorname{dist}(p, \widehat{T}_{p_r} M_r) = r$$

*(v) For every compact subset* $K \subset M$ *there is an* $r_1 > 0$ *such that for all* $r \in [0, r_1]$ *the set* $K_r := f_r(K) \subset M_r$ *lies strictly on one side of its tangent hyperplanes* $\widehat{T}_x M_r$ *(*$x \in K_r$*). Further we have*

$$\operatorname{dist}(p, K_r) = \operatorname{dist}(p, \widehat{T}_{p_r} M_r) = r$$

*for all* $p \in K$*.*
*In particular, the parallel hypersurface of a strictly convex and compact hypersurface is again strictly convex for a sufficiently small distance of the two hypersurfaces.*

In the following proof, we will need some basics in differential geometry, which are explained in Appendix C.1. We also need some notation: We denote by $D_v f(p)$ the directional derivative of a function $f \colon M_1 \to M_2$ of hypersurfaces in direction $v \in T_p M_1$ at $p$ (cf. Definition C.1.10) and by $L_p \colon T_p M \to T_p M$ the Weingarten map of a hypersurface $M$ at $p$ (cf. Definition C.1.12).

PROOF. (i) From the fact that $M$ is positively curved, we deduce that $D_v f_r(p) = I_n v + r D_v N(p) = (I_n - r L_p)(v)$, where $I_n$ denotes the $n \times n$ identity matrix and $v$ is considered as a vector in $\mathbb{R}^n$, is an invertible linear map from $T_p M$ to $T_{p_r} M_r$ if $r \in (-\infty, r_0]$ for some $r_0 > 0$. Therefore, by the inverse function theorem (cf. [Nar68], 2.2.10), the map $f_r$ is a local diffeomorphism, which implies that $M_r = f_r(M)$ is a smooth hypersurface in $\mathbb{R}^n$.
Claims (ii) and (iii) are special cases of a theorem proved in [Hic65], section 2.6 (see p. 35f).
(iv) This can be computed directly: Take $r < 0$ and take $p_r \in M_r$. We have for all $q_r \in M_r \setminus \{p_r\}$

$$\begin{aligned}
\langle q_r - p_r, N_r(p_r) \rangle &= \langle q + rN(q) - p - rN(p), N(p) \rangle \\
&= \langle q - p, N(p) \rangle + r \langle N(q) - N(p), N(p) \rangle \\
&= \langle q - p, N(p) \rangle + r \langle N(q), N(p) \rangle - r > 0
\end{aligned}$$

by the Cauchy-Schwartz inequality $|\langle N(q), N(p) \rangle| \leq \|N(q)\| \|N(p)\| = 1$. As for the distance: For all $q_r \in M_r$ we know that $\|q_r - p\| \geq r$ because $\|q_r - q\| = r$ and $M$ lies globally

on the inner side of the tangent hyperplane $\widehat{T}_q M$ whereas $q_r$ lies to the outer side. And on the other hand $\|p_r - p\| = r$ and thus we get the desired claim.
(v) The set $K$ lies locally strictly on one side of its tangent hyperplanes (as a subset of a positively curved hypersurface, cf. Remark C.1.17). Now, define

$$h_r : K \times K \to \mathbb{R}, (p,q) \mapsto h_r(p,q) = \langle q_r - p_r, N(p) \rangle$$

which is the signed distance of $q_r$ and $\widehat{T}_{p_r} M_r$ because $N_r$ is a unit normal vector field. From the fact that $K_{r_0}$ lies locally strictly on one side of the tangent hyperplanes and that $h_r$ is smooth for all $r \in (-\infty, r_0]$, we deduce that there is an open subset $U \subset K \times K$ containing the diagonal such that $h_{r_0}|_U \geq 0$ (and $h_{r_0}(p,q) = 0$ for $(p,q) \in U$ if and only if $p = q$). Since for $r < r_0$ we have

$$h_r(p,q) = \langle q_{r_0} - p_{r_0}, N(p) \rangle + (r - r_0)\langle N(q), N(p) \rangle - (r - r_0) \geq \langle q_{r_0} - p_{r_0}, N(p) \rangle = h_{r_0}(p,q)$$

by the Cauchy-Schwartz inequality $|\langle N(q), N(p) \rangle \leq \|N(q)\| \|N(p)\| = 1$, we conclude $h_r|_U \geq 0$ (and $h_r(p,q) = 0$ for $(p,q) \in U$ if and only of $p = q$) for all $0 < r \leq r_0$. Further, we conclude from the fact that $M$ is strictly convex that $h_0 > 0$ on $A := (K \times K) \setminus U$. This set is compact and therefore it follows that there is an $r_1 > 0$ such that for all $r \in [0, r_1]$ we have $h_r|_A > 0$ as well (because $h_r \to h_0$ as $r \to 0$). Everything put together, we have seen that $K_r$ lies globally strictly on one side of its tangent hyperplanes for $r \in [0, r_1]$.
Now the claim on the distances follows easily from this: Obviously, $\|p_r - p\| = r$ and thus $\operatorname{dist}(p, K_r) \leq r$. On the other hand, $K_r$ lies globally on the inner side of the tangent hyperplane $\widehat{T}_{p_r} M_r$ whereas $p$ lies on the outer side, i.e. $\|q_r - p\| \geq \|p_r - p\| = r$ for all $q_r \in K_r$. This gives the reverse inequality. □

**Definition 3.3.6.** Let $C \subset \mathbb{R}^n$ be a convex set with non-empty interior. We say that the *radii of curvature are bounded from below* by $r \in \mathbb{R}_{>0}$, if for all $p \in \partial K$ there is a $p' \in K$ such that $\operatorname{cl}(\mathrm{B}(p', r)) \subset K$ and $p \in \operatorname{cl}(\mathrm{B}(p', r))$.

With the following lemma we come close to finishing the first step, i.e. the construction of the convex extension of $C$ which we mentioned in the outline of the proof of Theorem 3.3.2.

**Lemma 3.3.7.** *Let $C \subset \mathbb{R}^n$ be a convex and compact set with non-empty interior. Let $M \subset \mathbb{R}^n$ be a strictly convex smooth hypersurface such that $M \cap \operatorname{int}(C) = \emptyset$ and $Z := \partial C \cap M$ is non-empty and compact. Further assume that $C$ lies strictly on one side of the affine hyperplane $\widehat{T}_p M$ for all $p \in M$ (i.e. $C$ lies on one side of the affine hyperplane $\widehat{T}_p M$ and $\widehat{T}_p M \cap C = \{p\}$ for all $p \in M$). Then there is a compact and convex set $K$ satisfying the following properties*

1. *$C \setminus Z \subset \operatorname{int}(K)$.*
2. *$\partial K \cap C = Z$.*
3. *The radii of curvature of $K$ are bounded from below by some $r > 0$.*

*4. There is an open subset $U \subset M$ such that $Z \subset U$, $Z \subset \mathrm{int}_U(U \cap K)$ and $U \cap \mathrm{int}(K) = \emptyset$.*

*Given an open set $O \subset \mathbb{R}^n$ with $C \subset O$, the set $K$ can be chosen to be contained in $O$.*

**Remark 3.3.8.** The hypothesis that $C$ lies strictly on one side of the affine tangent hyperplanes $\widehat{T}_pM$ in the above Lemma can be dropped after substituting $M$ by the intersection of $M$ and a suitable open set $U$ containing $C$, i.e. taking an open subset of $M$ containing $Z$ instead of the whole of $M$. A similar argument is given in the proof of Theorem 3.3.2 below - it can be easily adapted to see this.

PROOF OF LEMMA 3.3.7. Let $Z \subset U \subset M$ be an open set with compact closure (taken in $M$). Put

$$C_r = C \setminus \left( \bigcup_{0 \le s < r} U_s \right)$$

where $U_s$ denotes the parallel hypersurface $\{p + sN(p) : p \in U\}$ to $U$ defined in Lemma 3.3.5. We will show that this set is compact for all $r \in (0, \delta_1)$ and some $\delta_1 > 0$ and then use that to show that $C_r$ lies to the inside of the affine tangent hyperplanes $\widehat{T}_pM_r$ for all $p \in \mathrm{cl}(U_r) \subset M_r$ and all $r \in (0, \delta_2)$ (and some $\delta_2 > 0$). We will do this in part (i) of the following proof (cf. Figure 3.4). After having proved this (using mainly the implicit function theorem), we will be able to easily conclude the proof in part (ii):
(i) Since $C$ lies to the inside of the tangent hyperplanes to any point in $U$ and $U$ lies to the inside of the tangent hyperplane to any point in $U_s$ for $s < 0$, we have for all $a < 0$ and $r > 0$

$$C_r = C \setminus \left( \bigcup_{a < s < r} U_s \right)$$

In order to prove the compactness of $C_r$, we will show that $U_{(a,r)} := \bigcup_{a<s<r} U_s$ is open by using the implicit function theorem: Choose for every $p \in \mathrm{cl}(U)$ a neighbourhood $V_p \subset \mathbb{R}^n$ of $p$ and a function $g_p : V_p \to \mathbb{R}$ such that $\{x \in V_p : g_p(x) = 0\} = M \cap V_p$ and $N(p) = \frac{1}{\|g_p'(p)\|} g_p'(p)$. Then choose an open neighbourhood $W_p \subset V_p$ of $p$ with compact closure $\mathrm{cl}(W_p) \subset V_p$. By covering $\mathrm{cl}(U)$ by these neighbourhoods $W_p$ of $p$ and its compactness, we conclude $\mathrm{cl}(U) \subset W_{p_1} \cup \ldots \cup W_{p_m}$ for some $p_1, \ldots, p_m \in \mathrm{cl}(U)$ and some $m \in \mathbb{N}$. Put

$$\delta_1 = \max\left\{|\lambda| : \lambda < 0, \lambda \textit{ eigenvalue of } g''_{p_i}(y), y \in \mathrm{cl}(W_{p_i}), i = 1, \ldots, m\right\}^{-1}$$

and

$$c_1 = \max\left\{\lambda : \lambda > 0, \lambda \textit{ eigenvalue of } g''_{p_i}(y), y \in \mathrm{cl}(W_{p_i}), i = 1, \ldots, m\right\}^{-1}$$

if this set is non-empty and $c_1 = -\infty$ if this set is empty. Then for all $i = 1, \ldots, m$, the matrix $I_n + s g''_{p_i}(p)$ is positive definite for $s \in (-c_1, \delta_1)$ and $p \in \mathrm{cl}(W_{p_i})$ because $g''_{p_i}$ can be diagonalised simultaneously to $I_n$ and has at least $(n-1)$ negative eigenvalues which follows from the positive curvature of $M$ (cf. Proposition C.1.15).

Now, fix a point $p_{t_0} \in U_{t_0} \subset U_{(a,r)}$. Let $p \in U$ be a point such that $p + t_0 N(p) = p_{t_0}$. Let

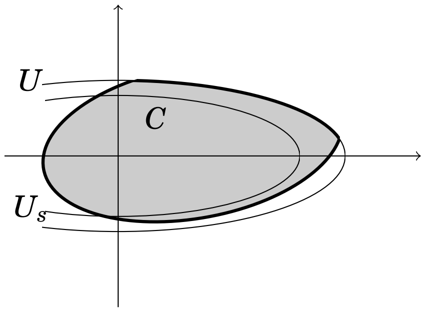


Figure 3.4.: Starting with the grey shaded area as our set $C$, we move $U$ to $U_s$ and the set $C_s$ as defined above is then the part of the grey shaded area which is limited by $U_s$.

$p_i \in \mathrm{cl}(U)$ be such that $p \in W_{p_i} =: W$ and $g := g_{p_i} : V_{p_i} \to \mathbb{R}$ be the function chosen above. Now, define the function

$$\Phi\colon \mathbb{R}^n \times W \times (-c_1, \delta_1) \to \mathbb{R}^{n+1}, (x, y, s) \mapsto \begin{pmatrix} y + sg'(y) - x \\ -g(y) \end{pmatrix}$$

The partial differential $(\mathrm{d}_2\Phi)$ of $\Phi$ with respect to the last $n+1$ variables, i.e. $(y, s)$, is

$$(\mathrm{d}_2\Phi)(x, y, s) = \begin{pmatrix} I_n + sg''(y) & -g'(y)^t \\ -g'(y) & 0 \end{pmatrix}$$

and the determinant is $\det((\mathrm{d}_2\Phi)(x,y,s)) = g'(y)\left(I_n + sg''(y)\right)^{-1}(g'(y))^t \det(I_n + sg''(y))$. Now put $B := \min\{\|g'_{p_i}(y)\| : y \in \mathrm{cl}(W_{p_i}), i = 1, \ldots, m\} > 0$ and take $a = -c_1 B$ and $r = \delta_1 B$. For $s_0 = \frac{t_0}{\|g'(p)\|}$ (the restrictions just made ensure $s_0 \in (-c_1, \delta_1)$) we have $\Phi(p_{t_0}, p, s_0) = 0$ and the rank of $(\mathrm{d}_2\Phi)(p_{t_0}, p, s_0)$ is full because $I_n + s_0 g''(p)$ is positive definite. By the implicit function theorem (cf. e.g. [Nar68], Theorem 1.3.5) there is a neighbourhood $O_1 \times O_2 \subset \mathbb{R}^n \times (W \times (-c_1, \delta_1))$ of $(p_{t_0}, p, s_0)$ such that for every $q_t \in O_1$ there is a unique $(q, s) \in O_2$ such that $\Phi(q_t, q, s) = 0$. That means simply that $q \in W \cap M$ and $q_t = q + sg'(q) = q + s\|g'(q)\|N(q)$ because $\frac{1}{\|g'(q)\|}g'(q) = N(q)$ by our choice of the sign of $g$ made above. Thus we conclude $q_t \in U_{s\|g'(q)\|} \subset U_{(a,r)}$ which implies that $U_{(a,r)}$ is open. Since $\delta_1$ can be chosen to be any number $0 < \delta \le \delta_1$ without changing the argument, we have seen that $U_{(a,r)}$ is open for $a = -c_1 B$ and arbitrary $0 < r \le \delta_1 B$. This in turn implies that $C_r$ is compact for all $0 < r \le \delta_1 B$.
In order to finish the first part, take $\delta$ to be the minimum of $r_0$, $\delta_1 B$ and the biggest positive number $r_1$ such that $\mathrm{cl}(U)_s$ lies strictly to the inside of the affine tangent hyperplanes $\widehat{T}_p M_s$ for all $s \in [0, r_1]$ (cf. Lemma 3.3.5(v)) and define the function

$$h_r\colon \mathrm{cl}(U) \times C_\delta \to \mathbb{R}, (p, x) \mapsto \langle x - p_r, N(p)\rangle$$

which is the signed distance of $x$ to $\widehat{T}_{p_r}M_r$ (at least for $r < r_0$ as in Lemma 3.3.5). Clearly, $h_0(p,x) > 0$ for all $(p,x) \in \mathrm{cl}(U) \times C_\delta$. Since $h_r$ converges pointwise to $h_0$ for $r \to 0$ and $\mathrm{cl}(U) \times C_\delta$ is compact, it converges uniformly and therefore there is a $\delta \geq \delta_2 > 0$ such that $h_r > 0$ on $\mathrm{cl}(U) \times C_\delta$ for all $r \in [0,\delta_2)$. This means that $C_\delta$ lies to the inside of $\widehat{T}_p M_s$ for all $p \in \mathrm{cl}(U)_s$ and $s \in [0,\delta_2]$. But, since

$$C_r = C_\delta \cup \left( \bigcup_{r \leq s < \delta} U_s \right)$$

this also implies that $C_r$ lies globally to the inside of the affine tangent hyperplanes $\widehat{T}_p M_r$ for all $p \in \mathrm{cl}(U)_r$ and all $r \in (0,\delta_2)$ because $U_s$ does for $s \in (r,\delta_2)$. This finishes the first part.
(ii) We will finish by the following construction: Fix some $r \in (0,\delta_2)$ and define $S = \mathrm{conv}(\mathrm{cl}(U)_r \cup C_r)$ and

$$K = S + \mathrm{cl}(B(0,r))$$

This set is convex (as the Minkowski sum of two convex sets) and compact (as the image of the compact set $S \times \mathrm{cl}(B(0,s))$ under the continuous map $+$) with non-empty interior (because the interior of the disc is non-empty). The radii of curvature of $K$ are bounded from below by $r$, which follows directly from the definition of the radius of curvature.
So it remains to prove properties 1, 2 and 4. We begin with property 1: Take $x \in C \setminus Z$. If $x \in S$, then $B(x,r) \subset K$ and it follows that $x$ is an interior point of $K$. If $x \notin S$, then there is an $s \in (0,r)$ such that $x \in U_s$. Then there is a $q \in U_r$ with $\mathrm{dist}(x,S) = \|q - x\| = r - s < r$ because $S$ contains $U_r$ and lies to the inside of its affine tangent hyperplanes. This means $x \in B(q,r) \subset \mathrm{int}(K)$. This shows property 1 and gives the inclusion $\partial K \cap C \subset Z$. The reverse inclusion is an easy distance argument: If $p \in U$, then $\mathrm{dist}(p,U_r) \geq \mathrm{dist}(p,S)$. From the fact that $S$ lies globally on the inside of the affine tangent hyperplanes $\widehat{T}_{p_r}U_r$ for all $p_r \in U_r$ we conclude equality $r = \mathrm{dist}(p,U_r) = \mathrm{dist}(p,S)$, which says $p \in \partial K$, since $\partial K = \{x \in \mathbb{R}^n : \mathrm{dist}(x,S) = r\}$. This gives the reverse inclusion for property 2, because $Z \subset U$, and proves property 4 because $U \subset K$ is open as a subset of $M$.
If we want to make sure that $K$ lies in an open set $O$ which contains $C$, then we have to slightly correct the construction: Let $d := \mathrm{dist}(\partial O, C) > 0$ be the distance of the boundary of $O$ and $C$ and put $U' := U \cap \{x \in \mathbb{R}^n : \mathrm{dist}(x,C) < \frac{d}{2}\}$. Then $\mathrm{conv}(\mathrm{cl}(U') \cup C) \subset \{x \in \mathbb{R}^n : \mathrm{dist}(x,C) \leq \frac{d}{2}\} = C + \mathrm{cl}(B(0,\frac{d}{2})) \subset O$. Take in step (ii) an $r \in (0,\min\{\delta_2,\frac{d}{2}\})$ and put as above $K = \mathrm{conv}(\mathrm{cl}(U')_r \cup C_r) + \mathrm{cl}(B(0,r))$. Then $K \subset \{x \in \mathbb{R}^n : \mathrm{dist}(x,C) < d\} \subset O$ since the distance function $d(.,C) \colon \mathbb{R}^n \to \mathbb{R}, x \mapsto \mathrm{dist}(x,C)$ is convex. □

Next, we finally complete the first step by the construction of the compact convex set with non-empty interior that we were talking about in the sketch of the proof above.

**Lemma 3.3.9.** *Let $K_1 \subset \mathbb{R}^n$ be a compact and convex set with non-empty interior and suppose that the radii of curvature of $K_1$ are bounded from below by $r > 0$. Let $M \subset \mathbb{R}^n$ be a strictly convex smooth hypersurface such that $M \cap \mathrm{int}(K_1) = \emptyset$ and $M \cap \partial K_1 \neq \emptyset$. Then, for every compact subset $Z \subset \mathrm{int}_M(M \cap \partial K_1)$ of the interior of $M \cap \partial K_1$ (in M), there is a*

*convex, compact set $K_2$ with non-empty interior such that $\partial K_2$ is a smooth and positively curved hypersurface and $Z \subset V \subset \partial K_2$ for an open set $V \subset M$.*
*Additionally, given a compact and convex set $C$ with $C \subset \operatorname{int}(K_1) \cup Z$ and an open set $O \subset \mathbb{R}^n$ with $K_1 \subset O$, we can choose the set $K_2$ such that $K_2 \subset O$, $C \subset K_2$ and $C \cap \partial K_1 = C \cap \partial K_2$.*

In [Gho04], a very similar result (cf. Proposition 3.3) is shown. In fact, we will use the same construction which is given there and only check that it is compatible with the additional properties, which we require here. In the proof, we will need the following result on convergence of a sequence of convex sets in the Hausdorff metric (for more information on the Hausdorff metric, cf. [Sch93], chapter 1.8):

**Remark 3.3.10.** Let $K_1$ be a compact and convex set with non-empty interior and $\{K_\varepsilon\}_{\varepsilon\in(0,1)}$ a family of compact and convex sets with non-empty interior which converges to $K_1$ in the Hausdorff metric, i.e. $\lim_{\varepsilon\to 0} K_\varepsilon = K_1$. Then the boundaries of the sets $K_\varepsilon$ converge to the boundary $\partial K_1$ of $K_1$:
For, if $x \in \partial K_1$ and $\delta > 0$, then $\mathrm{B}(x, 3\delta)$ contains a point $y$ with $\operatorname{dist}(y, K_1) = 2\delta$ (take a supporting hyperplane $H$ to $K_1$ at $x$ and take $y = x + 2\delta n$ for an outward unit normal vector of $H$) and so $y \notin K_\varepsilon$ for all $\varepsilon$ such that $d_H(K_\varepsilon, K_1) < \delta$. This yields that $\mathrm{B}(x, 2\delta) \cap \partial K_\varepsilon \neq \emptyset$ for small $\varepsilon$ which gives a sequence of points $\{x_\varepsilon\}$ such that $x_\varepsilon \in \partial K_\varepsilon$ and $x_\varepsilon \to x$ for $\varepsilon \to 0$. By [Sch93], Theorem 1.8.7, this means $x \in \lim \partial K_\varepsilon$. On the other hand, if $x \in \operatorname{int}(K_1)$ and $\delta > 0$ such that $\mathrm{B}(x, 3\delta) \subset K_1$ and suppose $x \in \lim \partial K_\varepsilon$, then there is an $\varepsilon' > 0$ such that for all $0 < \varepsilon \le \varepsilon'$ there is a $y_\varepsilon \in \partial K_\varepsilon$ with $\|x - y_\varepsilon\| < \delta$ and $d_H(K_\varepsilon, K_1) < \delta$. Again, take a supporting hyperplane to $K_\varepsilon$ at $y_\varepsilon$ and put $z_\varepsilon = y_\varepsilon + 2\delta n$ for the outward normal unit vector $n$ on the supporting hyperplane. Then $z_\varepsilon \in \mathrm{B}(y, 3\delta) \subset K_1$ and $d(z_\varepsilon, K_\varepsilon) = 2\delta > \delta$ which gives a contradiction to $d_H(K_\varepsilon, K_1) < \delta$.

Proof of Lemma 3.3.9. Let $h$ be the support function of $K_1$, i.e. the function $h_{K_1} : \mathbb{R}^n \to \mathbb{R}$, $p \mapsto \max\{\langle x, p\rangle : c \in K_1\}$ (cf. appendix C.2 for more details). We will use an integral transformation of $h$ to get the convex set $K_2$: For $\varepsilon > 0$ let $\theta_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}_{\ge 0}; \mathbb{R}_{\ge 0})$ be a smooth function with $\operatorname{supp}(\theta) \subset [\frac{\varepsilon}{2}, \varepsilon]$ and $\int_{\mathbb{R}^n} \theta_\varepsilon(\|x\|)\mathrm{d}x = 1$ (such a function exists, cf. [Nar68], Lemma 1.2.5). The integral transformation

$$\widetilde{h}_\varepsilon(p) := \int_{\mathbb{R}^n} h(p + \|p\| x)\theta_\varepsilon(\|x\|)\mathrm{d}x$$

of $h$ is a convex and positively homogeneous function (cf. [Sch93], Theorem 3.3.1) and thus determines a compact and convex set with non-empty interior $\widetilde{K}_\varepsilon$, the so-called Schneider transform of $K_1$ (cf. Theorem C.2.6, which is [Sch93], Theorem 1.7.1).
Put $U := \operatorname{int}_M(M \cap \partial K_1)$ and let $V \subset M$ be open with $Z \subset V$ and $\operatorname{cl}(V) \subset U$ compact. Put $U' = N(U)$ and $V' = N(V)$, where $N$ denotes the outward unit normal vector field on $M$, which makes $M$ negatively curved. Since $N : M \to S^{n-1}$ is a local diffeomorphism, these sets are open subsets of the sphere $S^{n-1} \subset \mathbb{R}^n$. Now, let $\overline{\phi} \in \mathscr{C}^\infty(S^{n-1}; \mathbb{R})$ be a smooth function with support $\operatorname{supp}(\overline{\phi}) \subset U'$ and $\overline{\phi}|_{V'} \equiv 1$ (again, the existence of such a function can be easily derived from [Nar68], Corollary 1.2.6, after applying e.g. a

suitable stereographic projection). Extend this function to the function $\phi\colon \mathbb{R}^n \to \mathbb{R}$ by

$$x \mapsto \begin{cases} \overline{\phi}(\frac{1}{\|x\|}x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Using the integral transform of $h$ defined above and this function $\phi$, we define $\overline{h}_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}$, $p \mapsto \overline{h}_\varepsilon(p) := \widetilde{h}_\varepsilon(p) + \phi(p)\big(h(p) - \widetilde{h}_\varepsilon(p)\big)$. We claim that this is the support function of a compact and convex set and that there is an $\varepsilon > 0$ such that this set, which is

$$K_\varepsilon := \{x \in \mathbb{R}^n : \forall\, p \in \mathbb{R}^n\ \langle x, p\rangle \leq \overline{h}_\varepsilon(p)\}$$

(cf. Remark C.2.7(i)) has all the claimed properties. Note that the function $\overline{h}_\varepsilon$ is in $\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\};\mathbb{R})$ because $\widetilde{h}_\varepsilon$ and $\phi$ are and the second summand only appears for $p \in \mathbb{R}^n$ such that $\frac{1}{\|p\|}p \in U'$ and $h$ is infinitely differentiable in such points by [Gho04] Lemma 3.1. By calculating

$$\begin{aligned} \frac{\partial}{\partial p_i}\widetilde{h}_\varepsilon(p) &= \int_{\mathbb{R}^n}\left(\frac{\partial}{\partial p_i}h(p + \|p\|x) + \frac{p_i}{\|p\|}\langle h'(p + \|p\|x), x\rangle\right)\theta_\varepsilon(\|x\|)\mathrm{d}x \\ \frac{\partial^2}{\partial p_i \partial p_j}\widetilde{h}_\varepsilon(p) &= \int_{\mathbb{R}^n}\left(\frac{\partial^2}{\partial p_i \partial p_j}h(p + \|p\|x) + \textit{terms of order } \geq 1 \textit{ in } x\right)\theta_\varepsilon(\|x\|)\mathrm{d}x \end{aligned}$$

we see that the derivatives of $\widetilde{h}_\varepsilon$ up to order 2 converge pointwise to the derivatives of $h$ for $\varepsilon \to 0$.

Now first, we want to show that $\overline{h}_\varepsilon$ is a support function. The fact that $h$ and $\widetilde{h}_\varepsilon$ are positively homogeneous and that $\phi(\lambda p) = \phi(\frac{\lambda}{\|\lambda p\|}p) = \phi(p)$ holds for all $\lambda > 0$, yield that $\overline{h}_\varepsilon$ is positively homogeneous, too. In order to show that $\overline{h}_\varepsilon$ is convex we check that the Hessian of $\overline{h}$ is positive semi-definite in all $p \in S^{n-1}$. We will do this in exactly the same way as in the proof of [Gho04], Proposition 3.3, using [Gho04], Lemmas 3.1 and 3.2. For $p \in \operatorname{cone}(S^{n-1} \setminus U')$ this follows from the fact $\overline{h}_\varepsilon|_{S^{n-1}\setminus U'} = \widetilde{h}_\varepsilon|_{S^{n-1}\setminus U'}$ and the convexity of $\widetilde{h}_\varepsilon$. Since $M$ is positively curved we know by [Gho04], Lemma 3.1, that $v^t h''(p)v > 0$ for all $v \in T_pS^{n-1}$ and all $p \in \operatorname{cl}(U')$. But by construction, the function $\overline{h}_\varepsilon(p)$ and its derivatives up to order 2 converge to $h(p)$ and its derivatives up to order 2 for all $p \in S^{n-1}$. In particular, this yields that $\overline{h}_\varepsilon$ converges to $h$ in the $\mathscr{C}^2$-norm on $\operatorname{cl}(U')$

$$\|f\|_{\mathscr{C}^2(\operatorname{cl}(U'))} = \sum_{|\alpha|\leq 2}\frac{1}{\alpha!}\max\left\{\left|\frac{\partial^\alpha}{\partial x^\alpha}f(x)\right| : x \in \operatorname{cl}(U')\right\}$$

So there is an $\varepsilon_0 > 0$ such that $v^t\overline{h}''_\varepsilon(p)v > 0$ for all $v \in T_pS^{n-1}$, all $\varepsilon \in (0, \varepsilon_0)$ and all $p \in \operatorname{cl}(U')$. By positive homogeneity of $\overline{h}_\varepsilon$ it follows that $\overline{h}''_\varepsilon(p)$ is positive semi-definite for all $p \in \operatorname{cl}(U')$:

$$0 = \frac{\mathrm{d}^2}{\mathrm{d}^2 t}t\overline{h}(p) = \frac{\mathrm{d}^2}{\mathrm{d}^2 t}\overline{h}(tp) = p^t\overline{h}''(tp)p$$

for all $t > 0$ and $p$ is a normal vector to the hyperplane $T_pS^{n-1}$. So $\overline{h}_\varepsilon$ is the support function of a compact and convex set with non-empty interior for all $\varepsilon \in (0, \varepsilon_0)$.

Next, we want to show that the boundary of the set defined by $\overline{h}_\varepsilon$ is a smooth and positively curved hypersurface for $\varepsilon \in (0,\varepsilon_0)$. Again by [Gho04], Lemma 3.1, we need to check that $\overline{h}_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\};\mathbb{R})$ and $v^t\overline{h}''_\varepsilon(p)v > 0$ for all $v \in T_pS^{n-1}$ and $p \in S^{n-1}$. The order of differentiability and the second condition for $p \in U'$ have already been done above. Since the functions $\overline{h}_\varepsilon$ and $\widetilde{h}_\varepsilon$ coincide in all $p \in \text{cone}(S^{n-1} \setminus U')$, it suffices to check this condition for $\widetilde{h}_\varepsilon$. This follows from [Gho04], Lemmas 3.1 and 3.2, by the hypothesis that the radii of curvature of $K_1$ are bounded from below by $r > 0$.
And finally we check that $Z \subset \partial K_\varepsilon$: Let $\overline{N}$ be the outward unit normal vector field on $\partial K_\varepsilon$. Since $V \subset \partial K_1$ is smooth, the support function at $p$ is $h(p) = \langle N^{-1}(p), p\rangle$ (cf. Remark C.2.12) and we conclude with the aid of [Sch93], Corollary 1.7.3, the following equality

$$N^{-1}(p) = h'(p)^t = \overline{h}'_\varepsilon(p)^t = \overline{N}^{-1}(p)$$

for all $p \in V'$ (note that the normal vector field is a local diffeomorphism because of the positive curvature of $M \supset V$). This implies $Z \subset V \subset \overline{N}^{-1}(V') \subset \partial K_\varepsilon$.
This completes the proof of the original version [Gho04], Proposition 3.3. We now check the addendum:
Let $C$ be a compact and convex set with non-empty interior such that $C \subset \text{int}(K_1) \cup Z$. Recall that we have chosen $Z \subset V \subset U = \text{int}_M(M \cap \partial K_1)$. We have just seen $V \subset \partial K_1 \cap \partial K_\varepsilon$ for all $\varepsilon \in (0,\varepsilon_0)$ and as $V$ is open in $M$ and therefore a set $V \subset M$ as claimed. It also implies that $\partial K_1 \setminus V$ and $\partial K_\varepsilon \setminus V$ are compact. Put $S := C \cap \partial K_1$. Obviously, $S \subset Z \subset \partial K_\varepsilon$ and therefore $C \cap (\partial K_1 \setminus V) = \emptyset$. The sets $\{K_\varepsilon\}$ converge to $K_1$ in the Hausdorff metric of sets because the support functions $\{\overline{h}_\varepsilon\}$ converge to the support function $h$ of $K_1$ (cf. [Sch93], Theorem 1.8.11). Since $K_\varepsilon$ and $K_1$ are convex, the boundaries also converge $\partial K_\varepsilon \to \partial K_1$ (cf. Remark 3.3.10). Therefore $\partial K_\varepsilon \setminus V$ converges to $\partial K_1 \setminus V$ and since $C \cap (\partial K_1 \setminus V) = \emptyset$ this implies $C \cap (\partial K_\varepsilon \setminus V) = \emptyset$ for all $0 < \varepsilon < \varepsilon_0$ and some $\varepsilon_0 > 0$. Thus we conclude $C \cap \partial K_\varepsilon = C \cap V$ from $V \subset \partial K_\varepsilon$ and this implies $S = C \cap V \subset \partial C \cap \partial K_\varepsilon \subset C \cap \partial K_\varepsilon = C \cap V \subset C \cap U = S = C \cap \partial K_1$ for $0 < \varepsilon < \varepsilon_0$.
It remains to show that $C \subset K_\varepsilon$: Suppose $C \not\subseteq K_\varepsilon$. Then $\text{int}(C) \not\subseteq \text{int}(K_\varepsilon)$ because $K_\varepsilon$ is closed. So take $x \in \text{int}(C) \cap \text{int}(K_\varepsilon)$, which is non-empty for $\varepsilon \in (0,\varepsilon_1)$ and some $0 < \varepsilon_1 < \varepsilon_0$, and $y \in \text{int}(C) \setminus K_\varepsilon$; then $\text{int}(C) \supset [x,y] \not\subseteq K_\varepsilon$ (where $[x,y]$ denotes the set $\{x + \lambda(y-x) \colon \lambda \in [0,1]\}$) and therefore $\emptyset \neq [x,y] \cap \partial K_\varepsilon \ni p$. It follows $p \in \text{int}(C)$, i.e. $p \notin V$ and therefore $C \cap (\partial K_\varepsilon \setminus V) \neq \emptyset$ which contradicts what we have just seen.
By choosing $\varepsilon$ maybe even smaller, we also satisfy the condition $K_2 \subset O$ for a given open set $K_1 \subset O \subset \mathbb{R}^n$ because $d_H(K_1, K_\varepsilon) \to 0$ $(\varepsilon \to 0)$. This finishes the proof of this Lemma. □

Taking the last three Lemmas together, we have completed the first step of our strategy to prove Theorem 3.3.2. So we turn to the second step and continue with the construction of the defining function $G_i$ of $K_i$ and the quotient $\frac{G_i}{g_i}$ as mentioned in the outline above.

**Definition 3.3.11.** Let $K \subset \mathbb{R}^n$ be a convex and compact set with non-empty interior and suppose that $0 \in \text{int}(K)$. The *Minkowski functional* to the set $K$ is defined to be the map

$$p_K \colon \mathbb{R}^n \to \mathbb{R},\ x \mapsto \begin{cases} \min\{\lambda > 0 \colon \frac{1}{\lambda}x \in K\} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

**Proposition 3.3.12.** *Let $K \subset \mathbb{R}^n$ be a compact and convex set with non-empty interior and smooth and positively curved boundary. Assume $0 \in \operatorname{int}(K)$ and denote by $p_K$ the Minkowski functional of $K$.*
*(i) There is an $\varepsilon > 0$ and an open set $U \subset \mathbb{R}^n$ such that $K \subset U$ and the function $G \colon \mathbb{R}^n \to \mathbb{R}$, $x \mapsto (1 - \varepsilon \sum_{i=1}^n x_i^2)(1 - p_K(x)^3)$ satisfies the following properties:*

1. *$G$ is twice differentiable everywhere, in fact $G \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\}) \cap \mathscr{C}^2(\mathbb{R}^n)$*
2. *$K = \{x \in U \colon G(x) \geq 0\}$ and $\partial K = \{x \in U \colon G(x) = 0\}$*
3. *$G$ is regular at all points $x \in \partial K$, i.e. $G'(x) \neq 0$.*
4. *The Hessian of $G$ is negative definite at all $x \in K$.*

*(ii) Let $K \subset O \subset \mathbb{R}^n$ be open and let $g \in \mathscr{C}^\infty(O;\mathbb{R})$. Write $\mathrm{Z}(g) = \{x \in O \colon g(x) = 0\}$. Let $W \subset O$ be an open set which contains $\operatorname{int}_{\partial K}(\partial K \cap \mathrm{Z}(g))$ such that $g(x) \neq 0$ for all $x \in W \setminus \partial K$. Further assume that $g'(x) \neq 0$ for all $x \in \partial K \cap \mathrm{Z}(g)$ and $G(x)g(x) \geq 0$ for all $x \in W$. Then the quotient*

$$w \colon W \to \mathbb{R},\ x \mapsto w(x) := \frac{G(x)}{g(x)}$$

*is well defined, positive and of class $w \in \mathscr{C}^2(W;\mathbb{R})$.*

PROOF. (i) Since the boundary of $K$ is assumed to be smooth, the Minkowski functional $p_K \colon \mathbb{R}^n \to \mathbb{R}$ of $K$ is of class $\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\};\mathbb{R}) \cap \mathscr{C}(\mathbb{R}^n;\mathbb{R})$ and $p_K'(x) \neq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ (cf. Lemma C.2.11). Therefore the function $\widetilde{G} \colon \mathbb{R}^n \to \mathbb{R}$, $x \mapsto (1 - p_K(x)^3)$ satisfies $\widetilde{G} \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\};\mathbb{R}) \cap \mathscr{C}(\mathbb{R}^n;\mathbb{R})$ and $\widetilde{G}'(x) \neq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. By $|p_K(x)| \leq c\|x\|$ for some $c > 0$ (take e.g. $c = \max\{|p_K(x)| \colon x \in \mathrm{S}^{n-1}\}$ and use positive homogeneity on both sides) we deduce from $p_K \in \mathscr{C}(\mathbb{R}^n;\mathbb{R})$ that $p_K^3$ is twice differentiable at $0 \in \mathbb{R}^n$, i.e. $p_K^3 \in \mathscr{C}^2(\mathbb{R}^n;\mathbb{R})$. We calculate the second derivative of $\widetilde{G} = \psi \circ p_K$ for $\psi \colon \mathbb{R} \to \mathbb{R}$, $t \mapsto 1 - t^3$:

$$\begin{aligned} \widetilde{G}' &= (\psi' \circ p_K)p_K', \text{ i.e. } \widetilde{G}'(x) = -3p_K(x)^2 p_K'(x) \\ \widetilde{G}'' &= (\psi'' \circ p_K)p_K'^t p_K + (\psi' \circ p_K)p_K'', \text{ i.e. } \widetilde{G}''(x) = -6p_K(x)\left(p_K'^t p_K'\right) - 3p_K(x)^2 p_K''(x) \end{aligned}$$

From the facts that $-p_K$ is strictly quasiconcave in every $x \in \mathbb{R}^n \setminus \{0\}$ (cf. Corollary C.2.13) and that $p_K''(x) \geq 0$ ($x \in \mathbb{R}^n$) we deduce that $\widetilde{G}''(x) < 0$ for all $x \in K \setminus \{0\}$ by Lemma A.2.1(iii) (note that the constant $M$ there can be chosen arbitrarily small because $p_K''$ is already positive semi-definite).
The origin can be dealt with in the following way: Define for all $\varepsilon > 0$ the function $G_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}$, $x \mapsto (1 - \varepsilon \sum_{i=1}^n x_i^2)\widetilde{G}(x)$. Then $G_\varepsilon''(0) = -2\varepsilon I_n$ because $\widetilde{G}'(0) = 0$ and $\widetilde{G}''(0) = 0$. On the other hand, there is for all $x \in K$ an $\varepsilon_x > 0$ such that for all $\varepsilon \in (0, \varepsilon_x)$ the matrix $G_\varepsilon''(y)$ is negative definite for all $y$ in a suitable neighbourhood of $x$; since $K$ is compact

this yields an $\varepsilon > 0$ such that $G''_\varepsilon(x) < 0$ for all $x \in K$ (and $U \subset \mathrm{B}(0, \frac{1}{\varepsilon})$). Then $G = G_\varepsilon$ is a function as we claimed to exist: Property 2 holds for $\widetilde{G}$ by properties of the Minkowski functional (cf. Remark C.2.3(iii)) and we have seen properties 1,3 and 4 for $\widetilde{G}$ and seen to it that none of them is disturbed by the multiplication with $(1 - \varepsilon \sum X_i^2)$.
(ii) Take $x \in \mathrm{int}_{\partial K}(\partial K \cap \mathrm{Z}(g))$ in the relative interior of $\partial K \cap \mathrm{Z}(g)$ as a subset of the hypersurface $\partial K$. Since $g'(x) \neq 0$ and $G'(x) \neq 0$ are both normal to $T_x \partial K$ and $G(x)g(x) \geq 0$ on $W$, they are collinear, i.e. $\frac{1}{\|g'(x)\|} g'(x) = \frac{1}{\|G'(x)\|} G'(x)$. By the implicit function theorem we can find an infinitely differentiable change of coordinates $\Phi \colon V \to \mathbb{R}^n$ for a neighbourhood $V \subset W$ of $x$ such that $\Phi \in \mathscr{C}^\infty(V; \mathbb{R}^n)$, $\Phi(x) = 0$, $\Phi(\partial K \cap V) = \{(\Phi_1(y), \dots, \Phi_{n-1}(y), 0) \colon y \in V \cap \partial K\}$ and $\Phi(\frac{1}{\|g'(x)\|} g'(x)) = a e_n$ for some $a \in \mathbb{R} \setminus \{0\}$. For, if $\varphi$ is an implicit solution of the equation $G(y) = 0$ in a neighbourhood $V$ of $x$ and we assume $\frac{\partial}{\partial x_n} G(x) e_n = G'(x) \neq 0$ after a linear change of coordinates, then $V \cap \partial K = \{(y_1, \dots, y_{n-1}, \varphi(y_1, \dots, y_{n-1})) \colon (y_1, \dots, y_{n-1}) \in \pi(V)\}$; now define $\Phi \colon V \to \mathbb{R}^n$ by $(y_1, \dots, y_n) \mapsto (y_1 - x_1, \dots, y_{n-1} - x_{n-1}, y_n - \varphi(y_1, \dots, y_n))$. By Taylor expansion at $x = 0$, we get in these new coordinates:

$$\begin{aligned} G(y) &= G(0) + G'(0)y + R_G(y) = G'(0)y + R_G(y) \\ g(y) &= g(0) + g'(0)y + R_g(y) = g'(0)y + R_g(y) \end{aligned}$$

with $G'(0) = a_G e_n$, $g'(0) = a_g e_n$, $a_G a_g > 0$ and

$$\begin{aligned} R_G(y) &= \frac{1}{2} y^t G''(\eta_G) y \\ R_g(y) &= \frac{1}{2} y^t g''(\eta_g) y \end{aligned}$$

for some $\eta_G, \eta_g \in [x, y]$. The remainder terms are $R_G, R_g \in \mathscr{C}^\infty(\Phi(V); \mathbb{R})$. We now write $y = (y', t)$ and sort the Taylor expansion of $G$ and $g$ by degree in $t$:

$$\begin{aligned} G(y) = G(y', t) &= G_1(y')t + G_2(y')t^2 \\ g(y) = g(y', t) &= g_1(y')t + g_2(y')t^2 \end{aligned}$$

where $g_1(0) = a_g \neq 0$ (and $G_1(0) = a_G \neq 0$). We do not get any constant terms in $t$ in the expansion of $g$ and $G$ for sufficiently small $t$ because $g(y', 0) = 0 = G(y', 0)$ for all $y'$ in a neighbourhood of the origin in $\mathbb{R}^{n-1}$. So it follows

$$\frac{G(y)}{g(y)} = \frac{G_1(y') + G_2'(y)t}{g_1'(y) + g_2'(y)t}$$

and numerator as well as denominator of this fraction are infinitely differentiable functions and the denominator does not vanish in $(y', t) = 0$. So the quotient is well-defined in a neighbourhood of 0 and as the quotient of $\mathscr{C}^\infty$-functions again a $\mathscr{C}^\infty$-function. It follows that $w$ is well-defined and infinitely differentiable at $x$. By $a_g a_G > 0$ we also get $w(x) > 0$. □

Finally, we have finished with the technical preparations for the proof of

**Theorem** (Theorem 3.3.2). *Let $C = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_r(x) \geq 0\}$ be compact and convex with non-empty interior. Assume that for all $i = 1, \dots r$ one of the following two conditions holds:*

1. *The matrix* $-g_i''$ *is a sum of squares in* $\mathrm{Sym}_{d\times d}(\mathbb{R})[X_1,\dots,X_n]$.

2. *The polynomial* $g_i$ *does not vanish in any interior point of* $C$ *and is strictly quasi-concave in every point* $x \in \mathrm{Z}(g_i)\cap\partial C$. *Further, there is an open set* $U_i \subset \mathbb{R}^n$ *such that* $\mathrm{Z}(g_i)\cap\partial C \subset U_i \cap \mathrm{Z}(g_i)$ *and* $U_i \cap \mathrm{Z}(g_i)$ *is a smooth and strictly convex hypersurface in* $\mathbb{R}^n$ *(in particular,* $g_i'(x) \neq 0$ *for all* $x \in U_i \cap \mathrm{Z}(g_i)$*).*

*If the quadratic module* $\mathrm{QM}(g_1,\dots,g_r)$ *is archimedean, then the Lasserre-relaxation for* $C$ *by the quadratic module* $\mathrm{QM}(g^\alpha : |\alpha| \le 2)$ *is exact.*

PROOF OF THEOREM 3.3.2. After an affine change of coordinates, we assume $0 \in \mathrm{int}(C)$. Fix an index $i \in \{1,\dots,r\}$ such that $g_i$ does not satisfy condition 1. We will construct a substitute for this polynomial as explained in the sketch of this proof above:
First, we need to show that $C$ lies strictly on the inner side of the affine tangent hyperplanes $\widehat{T}_p(U_i \cap \mathrm{Z}(g_i))$ ($p \in U_i \cap \mathrm{Z}(g_i) =: M_i$) for some maybe smaller open set $U_i \subset \mathbb{R}^n$. Suppose there is no such open set. Since $C$ lies strictly to the inner side of $\widehat{T}_p M_i$ for all $p \in \mathrm{Z}(g_i)\cap C =: Z_i$ ($C \subset \mathscr{S}(g_i)$ and $\mathscr{S}(g_i)$ lies locally strictly on one side of these tangent hyperplanes), there is a sequence $\{p_n\}_{n\in\mathbb{N}} \subset M_i \setminus Z_i$ such that $p_n \to p \in Z_i$ and $\widehat{T}_{p_n} M_i \cap C \neq \emptyset$. For all $n \in \mathbb{N}$, choose $x_n \in \widehat{T}_{p_n} M_i \cap C \neq \emptyset$. Since $C$ is compact, there is a convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$. For simplicity of notation, we assume that $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in C$. This limit lies in $\widehat{T}_p M_i$ because $0 = \langle x_n - p_n, N(p_n)\rangle \to \langle x-p, N(p)\rangle$. Therefore we have $x = p$. Since $\langle p - x_n, N(p)\rangle < 0$ ($\widehat{T}_p M_i$ is a supporting hyperplane to $C$) for all $n \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $\langle p - x_n, N(p_n)\rangle < 0$ ($\langle ., N(p_n)\rangle$ converges uniformly on $C$ to $\langle ., N(p)\rangle$) and from this we get $0 > \langle p - x_n, N(p_n)\rangle = \langle p - p_n, N(p_n)\rangle + \langle p_n - x_n, N(p_n)\rangle = \langle p - p_n, N(p_n)\rangle$ which is a contradiction to the hypothesis that $M_i$ is strictly convex (cf. Remark 3.3.4(iii)).
Therefore $C$ lies strictly on the inner side of $\widehat{T}_p M_i$ for all $p \in M_i$. Let $O \subset \mathbb{R}^n$ be an open set such that $\mathrm{Z}(g_i)\cap O = M_i = \mathrm{Z}(g_i)\cap U_i$ and $C \subset O$. Now apply Lemma 3.3.7 for $M = M_i$ and therefore $Z = Z_i$ and get a compact and convex set $K_1 \subset \mathbb{R}^n$ with the properties $C \subset K_1 \subset O$, $C \setminus Z_i \subset \mathrm{int}(K_1)$, $\partial K_1 \cap C = Z_i$ and the radii of curvature of $K_1$ are bounded from below by some $r > 0$.
To this set $K_1$ we apply Lemma 3.3.9 for $M = M_i \cap U = U$ for a set $U$ as given by property 4 of 3.3.7 and $Z = Z_i$ and end up with a compact and convex set $K_2$ such that $C \subset K_2 \subset O$, $Z_i = C \cap \partial K_1 = C \cap \partial K_2$ and the boundary of $K_2$ is a smooth and positively curved hypersurface in $\mathbb{R}^n$ which contains an open (in $\partial K_2$) neighbourhood of $Z_i$. We are now in a situation where we can apply Proposition 3.3.12: Take as $W$ a neighbourhood of $C$ with the required properties in the hypothesis there (e.g. $W = (\mathrm{int}(K)\cap\{x \in O : g(x) > 0\}) \cup \bigcup_{x\in V} W_x$ for suitable neighbourhoods of $x \in V \subset M$, where $G$ as well as $g$ change sign). Then we get a function

$$w : S \to \mathbb{R},\ x \mapsto \frac{G(x)}{g_i(x)}$$

which is well-defined on $S \supset C$, positive on $\mathrm{int}(K_2)\cup\mathrm{int}_{\partial K_2}(K_2 \cap M) \supset C$ and $\mathscr{C}^2(S;\mathbb{R})$. By the Weierstraß approximation theorem (cf. [Nar68], Theorem 1.6.2) there is for all $\varepsilon > 0$

a polynomial $h_\varepsilon \in \mathbb{R}[X_1,\dots,X_n]$ such that

$$\sum_{|\alpha|\le 2} \frac{1}{\alpha!} \max\left\{\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}(w-h)(x)\right| : x \in C\right\} = \|w-h_\varepsilon\|_{\mathscr{C}^2(C)} < \varepsilon$$

Since $\|G-h_\varepsilon w\|_{\mathscr{C}^2} \le \|G-wg_i\|_{\mathscr{C}^2} + \|g_i\|_{\mathscr{C}^2}\|w-h_\varepsilon\|_{\mathscr{C}^2}$ we can choose an $\varepsilon>0$ such that $h_\varepsilon(x)>0$ for all $x\in C$ ($w(x)>0$ for all $x\in C$) and the Hessian of $h_\varepsilon g_i$ is negative definite in all $x \in C$ (the Hessian of $G$ is negative definite on $C$). Put $h_i := h_\varepsilon$ for such an $\varepsilon>0$ and put $p_i := h_i g_i$. Note that $h_i$ is in $\mathrm{QM}(g_1,\dots,g_r)$, which is archimedean, by Putinar's Positivstellensatz ( [PD01], Theorem 5.3.8) and therefore $p_i \in \mathrm{QM}(g^\alpha : |\alpha| \le 2)$.
We do this for all $i \in \{1,\dots,r\}$ such that $g_i$ does not satisfy 1 and put $p_i := g_i$ for all $i$ such that $g_i$ satisfies 1. Then the hypothesis of our basic version Theorem 3.1.6 is met and we conclude that the Lasserre-relaxation by the quadratic module $\mathrm{QM}(g^\alpha : |\alpha| \le 2)$ is exact. □

**Remark 3.3.13.** The author conjectures that the following is true: Assume that $C = \{x \in \mathbb{R}^n : g_1(x) \ge 0,\dots,g_r(x)\ge 0\}$ is compact and convex with non-empty interior. Further assume that $g_i$ does not vanish in any interior point of $C$ and is strictly quasiconcave as a function on $C \cap \mathrm{Z}(g_i)$. Then there is an open set $U_i \subset \mathbb{R}^n$ containing $C$ such that $U_i \cap \mathrm{Z}(g_i)$ is a strictly convex hypersurface.
If that were true, it would greatly simplify condition 2 of the hypothesis in Theorem 3.3.2, because it says mainly that the first part of this condition 2, which is relatively easy to check, implies the second.
It is clear, that there is an open set $V_i$ containing $C$ such that $V_i \cap \mathrm{Z}(g_i)$ is a positively curved and smooth hypersurface (because this is an open condition on $g_i''$, resp. $g_i'$) and therefore locally on the inside of its affine tangent hyperplanes. The problem is the global condition, i.e. conditions 2 and 3 of Definition 3.3.3, in the case that $C \cap \mathrm{Z}(g_i)$ is not connected.

Unfortunately, these results do not cover all sets which admit an exact Lasserre-relaxation, as the following example (due to Netzer, Plaumann and Schweighofer) shows.

**Example 3.3.14** (cf. [NPS10], Example 3.7)**.** (i) Consider $g_1 = X_1$, $g_2 = 1-X_2$, $g_3 = X_2 - X_1^{2n+1} \in \mathbb{R}[X_1,X_2]$ for some $n \in \mathbb{N}$ and the set $C = \{(x_1,x_2)\in\mathbb{R}^2 : g_1(x_1,x_2)\ge 0, g_2(x_1,x_2) \ge 0, g_3(x_1,x_2)\ge 0\}$ (cf. Figure 3.5) which is convex and compact.
The polynomial $g_3$ is the interesting thing here: It is concave as a function on $C$ but its Hessian matrix evaluated in the origin $(0,0) \in \mathrm{Z}(g_3)\cap \partial C$ is the zero matrix. Thus the polynomial is not strictly quasiconcave on $C$ either and $-g_3''$ is not a sum of squares because the first diagonal entry $-\frac{\partial^2}{\partial X_1^2} g_3 = (2n+1)2nX_1^{2n-1}$ of $-g_3''$ is not a sum of squares. So this polynomial does not satisfy any of the assumptions in the above corollaries on the exactness of the Lasserre-relaxation and yet the Lasserre-relaxation is exact as can be shown by direct calculation:

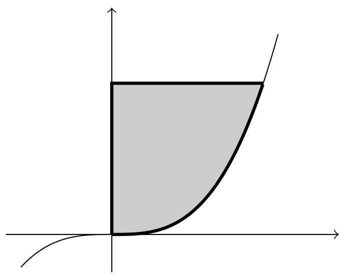

Figure 3.5.: The grey area is a projection of a spectrahedron defined in terms of polynomials by $X_1$, $1-X_2$ and $X_2-X_1^3$. It has an exact Lasserre-relaxation although this cannot be deduced from any result in this work.

The tangent $\ell_a$ to $\mathrm{Z}(g_3)$ at $(a, a^{2n+1})$ is

$$\begin{aligned}\ell_a &= \frac{\partial g_3}{\partial X_1}(a,a^{2n+1})(X_1-a)+\frac{\partial g_3}{\partial X_2}(a,a^{2n+1})(X_2-a^{2n+1})\\ &= -(2n+1)a^{2n}(X_1-a)+X_2-a^{2n+1}\\ &= X_1^{2n+1}-(2n+1)a^{2n}X_1+2na^{2n+1}+(X_2-X_1^{2n+1})\end{aligned}$$

The polynomial $f := X_1^{2n+1}-(2n+1)a^{2n}X_1+2na^{2n+1} \in \mathbb{R}[X_1]$ in one variable is non-negative on $[0,\infty)$ as it attains its minimum on $[0,\infty)$ at $X_1 = a$ ($f' = (2n+1)X_1^{2n}-(2n+1)a^{2n}$ and $f'' = (2n+1)2nX_1^{2n-1}$) and $f(a) = 0$. Now [KMS05], Theorem 4.1, tells us that $f \in \mathrm{QM}(X_1)_{2n+1} = \mathrm{PO}(X_1)_{2n+1} \subset \mathbb{R}[X_1]_{2n+1}$ because $\{X_1\}$ is the natural set of generators for $[0,\infty)$. Thus we get $\ell_a \in \mathrm{QM}(g_1,g_2,g_3)_{2n+1} \subset \mathrm{PO}(g_1,g_2,g_3)_{2n+1} \subset \mathbb{R}[X_1,X_2]_{2n+1}$.
This gives the exactness of the Lasserre-relaxation of degree $2n+1$ because by Theorem 2.2.4 we have

$$\mathrm{cl}(C_{2n+1}) \subset g_1^{-1}[0,\infty)\cap g_2^{-1}[0,\infty)\cap\bigcap\{\ell_a^{-1}[0,\infty): a\in[0,1]\} = C$$

From Corollary 2.2.6 we know that all linear polynomials which are non-negative on $C$ lie in $\mathrm{QM}(g_1,g_2,g_3)_{2n+1}$. This can also be seen by using Farkas's Lemma:
Take a linear polynomial $\ell \in \mathbb{R}[X_1,X_2]_1$ which is non-negative on $C$ and satisfies $\ell(u) = 0$ for some $u \in C$. If $\ell$ is equal to $g_1$, $g_2$ or any of the $\ell_a$ for $a \in [0,1]$, then we have already seen $\ell \in \mathrm{QM}(g_1,g_2,g_3)_{2n+1}$. Otherwise, we have $u \in \{(0,0),(0,1),(1,1)\}$. If $u = (0,0)$, then the polynomial $\ell$ is non-negative on the polyhedron $g_1^{-1}([0,\infty))\cap\ell_0^{-1}([0,\infty))$ and by Corollary 2.2.9, $\ell$ is a positive combination of $g_1$ and $\ell_0$ and in particular also in $\mathrm{QM}(g_1,g_2,g_3)_{2n+1}$. The other two cases for $u$ are completely analogous.
(ii) If we take in part (i) instead of $g_3$ the polynomial $X_2-X_1^{2n} \in \mathbb{R}[X_1,X_2]$ for some $n\in\mathbb{N}$, then the Hessian matrix of this polynomial is a sum of squares and the Lasserre-relaxation for $C$ is exact by, for example, Corollary 3.1.9.

## 3.4. Short Summary

In this last section of the chapter, we want to give a short summary, repeat the important results of this chapter and point out the distinguishing examples: Let $C = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$ be a compact and convex set defined by polynomials $g_1, \ldots, g_r \in \mathbb{R}[X_1, \ldots, X_n]$.

The basic result is

**Theorem** (3.1.6)**.** *Let $C = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$ be a convex and compact set. Let $p_1, \ldots, p_s \in \mathrm{QM}(g_1, \ldots, g_r)$ and assume that there is an open set $U \subset \mathbb{R}^n$ with the property $\{x \in U : p_1(x) \geq 0, \ldots, p_s(x) \geq 0\} = C$. Further assume that for all $i \in \{1, \ldots, s\}$ one of the following two conditions holds:*

1. *The matrix $-p_i'' \in \mathrm{Sym}_{n \times n}(\mathbb{R}[X_1, \ldots, X_n])$ is a sum of squares.*
2. *For all $x \in \mathrm{Z}(p_i) \cap \mathrm{cl}(\mathrm{Ex}(C))$ the matrix $p_i''(x) \in \mathrm{Sym}_{n \times n}(\mathbb{R})$ is negative definite and $p_i$ is concave as a function on $C$.*

*If $\mathrm{QM}(g_1, \ldots, g_r)$ is archimedean, then there is a number $N \in \mathbb{N}$ such that every linear polynomial $\ell \in \mathbb{R}[X_1, \ldots, X_n]_1$ which is non-negative on $C$ lies in the truncated quadratic module $\mathrm{QM}(g_1, \ldots, g_r)_N$. In particular, the Lasserre-relaxation for $C$ is exact.*

We formulated the direct Corollary 3.1.9, where we chose in the above theorem $s = r$ and $p_i = g_i$. We gave an example illustrating the difference between the two different conditions in the hypothesis of our basic theorem in Example 3.2.7(i).
In the second section, we reduced the following theorem to our basic version by constructing substitutes of the defining polynomials of $C$ by elementary means:

**Theorem** (3.2.5)**.** *Let $C$ be convex and compact. Assume that one of the three following conditions is satisfied for all $i = 1, \ldots, r$:*

1. *The matrix $-g_i'' \in \mathrm{Sym}_{n \times n}(\mathbb{R}[X_1, \ldots, X_n])$ is a sum of squares.*
2. *For all $x \in \mathrm{Z}(g_i) \cap \mathrm{cl}(\mathrm{Ex}(C))$ the matrix $g_i''(x) \in \mathrm{Sym}_{n \times n}(\mathbb{R})$ is negative definite and $g_i$ is concave as a function on $C$.*
3. *The polynomial $g_i$ is strictly quasiconcave as a function on $C$.*

*If $\mathrm{QM}(g_1, \ldots, g_r)$ is archimedean, then the Lasserre-relaxation for $C$ by the quadratic module $\mathrm{QM}(g^\alpha : |\alpha| \leq 2)$ (i.e. we take the quadratic module generated by $g_1, \ldots, g_r$ and all the products of two of them) is exact.*

We show by giving Example 3.2.7(ii) that this Theorem 3.2.5 is indeed more general than Corollary 3.1.9.

In the third part, we used more advanced techniques to reduce the proof of our probably most general result to the basic version:

**Theorem** (3.3.2)**.** *Let $C$ be compact and convex with non-empty interior. Assume that for all $i = 1, \ldots r$ one of the following two conditions holds:*

1. *The matrix* $-g_i''$ *is a sum of squares in* $\mathrm{Sym}_{d\times d}(\mathbb{R})[X_1,\dots,X_n]$.

2. *The polynomial* $g_i$ *does not vanish in any interior point of* $C$ *and is strictly quasi-concave in every point* $x \in \mathrm{Z}(g_i) \cap \partial C$. *Further, there is an open set* $U_i \subset \mathbb{R}^n$ *such that* $\mathrm{Z}(g_i) \cap \partial C \subset U_i \cap \mathrm{Z}(g_i)$ *and* $U_i \cap \mathrm{Z}(g_i)$ *is a smooth and strictly convex hypersurface in* $\mathbb{R}^n$ *(in particular,* $g_i'(x) \neq 0$ *for all* $x \in U_i \cap \mathrm{Z}(g_i)$*).*

*If the quadratic module* $\mathrm{QM}(g_1,\dots,g_r)$ *is archimedean, then the Lasserre-relaxation for* $C$ *by the quadratic module* $\mathrm{QM}(g^\alpha : |\alpha| \leq 2)$ *is exact.*

Here, the author conjectures, that this Theorem 3.3.2 can be applied in almost all examples where Theorem 3.2.5 is applicable (as explained in Remark 3.3.13).

# A. Concave Functions, Convex Sets and Lagrange Multipliers

In this appendix, we will assemble some well-known facts on concave and quasiconcave functions which we use in the preceding. Some of these facts can be found in standard textbooks on Analysis or (partly as exercises) in e.g. [BV04]. We will first look at convex and concave functions and later turn to the generalisation of quasiconcave functions and the partly independent notion of strictly quasiconcave functions.

## A.1. Concave and Quasiconcave Functions

**Definition A.1.1.** Let $D \subset \mathbb{R}^n$ be a convex set.
A function $f : D \to \mathbb{R}$ is called *concave* if for all $x, y \in D$ and $t \in [0,1]$ the following inequality holds

$$f(x + t(y - x)) \geq f(x) + t(f(y) - f(x)) = (1 - t)f(x) + tf(y)$$

**Lemma A.1.2.** *(i) Let $a < b$, let $f : (a,b) \to \mathbb{R}$ be a differentiable function. If $f$ is concave, then we have for all $x, y \in (a,b)$, $x < y$*

$$f'(x) \geq \frac{f(y) - f(x)}{y - x}$$

*The function $f$ is concave if and only if its derivative is monotonically decreasing. So in particular, if $f$ is twice differentiable, then it is concave if and only if its second derivative is non-positive.*
*(ii) Let $\emptyset \neq D \subset \mathbb{R}^n$ be an open and convex set. Let $f : D \to \mathbb{R}$ be a differentiable map. Then again, if $f$ is concave, then we have for all $x, y \in D$*

$$f(y) \leq f(x) + f'(x)(y - x)$$

*And again, if $f$ is twice differentiable, then it is concave if and only if the Hessian of $f$ is negative semi-definite.*

PROOF. (i) Let $x, y \in (a,b)$, $x < y$. Put $z = x + t(y - x)$ for $t \in (0,1)$. Then we have $t = \frac{z-x}{y-x}$.
Now concavity of $f$ is equivalent to the following inequality (mind $y - x > 0$)

$$\begin{aligned} f(z) &\geq \left(1 - \frac{z - x}{y - x}\right) f(x) + \frac{z - x}{y - x} f(y) \\ \Leftrightarrow 0 &\geq (z - x)f(y) - zf(x) + yf(x) - (y - x)f(z) \qquad \text{(A.1)} \end{aligned}$$

By expanding the expressions on the right-hand side and adding $xf(x) - xf(x)$ we see that this inequality is equivalent to the following (again, mind $y - x > 0$ and $z - x > 0$)

$$\frac{f(z) - f(x)}{z - x} \geq \frac{f(y) - f(x)}{y - x} \tag{A.2}$$

By taking the limit for $z \to x$, we get the claim $f'(x) \geq \frac{f(y)-f(x)}{y-x}$.
Inequality A.1 is also equivalent to (add $yf(y) - yf(y)$)

$$\frac{f(y) - f(x)}{y - x} \geq \frac{f(y) - f(z)}{y - z} \tag{A.3}$$

From inequalities A.2 and A.3 we deduce, that the derivative of $f$ is monotonically decreasing.
Conversely, if the derivative of $f$ is monotonically decreasing, take $x, y \in (a, b)$ and $t \in (0, 1)$. Again, put $z = x + t(y - x)$. By the mean value theorem, we find two points $x < w_1 < z$ and $z < w_2 < y$ such that the following two equalities hold

$$\frac{f(z) - f(x)}{z - x} = f'(w_1) \tag{A.4}$$

$$\frac{f(y) - f(z)}{y - z} = f'(w_2) \tag{A.5}$$

By monotonicity of $f'$, i.e. $f'(w_1) \geq f'(w_2)$, we get

$$\frac{f(z) - f(x)}{z - x} \geq \frac{f(y) - f(z)}{y - z}$$

This inequality is equivalent to the concavity of $f$.
The claim on the second derivative being non-positive if and only if $f$ is concave is a direct consequence of the monotonicity of $f'$.
(ii) We will reduce to the one-dimensional case (i): Take $x, y \in D$. Since $D$ is convex and open, there is a $r > 0$ such that $\{x + t(y - x) : t \in (-r, 1 + r)\} \subset D$. Now we consider $\widetilde{f} : (-r, 1 + r) \to \mathbb{R}$, $t \mapsto f(x + t(y - x))$. Then $\widetilde{f}$ is concave and by part (i) we get the first part of the claim of (ii) by the identity

$$\frac{\widetilde{f}(1) - \widetilde{f}(0)}{1 - 0} \leq \widetilde{f}'(t) = f'(x + t(y - x))(y - x)$$

For the second part of (ii) take $x \in D$. As $D$ is open, there is a $r > 0$ such that $B(0, r) \subset D$. For all $v \in \mathbb{R}^n$, $\|v\| = 1$, consider the map $\widetilde{f_v} : (-r, r) \to \mathbb{R}$, $t \mapsto f(x + tv)$. It is concave and again by part (i) we get

$$0 \geq \widetilde{f}''(0) = v^t f''(x) v$$

As this inequality holds for all $v \in \mathbb{R}^n$, $\|v\| = 1$, it follows that the Hessian of $f$ is negative semi-definite. The converse follows in the same way from the converse of part (i) (Take $x, y \in D$ and consider $\widetilde{f} : \{x + t(y - x) : t \in [0, 1]\} \to \mathbb{R}$, $\widetilde{f}(x + t(y - x)) = f(x + t(y - x))$, the restriction of $f$ to this line segment). □

**Remark A.1.3.** Let $\emptyset \neq D \subset \mathbb{R}^n$ be an open set and $f: D \to \mathbb{R}$ be a diffenrentiable function. Suppose $f$ is concave on a closed and convex set $C \subset D$. Then for all $x, y \in C$ the inequality

$$f(y) \leq f(x) + f'(x)(y - x)$$

holds. In particular, if $f(y) > 0$ for some $y \in C$, then we deduce from the above inequality that $f'(x) \neq 0$ for all $x \in C$ such that $f(x) = 0$, i.e. the gradient of $f$ does not vanish in zeros of $f$ on $C$.

**Definition A.1.4.** Let $D \subset \mathbb{R}^n$.
(a) A function $f: D \to \mathbb{R}$ is called *quasiconcave* if $D$ is convex and for all $a \in \mathbb{R}$ the set $D_a := \{x \in D : f(x) \geq a\}$ is convex. It is said to be *quasiconvex* if $-f$ is quasiconcave.
(b) Let $D$ be open. A twice differentiable function $f: D \to \mathbb{R}$ is called *strictly quasiconcave* in $x \in D$ if for all $y \in \mathbb{R}^n$, $y \neq 0$ such that $f'(x)y = 0$ the following inequality holds

$$y^t f''(x) y < 0$$

It is said to be *strictly quasiconcave* on $D$ if it is strictly quasiconcave in all points $x \in D$ of $D$. It is called *strictly quasiconvex* at $x \in D$ if $-f$ is strictly quasiconcave at $x$.

**Remark A.1.5.** (i) Let $D \subset \mathbb{R}^n$ be a convex set. A function $f: D \to \mathbb{R}$ is quasiconcave (resp. strictly quasiconcave) if and only if its restrictions to the intersection of every line in $\mathbb{R}^n$ and $D$ is the same (cf. for example the proof of Lemma A.2.1 below, where this is used). So it is in theory sufficient to know all quasiconcave (resp. strictly quasiconcave functions) from $\mathbb{R}$ to $\mathbb{R}$.
(ii) Note that, if the gradient of the function $f$ in the above definition vanishes in the point $x \in D$, then the definition means simply that the Hessian of the polynomial in that point is positive definite.

**Remark A.1.6.** (i) Every monotonic function from $\mathbb{R}$ to $\mathbb{R}$ is quasiconcave.
(ii) Every concave function is quasiconcave. The converse is in general false: Consider for example the function $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$. This function is monotonically increasing, thus quasiconcave, but it is not concave on $(0, \infty)$.
(iii) A concave function need not be strictly quasiconcave; e.g. the polynomial $f = -X^4$ is concave on $\mathbb{R}$ but it is not strictly quasiconcave in $0 \in \mathbb{R}$.
(iv) A strictly quasiconcave function need not be concave either; e.g. the polynomial $f = X_1 X_2$ is strictly quasiconcave as a function on $(0, \infty) \times (0, \infty) \subset \mathbb{R}^2$ but its Hessian is indefinite.
(v) The set $\{x \in \mathbb{R}^n : g(x) \geq 0\}$ for a polynomial $g \in \mathbb{R}[X_1, \ldots, X_n]$ can of course be convex without the polynomial being convex as a function:
Consider the polynomial $g = -X^4 + X^3 + 4X = -X(X-2)(X^2 + X + 2) \in \mathbb{R}[X]$. The real zeros of this polynomial are $X = 0$ and $X = 2$, both of order 1. So the polynomial is non-negative on $[0, 2]$. The second derivative $g'' = -12X^2 + 6X$ has the two real zeros $X = 0$ and $X = \frac{1}{2}$, so it changes sign on $[0, 2]$. This means, that $g$ is not concave on $[0, 2]$.

## A.2. Convex Sets

In this subsection, we will look at the converse of the first part: What can we say about functions defining a convex set. Unfortunately, the necessary conditions on the defining functions (we will restrict our attention to polynomials which is not necessary) are not very useful for us in the preceding sections. We start by relating supporting hyperplanes of a convex set defined by one function to the gradient of this function and by further investigating the condition of (strict) quasiconcavity:

**Lemma A.2.1.** *Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$ be a twice differentiable function.*
*(i) Let $x_0 \in D$ be a point such that $D_{f(x_0)} := \{x \in D : f(x) \geq f(x_0) =: a\}$ is convex and $f'(x_0) \neq 0$. Then the affine tangent space $H := \{x \in \mathbb{R}^n : f'(x_0)x = f'(x_0)x_0\}$ to the set $\{x \in D : f(x) = f(x_0)\}$ is a supporting hyperplane of the convex set $D_a$ at $x_0$. In particular, this holds true for all $x \in D$ if $f$ is a quasiconcave function and $D$ is convex.*
*(ii) The condition $y^t f''(x_0)y \leq 0$ holds at $x_0 \in D$ for all $y \in \mathbb{R}^n$ such that $f'(x_0)y = 0$ if and only if there is an $M \leq 0$ such that*

$$f''(x_0) + Mf'(x_0)^t f'(x_0) \leq 0$$

*If $f$ is quasiconcave, then these two conditions are satisfied for all $x \in D$.*
*(iii) The function $f$ is strictly quasiconcave in a point $x_0 \in D$ if and only if there is an $M \leq 0$ such that*

$$f''(x_0) + Mf'(x_0)^t f'(x_0) < 0$$

*In particular, if $f$ is continuously twice differentiable, then the set of all points where $f$ is strictly quasiconcave is open.*
*(iv) Let $D$ be convex. If $f$ is strictly quasiconcave, then it is also quasiconcave.*

PROOF. (i) Put $H := \{x \in \mathbb{R}^n : f'(x_0)x = f'(x_0)x_0\}$. Let $x_1 \in D_a$. By convexity of $D_a$ we get for all $t \in [0,1]$

$$a \leq f(x_0 + t(x_1 - x_0)) = f(x_0) + tf'(x_0)(x_1 - x_0) + R(t)$$

by Taylor expansion for a remainder term $R$ with the property $\lim_{t\to 0} \frac{R(t)}{t} = 0$. So for sufficiently small $t \in [0,1]$ we get

$$\begin{aligned} f'(x_0)(x_1 - x_0) &\geq 0 \\ f'(x_0)x_1 &\geq f'(x_0)x_0 \end{aligned}$$

which means, that $D_a$ is contained in the closed half-space $\{x \in \mathbb{R}^n : f'(x_0)x \geq f'(x_0)x_0\}$. If $f$ is quasiconcave, put $a := f(x_0)$ and $D_a := \{x \in D : f(x) \geq a\}$, then $D_a$ is convex and the second part follows by the first.
(ii) Without loss of generality we can assume $f'(x_0) \neq 0$ (else, there is nothing to prove). Now, if for all $y \in \mathbb{R}^n$ such that $f'(x_0)y = 0$ the condition $y^t f''(x_0)y \leq 0$ holds, we set $v_1 := f'(x_0)$ and complete to an orthogonal basis $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$. Then we have $f'(x_0)v_i = 0$ for all $2 \leq i \leq n$. Now choose $M \leq 0$ such that

$$v_1^t f''(x_0)v_1 + Mv_1^t f'(x_0)^t f'(x_0)v_1 \leq 0$$

Then the desired condition follows for every $y \in \mathbb{R}^n$ by expansion in the chosen basis of $\mathbb{R}^n$. The converse is trivial.
If $f$ is quasiconcave, take $x_0 \in D$ and $x_1 \in \{x \in D : f(x) \geq f(x_0)\}$. Consider the function $\tilde{f} : [0,1] \to \mathbb{R}$, $t \mapsto f(x_0 + t(x_1 - x_0))$ - i.e. the restriction of $f$ to the line segment in between $x_0$ and $x_1$. We have $\tilde{f}'(t) = f'(x_0 + t(x_1 - x_0))(x_1 - x_0)$ and $\tilde{f}''(t) = (x_1 - x_0)^t f''(x_0 + t(x_1 - x_0))(x_1 - x_0)$. Thus it is sufficient to prove the claim for $\tilde{f}$:
If there were $t_0 \in [0,1]$ such that $\tilde{f}'(t_0) = 0$ and $\tilde{f}''(t_0) > 0$, then $t_0$ would be a local minimiser of $\tilde{f}$, i.e. for sufficiently small $\delta > 0$ we would have $\tilde{f}(t_0 - \delta) > \tilde{f}(t_0)$ as well as $\tilde{f}(t_0 - \delta) > \tilde{f}(t_0)$. But then the set $\{t \in [0,1] : f(t) \geq f(t_0) + \varepsilon\}$ would not be convex for a suitably small $\varepsilon > 0$ (not even connected) which is a contradiction to the assumption of $\tilde{f}$ being quasiconcave.
The proof of (iii) is analogous to the proof of (ii). The openness of the set of points where $f$ is strictly quasiconcave follows from the fact, that the eigenvalues of the matrix $f''(x) + Mf'(x)^t f'(x)$ depend continuously on its coefficients which in turn depend continuously on $x$.
(iv) Let $f$ be a strictly quasiconcave function on the convex set $D$. Let $a \in \mathbb{R}$ such that $D_a := \{x \in D : f(x) \geq a\} \neq \emptyset$ and take $x_1, x_2 \in D_a$. Again consider the function $\tilde{f} : [0,1] \to \mathbb{R}$, $t \mapsto f(x_1 + t(x_2 - x_1))$. As we have $\tilde{f}'(t) = f'(x_1 + t(x_2 - x_1))(x_2 - x_1)$ and $\tilde{f}''(t) = (x_2 - x_1)^t f''(x_1 + t(x_2 - x_1))(x_2 - x_1)$, we get by assumption for all $t \in [0,1]$: $\tilde{f}'(t) = 0 \Rightarrow \tilde{f}''(t) < 0$. This means that there is at most one $t_0 \in [0,1]$ such that $\tilde{f}'(t_0) = 0$.
If $\tilde{f}'(t) \neq 0$ for all $t \in [0,1]$, then $\tilde{f}$ is strictly monotonic on $[0,1]$ and thus $\{x_1 + t(x_2 - x_1) : t \in [0,1]\} \subset D_a$.
If there is a $t_0 \in [0,1]$ such that $\tilde{f}'(t_0) = 0$, then we know by $\tilde{f}''(t_0) < 0$ that $\tilde{f}(t) > 0$ for all $t \in [0, t_0)$ and $\tilde{f}(t) < 0$ for all $t \in (t_0]$. This implies $\tilde{f} \geq a$ on $[0,1]$ as the global minima of $\tilde{f}$ are attained on the boundary. Thus we also have $\{x_1 + t(x_2 - x_1) : t \in [0,1]\} \subset D_a$.
So we get that $D_a$ is convex for all $a \in \mathbb{R}$ which means that $f$ is quasiconcave. □

**Remark A.2.2.** In the above Lemma A.2.1(ii) and (iii) it suffices to choose $M$ (strictly) greater in absolute value than the biggest non-negative eigenvalue of $f''(x)$ for fixed $x \in D$. Thus $M$ can be chosen to depend continuously on $x$ and we get:
A twice differentiable function $f$ is strictly quasiconcave on a compact set $C \subset \mathbb{R}^n$ if and only if there is a $M < 0$ such that for all $x \in C$ the following condition holds

$$f''(x) + Mf'(x)^t f'(x) < 0$$

We are now ready to prove the main result:

**Theorem A.2.3** (cf. [HN09], Theorem 3.5)**.** *Let $C = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$ be a set defined by irreducible polynomials $g_1, \ldots, g_r \in \mathbb{R}[X_1, \ldots, X_n]$. Let $u \in \partial C$ be a point such that $U \cap C$ is convex for some neighbourhood $U$ of $u$. If a polynomial $g_i \in \{g_1, \ldots, g_r\}$ satisfies the conditions*

1. $g_i(u) = 0$;

2. $g_i'(u) \neq 0$;

3. *$Z(g_i) \cap (V \setminus \{u\}) \neq \emptyset$ for all neighbourhoods $V$ of $u$, i.e. $u$ is not an isolated zero of $g_i$,*

*then the inequality*

$$y^t g_i''(u) y \leq 0$$

*holds for all $y \in \mathbb{R}^n$ such that $g_i'(u)y = 0$.*

PROOF. Put $I(u) := \{i \in \{1,\ldots,r\} \colon g_i(u) = 0\}$ and assume that all defining polynomials are pairwise distinct.
First assume that $|I(u)| = 1$, say $I(u) = \{i\}$, i.e. fix $i \in \{1,\ldots,r\}$ such that $g_i(u) = 0$. Then there is a neighbourhood $U'$ of $u$ such that $U' \cap C = \{x \in U' \colon g_i(x) \geq 0\}$ and $U' \cap C$ is convex. Therefore by Lemma A.2.1 the affine hyperplane $H := \{x \in \mathbb{R}^n : g_i'(u)x \geq g_i'(u)u\}$ is a supporting hyperplane to $U' \cap C$. Thus $g_i(u + \frac{\varepsilon}{\|y\|} y) \leq 0$ for all $y \in \mathbb{R}^n$ such that $g_i'(u)y = 0$ and all $\varepsilon \in \mathbb{R}$ such that $u + \frac{\varepsilon}{\|y\|} y \in U'$. So Taylor expansion of $g_i$ at $u$ up to order 2 gives the claim in this case.
We now consider the general case $|I(u)| \geq 1$: For all $i \in I(u)$ there is a sequence $\{u_k^i\}_{k \in \mathbb{N}} \subset Z(g_i) \cap \partial C \setminus (\bigcup_{j \neq i} Z(g_j))$ (i.e. a sequence of zeros of $g_i$ on the boundary of $C$ where no other defining polynomial vanishes) such that $u_k^i \to u$ $(k \to \infty)$. For if that were not true, then there would be an index $j$ and a polynomial $g_j$ that vanishes on $V \cap Z(g_i)$ for some neighbourhood $V$ of $u$ and then, by Artin-Lang's Theorem (in the version [Sch08, Kapitel I], Satz 7.11), $g_i = g_j$ (at least up to a positive constant). Since $g_i'(u) \neq 0$ there is an index $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$ the gradient $g_i'(u_k^i)$ is non-zero. Therefore we know by the first part of the proof, that the claim holds for these points, i.e. that $y g_i''(u_k^i) y \leq 0$ for all $y \in \mathbb{R}^n$, $g_i'(u_k^i)y = 0$, $k \geq n_0$. This implies the claim for $u$ because $g_i'(u_k^i) \to g_i'(u)$ $(k \to \infty)$ (and therefore, there is a sequence $\{y_k\}_{k \in \mathbb{N}}$ for all $y \in \mathbb{R}^n$ such that $g_i'(u)y = 0$ with the properties $y_k \to y$ $(k \to \infty)$ and $g_i'(u_k^i) y_k = 0$ for all $k \in \mathbb{N}$). □

**Remark A.2.4.** (i) Although the nonsingular points are dense in $Z(g_i)$, we cannot use the same approximation method, which we used in the above proof, to show that $g_i''(u) \leq 0$ holds in all points of $Z(g_i) \cap \partial C$ (so, in particular, the singular points). The reason is, of course, that the tangent space of a nonsingular point lacks a dimension. In fact, the claim is false in this case. Consider the example $g_1 := X_2^2 - X_1^3, g_2 := X_2 \in \mathbb{R}[X_1, X_2]$: $0 \in \mathbb{R}^2$ is a point on the boundary and a singular point of $g_1$ but the Hessian of $g_1$ at 0 is

$$g_1''(0) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \geq 0$$

and obviously has the positive eigenvalue 2. (The fact that the set defined by $g_1$ and $g_2$ is not compact does not change anything. One can add, for example, the defining polynomial $g_3 := 1 - X_1^2 - X_2^2 \in \mathbb{R}[X_1, X_2]$, get a compact set and remain with the same problem.)
(ii) We cannot drop assumption 3 in the above Theorem either. If a zero on the boundary of $C$ of one of the defining polynomials is isolated, then this polynomial is redundant

at this point, i.e. there is a neighbourhood $U$ of this isolated zero such that $C \cap U$ is defined by the other polynomials. Therefore it has no consequences for it. Consider the example $g_1 := 1 - X_1^2 - X_2^2, g_2 := (X_1 - 2)^2 + X_2^2 - 1 \in \mathbb{R}[X_1, X_2]$, i.e. the unit disk in $\mathbb{R}^2$ with a redundant defining polynomial $g_2$ (defining the exterior of the circle around $(2,0)$ with radius 1). Then $(1,0)$ is an isolated zero of $g_2$ on the boundary of $C = \{(x_1, x_2) \in \mathbb{R}^2 : g_1(x_1,x_2) \geq 0, g_2(x_1,x_2) \geq 0\}$ and $g_2$ does not satisfy the claim in this point ($g_2''(1,0)$ has the eigenvalue 2 (with algebraic and geometric multiplicity 2)).
(iii) The assumption on the polynomials to be irreducible can be substituted by the assumption that there is a sequence of zeros of $g_i$ on the boundary with the properties that it converges to $u$ and that no other polynomial vanishes in the elements of the sequence (this is a generalisation that also holds for twice continuously differentiable functions as defining functions). This is how [HN09], Theorem 3.5, is essentially stated. As we show in the proof, this condition is met, when all of the defining polynomials are irreducible - so it is slightly more general than our statement here.

## A.3. Optimisation and Lagrange Multipliers

In this section of the appendix, we present some results on the existence of Lagrange multipliers and some examples, where no Lagrange multipliers exist. Again, the facts presented here, are well known. We stick to the book [HF] for our presentation. We begin with some basic notions of optimisation theory.

**Definition A.3.1.** Let $D \subset \mathbb{R}^n$.
(a) An *optimisation problem* in standard form with domain $D$ is an optimisation problem of the form

$$\inf\{f(x) : x \in D, g_1(x) \geq 0, \ldots, g_r(x) \geq 0, h_1(x) = 0, \ldots, h_s(x) = 0\}$$

with the so-called objective function $f : D \to \mathbb{R}$ and the constraints $g_i : D \to \mathbb{R}$ $(i = 1, \ldots, r)$ and $h_i : D \to \mathbb{R}$ $(i = 1, \ldots, s)$. We write $p^* \in \mathbb{R}$ for the optimal value, if it is attained.
(b) A point $x \in D$ for which all constraints hold is called a *feasible point* (for the corresponding problem).
(c) A feasible point $x^* \in D$ is called a *minimiser* if the optimal value is attained at this point, i.e. if we have $f(x^*) = p^*$.

**Definition A.3.2.** For an optimisation problem in standard form we define the *linearising cone* of the problem at $x_0$ to be

$$\mathfrak{C}_l(P, x_0) := \{v \in \mathbb{R}^n : g_i'(x_0)v \geq 0, \text{ if } g_i(x_0) = 0, \text{ and } h_j'(x_0)v = 0 \text{ for all } j = 1, \ldots, s\}$$

**Remark A.3.3.** The linearising cone of an optimisation problem at any feasible point is a closed and convex cone in $\mathbb{R}^n$.

**Definition A.3.4.** For a set $S \subset \mathbb{R}^n$ we define the *tangent cone* of $S$ at $x_0 \in S$ to be

$$\mathfrak{C}_t(S, x_0) := \{v \in \mathbb{R}^n : \text{there are sequences } (x_k)_{k \in \mathbb{N}} \subset S \text{ and } (a_k)_{k \in \mathbb{N}} \subset \mathbb{R} \text{ such that } a_k \to +0 \text{ and } \frac{1}{a_k}(x_k - x_0) \to v\}$$

**Remark A.3.5.** (i) Examples and pictures of tangent cones to sets can be found in Chapter 2.2 of [HF].
(ii) If the set $S$ in question is closed and convex, then this cone is also closed and convex. These statements can be found in [HF] (cf. Lemma 2.2.3 and Exercice 9 of Chapter 2).

Next, we cite the abstract and most important, known (to the author) result for the existence of Lagrange multipliers, the Theorem of Karush-Kuhn-Tucker.

**Theorem A.3.6** ( [HF], Opt., Theorem 2.2.5). *Let $D \subset \mathbb{R}^n$ be open and let* $\inf\{f(x)\colon x \in D, g_1(x) \geq 0, \ldots, g_r(x) \geq 0, h_1(x) = 0, \ldots, h_s(x) = 0\}$ *be an optimisation problem in standard form with domain $D$. Let $x^* \in \mathbb{R}^n$ be a feasible point which is a minimiser. Let the dual cones of the linearising cone at $x^*$ and the tangent cone to the set of feasible points at $x^*$ be equal. Then there exist real, non-negative constants, called Lagrange multipliers, $\lambda_1^*, \ldots, \lambda_r^* \geq 0$ and $\mu_1^*, \ldots, \mu_s^* \in \mathbb{R}$ such that the following conditions hold*

$$\begin{aligned} \lambda_i^* g_i(x^*) &= 0 \text{ for all } i = 1, \ldots, r \\ f'(x^*) - \sum \lambda_i^* g_i'(x^*) - \sum \mu_j^* h_j'(x^*) &= 0 \end{aligned}$$

**Remark A.3.7.** If the set of feasible points $S$ is convex, then $\mathfrak{C}_l(P, x_0)$ and $\mathfrak{C}_t(S, x_0)$ are both closed and convex cones. Thus the condition $\mathfrak{C}_l(P, x_0)^\vee = \mathfrak{C}_t(S, x_0)^\vee$ is equivalent to $\mathfrak{C}_l(P, x_0) = \mathfrak{C}_t(S, x_0)$ (by Lemma 2.1.14(v))

Now there are many conditions, so-called constraint qualifications, which assure the equality of the dual cones which is assumed in Theorem A.3.6. We will now present three of them.

**Definition A.3.8.** Let $D \subset \mathbb{R}^n$ be an open set. Let $\inf\{f(x)\colon x \in D, g_1(x) \geq 0, \ldots, g_r(x) \geq 0, h_1(x) = 0, \ldots, h_s(x) = 0\}$ be an optimisation problem in standard form with domain $D$ and let $x_0$ be a feasible point.
(a) We say that the *Slater constraint qualification* is satisfied if the functions $g_i : D \to \mathbb{R}$ are concave for all $i = 1, \ldots, r$ and there is a feasible point $\tilde{x}$ such that $g_i(\tilde{x}) > 0$ for all constraint functions which are not linear polynomials.
(b) The *Mangasarian-Fromowitz constraint qualification* requires the existence of a vector $v \in \mathbb{R}^n$ with the following two properties:

$$\begin{aligned} g_i'(x_0)v &\geq 0 \text{ for all linear polynomials with } g_i(x_0) = 0 \\ g_i'(x_0)v &> 0 \text{ for all other constraint functions with } g_i(x_0) = 0 \end{aligned}$$

(c) The *Abadie constraint qualification* holds if the linearising cone of the optimisation problem at $x_0$ is equal to the tangent cone to the set of feasible points at $x_0$, i.e. $\mathfrak{C}_l(P, x_0) = \mathfrak{C}_t(S, x_0)$ (where $S$ denotes the set of feasible points).

**Remark A.3.9.** (i) The Slater constraint qualification implies the Mangasarian-Fromowitz constraint qualification. For if we take a feasible point $x_0 \in \mathbb{R}^n$ we get

$$\begin{aligned} g_i'(x_0)(\tilde{x} - x_0) &= g_i(\tilde{x}) - g_i(x_0) = g_i(\tilde{x}) \geq 0 \text{ for all linear polynomials} \\ g_i'(x_0)(\tilde{x} - x_0) &\geq g_i(\tilde{x}) - g_i(x_0) = g_i(\tilde{x}) > 0 \text{ for all other constraint functions} \end{aligned}$$

by Lemma A.1.2(ii).
In turn, the Mangasarian-Fromowitz constraint qualification implies the Abadie constraint qualification (cf. [HF], p. 55f. for a proof). Thus it is obvious that the assumption on the cones in Theorem A.3.6 is satisfied if the Slater constraint qualification holds for any feasible point or any of the other two conditions is satisfied for the minimiser in question.
(ii) The Mangasarian-Fromowitz constraint qualification is a weak assumption. Nevertheless there are examples of polynomial constraint functions where the constraint qualification is not satisfied:
Consider $g_1(X_1,X_2)=1-X_1^2-(X_2-1)^2$ and $g_2(X_1,X_2)=1-X_1^2-(X_2+1)^2$ both in $\mathbb{R}[X_1,X_2]$. The only feasible point under these constraints is $C=\{(0,0)\}\subset\mathbb{R}^2$ but $g_1'(0,0)=(0,+2)$ and $g_2'(0,0)=(0,-2)$ which means, that the Mangasarian-Fromowitz constraint qualification is not satisfied in $(0,0)\in\partial C$.

We will now give some examples where there are no Lagrange-multipliers for certain linear polynomials as objective functions:

**Examples A.3.10.** (i) Take $g_1:=1-X_1^2-(X_2-1)^2$ and $g_2:=1-X_1^2-(X_2+1)^2$, i.e. two circles with radius 1 and centre $(0,1)$, resp. $(0,-1)$. Therefore $C=\{(0,0)\}$. There are no Lagrange multipliers in this case for any linear polynomial $a_0+a_1X_1+a_2X_2\in\mathbb{R}[X_1,X_2]$ with $a_1\neq 0$ ($g_1'(0,0)=(0,+2)$ and ($g_2'(0,0)=(0,-2)$), although the polynomials $g_1$ and $g_2$ are concave as functions on $\mathbb{R}^2$. The Slater constraint qualification fails in this example because the set has empty interior.
(ii) Consider $g_1:=-X_1$ and $g_2:=X_1^3+X_1^2-X_2^2$. Then $g_1'=(-1,0)$ and $g_2'=(3X_1^2+2X_1,-2X_2)$, so $g_2'(0,0)=0$. In this case, there are no Lagrange multipliers for all linear polynomials which minimise on $C$ at $(0,0)$ and satisfy $a_2\neq 0$. For example, take $\ell=a_1X_1+a_2X_2$ such that $\frac{a_1}{a_2}\in(-\infty,1)$. The constraint qualifications fail in this case, because the polynomial $g_2$ is not concave and non-linear and singular at $(0,0)$.
(iii) The upper half of the cone defined by $g_1:=X_3^2-X_1^2-X_2^2$ and $g_2:=X_3\in\mathbb{R}[X_1,X_2,X_3]$ is another example. Again, the polynomial $g_1$ is not concave and non-linear and singular at $(0,0)$. And again, there are no Lagrange multipliers for all linear polynomials minimising on $C$ at $(0,0)$ which are not $X_3$, e.g. $\ell=X_3-\frac{1}{2}X_2$.

We state explicitly a very useful (immediate) corollary of the Karush-Kuhn-Tucker Theorem for our applications. It is easy to prove directly using Farkas's Lemma (which we will cite and derive from the Separation Theorem for convex sets just after this corollary), so we will explicitly prove it:

**Corollary A.3.11.** *Let $C=\{x\in\mathbb{R}^n: g_1(x)\geq 0,\dots,g_r(x)\geq 0\}$ be a convex, compact and basic closed semi-algebraic set defined by polynomials $g_1,\dots,g_r\in\mathbb{R}[X_1,\dots,X_n]$. Let $\ell\in\mathbb{R}[X_1,\dots,X_n]_1$ be a linear polynomial and $u\in\partial C$ be a minimiser for the optimisation problem* $\min\{\ell(x)\colon g_1(x)\geq 0,\dots,g_r(x)\geq 0\}$. *If the Mangasarian-Fromowitz constraint qualification holds at $u$, then we get Lagrange multipliers $\lambda_1,\dots,\lambda_r\geq 0$ such that*

$$\begin{aligned}\lambda_i g_i(u) &= 0 \text{ for all } i=1,\dots,r\\ \ell'(u)-\sum\lambda_i g_i'(u) &= 0\end{aligned}$$

PROOF. Assume without loss of generality that $\ell(u) = 0$. After applying the affine coordinate transformation $x \mapsto x - u$, $\mathbb{R}^n \to \mathbb{R}^n$ we can also assume without loss of generality that $u = 0 \in \mathbb{R}^n$ (note that the derivative of the coordinate transformation is the identity matrix). Assume that $g_i(u) = 0$ for all $i \in \{1,\dots,r_1\}$ and $g_i(u) > 0$ for all $i \in \{r_1+1,\dots,r\}$. Then put $b = \ell'(u)^t$ and $A = (\frac{\partial}{\partial x_i} g_j(u))_{i=1,\dots,n;\, j=1,\dots,r_1}$, i.e. we define the $j$-th column of $A$ to be the gradient of (the active polynomial) $g_j$ at $u$. Suppose that there is a vector $y \in \mathbb{R}^n$ such that $0 > y^t b = y^t \ell'(u)^t = \ell'(u)y = \ell(y)$ (note that by $u = 0$ it follows that $\ell$ is homogeneous of degree 1) and $(y^t A)_i \geq 0$. Let $v \in \mathbb{R}^n$ be a vector satisfying the Mangasarian-Fromowitz constraint qualification, i.e. for all $i = 1,\dots,r_1$ we have $g_i'(u)v > 0$ if $\deg(g_i) > 1$ and $g_i'(u)v \geq 0$ if $\deg(g_i) = 1$. Now, for sufficiently small $\varepsilon > 0$ the inequality $(y+\varepsilon v)^t b < 0$ holds. And for all $\varepsilon > 0$ and $i \in \{1,\dots,r_1\}$ such that $\deg(g_i) > 1$ we have $0 < ((y^t + \varepsilon v)^t A)_i = (y+\varepsilon v)^t g_i'(u)^t = g_i'(y)(y+\varepsilon v)$. So from Taylor expansion of $g_i$ at $u$ and $g_i(u) \geq 0$ ($u \in C$), it follows that $g_i(\alpha(y+\varepsilon v)) \geq 0$ for sufficiently small $\alpha > 0$ and all $i = 1,\dots,r_1$. The case $\deg(g_i) = 1$ does not pose any problems because $0 \leq (y+\varepsilon v)^t g_i'(u) = g_i(y+\varepsilon v)$. If we choose $\alpha$ maybe even smaller, then we can also guarantee that $g_i(\alpha(y+\varepsilon v)) > 0$ remains positive for all $i = r_1+1,\dots,r$. So we have $w := \alpha(y+\varepsilon v) \in C$ and $\ell(w) < 0$ which is a contradiction to $0 = \ell(u) = \min\{\ell(x) : x \in C\}$. Therefore, by Farkas's Lemma ( [HF], p. 43; also cited below, cf. A.3.12), there is a vector $(\lambda_1,\dots,\lambda_r) \in \mathbb{R}^r_{\geq 0}$ such that $A(\lambda_1,\dots,\lambda_r)^t = b$, which is nothing but the claim

$$\sum_{i=1}^{r_1} \lambda_i g_i'(u) = \ell'(u)$$

Note that the complementary slackness is included here because we can choose the multipliers $\lambda_{r_1+1},\dots,\lambda_r$ to be 0. □

**Theorem A.3.12** (Farkas's Lemma or Theorem of the Alternative, cf. [HF], p. 43)**.** *Let $A \in \mathrm{M}_{k\times l}(\mathbb{R})$ be a real matrix and $b \in \mathbb{R}^k$. Then exactly one of the following claims holds:*
*(a) There is $x \in \mathbb{R}^l$ such that $x_1,\dots,x_l \geq 0$ and $Ax = b$.*
*(b) There is $y \in \mathbb{R}^k$ such that $y^t b < 0$ and $(y^t A)_i \geq 0$ for all $i = 1,\dots,l$.*

PROOF. Denote by $v_1,\dots,v_l$ the columns of $A$, i.e. $A = (v_1,\dots,v_l)$ and let $K$ be the convex cone $\{\sum_{i=1}^l a_i v_i : a_1 \geq 0,\dots,a_l \geq 0\}$ generated by these columns. Then there is a vector $x \in \mathbb{R}^l$ with the properties $x_1,\dots,x_l \geq 0$ and $b = Ax = x_1v_1 + \ldots + x_lv_l$ if and only if $b \in K$. If $b \notin K$, then there is by the Separation Theorem for convex sets (note that $K$ is, as a finitely generated cone, closed) a linear functional $\ell \in K^\vee$ such that $\ell(b) < 0$, i.e. there is a $y \in \mathbb{R}^k$ such that $y^t b < 0$ and $(y^t A)_i = y^t v_i \geq 0$ for all $i = 1,\dots,l$. So (a) is equivalent to $b \in K$ and (b) is equivalent to $b \notin K$ by the Separation Theorem and the proof is finished. □

**Remark A.3.13.** As we just saw in the proof, the given version of Farkas's Lemma is only a reformulation in terms of matrices of the special case of the Separation Theorem stating that a finitely generated convex cone and a point can be strictly separated by a hyperplane.

As a last remark on Lagrange multipliers, we will show that they are in general not continuous functions of the linear polynomial, which is to be optimised.

**Remark A.3.14.** In general, the Lagrange multipliers cannot be chosen to depend continuously on the linear polynomial they correspond to. If we consider for example the convex body (an ellipsoid) $C$ defined by the polynomials $g_1 := 1-2X_1^2-X_2^2-X_3^2$ and $g_2 := 1-X_1^2-2X_2^2-X_3^2 \in \mathbb{R}[X_1,X_2,X_3]$, then their gradient at $(0,0,1) \in \partial C$ is the same, namely $(0,0,-2)$. This is the reason why we will be unable to choose Lagrange multipliers continuously with respect to the corresponding linear polynomial: If we take the sequence $\{\frac{1}{n}X_1 - X_3\}_{n\in\mathbb{N}}$ of linear polynomials, then they minimise in points on the boundary of $C$ where only $g_1$ vanishes. So the Lagrange multipliers will be uniquely determined tuples of the form $(\lambda_1^n, 0)$ where $\lambda_1^n$ converges to $\frac{1}{2}$ as $n$ tends to $\infty$. But of course, this situation is symmetric in $X_1$ and $X_2$, i.e. if we take the sequence $\{\frac{1}{n}X_2 - X_3\}_{n\in\mathbb{N}}$ of linear polynomials, then they minimise in points on the boundary of $C$ where only $g_2$ vanishes and so the Lagrange multipliers will look like $(0,\lambda_2^n)$. And again, they are unique for all $n \in \mathbb{N}$ and we have $\lambda_2^n \to \frac{1}{2}$ $(n \to \infty)$. So a continuous choice of Lagrange multipliers is impossible.

# B. The Dimension of Semi-algebraic Sets and the Real Spectrum

In this section, we will use the real spectrum to prove Lemma 2.1.17(ii). We will rely on the correspondence of semi-algebraic subsets of $\mathbb{R}^n$ and constructible subset of the real spectrum $\mathrm{Sper}(\mathbb{R}[X_1,\ldots,X_n])$ of the ring of polynomials in $n$ variables over $\mathbb{R}$. This can be found in [BCR98], chapter 7.2.

**Definition B.1.15.** The *dimension* of a semi-algebraic set $S \subset \mathbb{R}^n$ is defined to be the algebraic dimension of its Zariski-closure

$$\dim(S) = \dim(\mathrm{clos}_Z(S))$$

**Remark B.1.16.** The dimension of $S$ is $n$, i.e. maximal, if the topological interior of $S$ is non-empty in the semi-algebraic (euclidean) topology (cf. [BCR98], Corollary 2.8.9 and Theorem 2.3.6)

By [BCR98], Proposition 7.2.2, there is to every semi-algebraic set $S \subset \mathbb{R}^n$ a uniquely determined constructible subset of $\mathrm{Sper}(\mathbb{R}[X_1,\ldots,X_n])$, which we will denote by $\widetilde{S}$, such that $\widetilde{S} \cap \mathbb{R}^n = S$. If $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0,\ldots,g_r(x) \geq 0\}$ is a basic-closed set defined by polynomials $g_1,\ldots,g_r \in \mathbb{R}[X_1,\ldots,X_n]$, then the same is true in the real spectrum, i.e. $\widetilde{S} = \{P \in \mathrm{Sper}(\mathbb{R}[X_1,\ldots,X_n]) : g_1 \in P,\ldots,g_r \in P\}$.

**Definition B.1.17.** Let $C \subset \mathrm{Sper}(\mathbb{R}[X_1,\ldots,X_n])$ be constructible. The *dimension* of $C$ is defined to be maximal length of a chain of specialisation in $C$.

**Proposition B.1.18** (cf. [BCR98], Propositions 7.5.6 and 7.5.8)**.** *The dimension of $S \subset \mathbb{R}^n$ as a semi-algebraic set is equal to the dimension of $\widetilde{S} \subset \mathrm{Sper}(\mathbb{R}[X_1,\ldots,X_n])$ as a constructible subset of the real spectrum.*

**Definition B.1.19.** Let $A$ be a commutative ring with unity, let $P \subset A$ be an ordering (a prime cone in the terminology of [BCR98]) of $A$. The *dimension* of $P$ is defined to be the Krull dimension of the ring $A/\mathrm{supp}(P)$ where we write $\mathrm{supp}(P) = P \cap (-P)$ for the support of $P$, a prime ideal of $A$.

Now we can proof Lemma 2.1.17(ii):

**Lemma** (Lemma 2.1.17)**.** *(ii) If $\mathrm{PO}(g_1,\ldots,g_r)_k$ is pointed for all $k \in \mathbb{N}$, then the interior of $C = \{x \in \mathbb{R}^n : g_1(x) \geq 0,\ldots,g_r(x) \geq 0\}$ is non-empty.*

PROOF. The assumption $\mathrm{PO}(g_1,\dots,g_r)_k$ pointed for all $k \in \mathbb{N}$ is equivalent to saying that the support $\mathrm{supp}(T)$ of the preordering $T = \mathrm{PO}(g_1,\dots,g_r) \subset \mathbb{R}[X_1,\dots,X_n]$ is the zero ideal.
Now we extend the preordering $T$ to a preordering $T'$ of the function field $\mathbb{R}(X_1,\dots,X_n)$ in the following way: A quotient $\frac{p}{q}$ of polynomials $p,q \in \mathbb{R}[X_1,\dots,X_n]$ $(q \neq 0)$ is in $T'$ if and only if $pq \in T$. This defines a proper preordering of the function field. By Theorem 1.1.9 in [PD01] the preordering $T'$ extends to an ordering $P'$ of the function field. By the functoriality of the real spectrum (cf. [BCR98], 7.1.7), the pullback $P$ of $P'$ to $\mathbb{R}[X_1,\dots,X_n]$ (i.e. $\phi^{-1}(P')$ for the map $\phi\colon \mathbb{R}[X_1,\dots,X_n] \to \mathbb{R}(X_1,\dots,X_n)$, $p \mapsto \frac{p}{1}$) is an ordering of the ring of polynomials such that $T \subset P$ and $\mathrm{supp}(P) = 0$. Thus we know that the dimension of $P$ is $n$.
As we have $T \subset P$, we get $P \in \widetilde{C}$ (where $C$ is the basic closed, semi-algebraic set defined by $T$). Thus Proposition 7.5.8 in [BCR98] gives us the following equality

$$\dim(C) = \dim(\widetilde{C}) = \max\{\dim(\alpha)\colon \alpha \in \widetilde{C}\} = n$$

By Remark B.1.16 this implies that the topological interior of $C$ in the semi-algebraic topology is non-empty which is the claim. □

The claim of Lemma 2.1.17 is in general false for quadratic modules, even if they are assumed to be archimedean. The reason is that an archimedean quadratic module with support (0) is in general not contained in an ordering with support (0):

**Example B.1.20.** Take the quadratic module $M = \mathrm{QM}(X_1,X_2,1-X_1,1-X_2,-X_1X_2) \subset \mathbb{R}[X_1,X_2]$. This is a proper module which is archimedean and has support (0):
The module is archimedean by [KMS05], Corollary 3.6: For $\{X,1-X\}$ is the set of natural generators for the set $[0,1]$, so $1-X^2$ is in the quadratic module $\mathrm{QM}(X,1-X)$ (which also happens to be equal to the preordering generated by the same polynomials) by [KMS05], Corollary 3.6. Thus $\mathrm{QM}(X_1,X_2,1-X_1,1-X_2)$ is already archimedean (it contains the polynomial $2-X_1^2-X_2^2 = (1-X_1^2)+(1-X_2^2)$).
In order to show that the support of $M$ is trivial, we define the following semi-ordering $S \subset \mathbb{R}[X_1,X_2]$: Let $0 \in S$. Now take a polynomial $p = \sum_{\nu\in\mathbb{N}_0^2} a_\nu X^\nu \in \mathbb{R}[X_1,X_2]$ and let $\nu = (\nu_1,\nu_2) \in \mathbb{N}_0^2$ be the smallest multi-index with respect to the lexicographic order such that $a_\nu \neq 0$. Then we have $p \in S$ if and only if

$$\begin{cases} a_\nu > 0 & \text{, if } \nu_1,\nu_2 \cong 0 \bmod 2 \\ a_\nu > 0 & \text{, if } \nu_1 \cong 1 \bmod 2, \nu_2 \cong 0 \bmod 2 \\ a_\nu > 0 & \text{, if } \nu_1 \cong 0 \bmod 2, \nu_2 \cong 1 \bmod 2 \\ a_\nu < 0 & \text{, if } \nu_1 \cong 1 \bmod 2, \nu_2 \cong 1 \bmod 2 \end{cases}$$

Direct verification of the definition shows that $S$ is indeed a semi-ordering (cf. [PD01], 6.1.2 and 5.5.3). This semi-ordering contains $M$, because it contains every generator of $M$, and the support of $S$ is (0) by definition.
And yet $M$ cannot be contained in an ordering of $\mathbb{R}[X_1,X_2]$ with support (0) because already the preordering generated by the polynomials $X_1$, $X_2$, $1-X_1$, $1-X_2$ and $-X_1X_2$ has a non-trivial support.

The set defined by $M$ is $[0,1] \times \{0\} \cup \{0\} \times [0,1] \subset \mathbb{R}^2$; it has empty interior although the cone $\mathrm{QM}(X_1, X_2, 1 - X_1, 1 - X_2, -X_1X_2)_k$ is pointed for all $k \in \mathbb{N}$.
Another example can be found in [Sch05a] (Example 3.2).

# C. Basics of Differential Geometry and Boundaries of Convex Sets

## C.1. Hypersurfaces in $\mathbb{R}^n$ and Curvature

In the preceding, we need some basic facts about (analytic) hypersurfaces in $\mathbb{R}^n$ such as the first and second fundamental form. These are basic concepts in differential geometry of embedded manifolds which are unfortunately hard to find in the literature. Most of the books do not deal with embedded (to $\mathbb{R}^n$) manifolds at all or they restrict their attention to curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$. The concepts are easily generalised to the case of hypersurfaces in $\mathbb{R}^n$. For the lack of an explicit reference, we include a short treatment of this matter here. We will mostly omit proofs and only give short indications on how to prove the result or a reference which mostly requires generalisation to higher dimensions to fit the claim.
The following Lemma is a well-known result in basic analysis - it follows by use of the implicit function theorem:

**Lemma C.1.1.** *Let $M \subset \mathbb{R}^n$ be a set. The following statements are equivalent:*
*(i) For all $p \in M$, there is an open set $U \in \mathbb{R}^{n-1}$ and a map $\phi \in \mathscr{C}^\infty(U;\mathbb{R}^n)$ such that the rank of the Jacobian of $\phi$ is $n-1$ in all $x \in U$ and $\phi(U) = M \cap V$ for an open neighbourhood $V \subset \mathbb{R}^n$ of $p$.*
*(ii) For all $p \in M$, there is an open neighbourhood $V \subset \mathbb{R}^n$ of $p$, an open set $U \subset \mathbb{R}^{n-1}$ and a map $f \in \mathscr{C}^\infty(U;\mathbb{R})$ such that $M \cap V = \mathrm{graph}(f) = \{(x, f(x)) \colon x \in U\}$.*
*(iii) For all $p \in M$, there is an open neighbourhood $V \subset \mathbb{R}^n$ of $p$ and a map $g \in \mathscr{C}^\infty(V;\mathbb{R})$ such that $M \cap V = \{x \in V : g(x) = 0\} =: \mathrm{Z}(g)$.*

**Definition C.1.2.** Let $M \subset \mathbb{R}^n$ be a set.
(a) It is called a *smooth hypersurface*, if it satisfies one of the statements in the above Lemma C.1.1.
(b) A map $\phi$ as in C.1.1(i) is called a *chart* of $M$ at $p$.

**Remark C.1.3.** Note that we do not assume the hypersurface to be connected, which leads to minor technical difficulties in section 3.3 but also gives slightly more general results.

**Definition C.1.4.** Let $M \subset \mathbb{R}^n$ be a smooth hypersurface and let $p \in M$ be a point. Write $I_\varepsilon := (-\varepsilon, \varepsilon) \subset \mathbb{R}$. The set

$$T_pM = \{\gamma'(0) \colon \gamma \in \mathscr{C}^\infty(I_\varepsilon; M), \gamma(0) = p\}$$

is called the *tangent space* of $M$ at $p$. It is an $\mathbb{R}$-vector space (cf. [Spi70], Chapter III). The affine tangent space $\widehat{T}_pM$ is the affine vector space $p + T_pM$.

**Remark C.1.5.** The restriction of the scalar product on $\mathbb{R}^n$ to $T_pM$ is called the first fundamental form of $M$ at $p$. It is, of course, still a symmetric bilinear form, written as follows

$$I_p \colon T_pM \times T_pM \to \mathbb{R}, (v,w) \mapsto \langle v,w\rangle$$

We also denote the canonical euclidean product on $\mathbb{R}^n$ by $\langle .,.\rangle$ because confusion does not seem likely.

**Proposition C.1.6.** *Let $M \subset \mathbb{R}^n$ be a smooth hypersurface.*
*(i) The tangent space to $M$ at any point $p \in M$ is an $\mathbb{R}$-vector space.*
*(ii) The tangent space to $M$ has dimension $n-1$ at all $p \in M$.*
*(iii) Let $p \in M$ and let $V \subset \mathbb{R}^n$ be a neighbourhood of $p$ such that $M \cap V = \mathrm{graph}(f)$ for a map $f \in \mathscr{C}^\infty(U;\mathbb{R})$ and $p = (u, f(u))$. Then the vectors $\{(e_i, \frac{\partial}{\partial x_i} f(u))\}_{i=1,\dots,n-1}$ (where $e_i$ denotes the i-th canonical base vector of $\mathbb{R}^{n-1}$) form a basis of $T_pM$.*

PROOF. (i) is proved for the special case of surfaces in $\mathbb{R}^3$ in [Blo97], Lemma 5.4.2 and the proof generalises to higher dimensions. To prove (iii), let $p$, $V$, $u$ and $f$ be as in the claim. Then the claim follows by calculating the derivatives of the curves

$$\gamma_i \colon (-\varepsilon, \varepsilon) \to M, t \mapsto (u + te_i, f(u + te_i))$$

where $e_i$ denotes the $i$-th canonical base vector in $\mathbb{R}^{n-1}$ ($i = 1,\dots,n-1$) and $\varepsilon$ is chosen such that $u + te_i$ is contained in $U$ for all $t \in (-\varepsilon, \varepsilon)$ and all $i = 1,\dots,n-1$. This implies (ii), because the fact that the gradient of a function describing $M$ locally as its zero set is normal to the tangent space (cf. Proposition C.1.9(i)) implies that the dimension is at most $n-1$. □

**Definition C.1.7.** (a) A *normal vector field* on a smooth hypersurface $M \subset \mathbb{R}^n$ is a map $N \colon M \to \mathbb{R}^n$ such that $N(p) \neq 0$ is perpendicular to all tangent vectors $v \in T_pM$ at $p$, i.e. $\langle v, N(p)\rangle = 0$. It is called *smooth*, if its composition with all charts is infinitely differentiable. It is called a *unit* normal vector field, if $\|N(p)\| = 1$ for all $p \in M$.
(b) A smooth hypersurface is called *orientable*, if there is a smooth normal vector field on $M$.

**Remark C.1.8.** (i) Every hypersurface is locally orientable (as we will see in the following proposition). We will therefore often see the tangent space $T_pM$ to a smooth hypersurface $M \subset \mathbb{R}^n$ at $p \in M$ as the hyperplane of $\mathbb{R}^n$ defined by a normal vector to $M$ at $p$ - but sometimes, we will see it as an $(n-1)$-dimensional $\mathbb{R}$-vector space.
(ii) Every compact (in the topology of $\mathbb{R}^n$) hypersurface in $\mathbb{R}^n$ is orientable (cf. [Gre67], Theorem (27.11)).

**Proposition C.1.9.** *Let $M$ be a smooth hypersurface and let $p \in M$ be a point.*
*(i) If $V \subset \mathbb{R}^n$ is a neighbourhood such that $M \cap V$ is the zero set of a function $g \in \mathscr{C}^\infty(V;\mathbb{R})$, then the map $N\colon M \cap V \to \mathbb{R}^n$, $x \mapsto \frac{1}{\|g'(x)\|} g'(x)$ is a unit normal vector field on $M \cap V$.*
*(ii) If $V \subset \mathbb{R}^n$ is a neighbourhood of $p$ such that $M \cap V = \mathrm{graph}(f)$ for a function $f \in \mathscr{C}^\infty(U;\mathbb{R})$, then the map $N\colon M \cap V \to \mathbb{R}^n$, $(x, f(x)) \mapsto \frac{1}{\|(f'(x),1)\|}(f'(x), -1)$ is a unit normal vector field on $M \cap V$.*

PROOF. (i) follows from calculating the derivative of the function $g \circ \gamma\colon I_\varepsilon := (-\varepsilon, \varepsilon) \to \mathbb{R}$, which is identically zero on $I_\varepsilon$, at 0 for any curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$. (ii) follows from Proposition C.1.6(iii). □

**Definition C.1.10.** (i) Let $f : M \to \mathbb{R}$ be a function. The *directional derivative* $D_v f(p)$ of $f$ at $p \in M$ in direction $v \in T_pM$ is defined to be

$$D_v f(p) = \frac{d}{dt}(f \circ \gamma)|_{t=0}$$

for a curve $\gamma \in \mathscr{C}^\infty((-\varepsilon,\varepsilon);M)$, $\gamma(0) = p$ and $\gamma'(0) = v$ (It is shown in [Blo97], Lemma 5.6.1 that this value does not depend on the choice of $\gamma$ in the special case of a surface in $\mathbb{R}^3$ - the proof generalises to higher dimensions).
(ii) We define the *directional derivative* of a function $f \in \mathscr{C}^\infty(M;\mathbb{R}^m)$ to be the vector of the directional derivatives of the component functions.

**Remark C.1.11.** (i) The same rules as for the usual derivative of a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ also apply to the directional derivative: If $a, b \in \mathbb{R}$, $v, w \in T_pM$ and $f, g \in \mathscr{C}^\infty(M;\mathbb{R})$, then

$$\begin{aligned} D_v(f+g)(p) &= D_v f(p) + D_v g(p) \\ D_v(fg)(p) &= (D_v f(p))g(p) + f(p)(D_v g(p)) \\ D_{av+bw} f(p) &= aD_v f(p) + bD_w f(p) \end{aligned}$$

If $F, F'\colon M \to \mathbb{R}^m$ are smooth vector fields on $M$, then we have

$$D_v\langle F, F'\rangle(p) = \langle D_v F(p), F'(p)\rangle + \langle F(p), D_v F'(p)\rangle$$

where $\langle .,. \rangle$ denotes the scalar product on $\mathbb{R}^m$.
These rules can be reduced to the corresponding rules for the derivation of maps on euclidean space, cf. [Blo97], Lemma 5.6.2 for an example (again in the special case of $n = 3$).
(ii) Let $N$ be a unit normal vector field on $M$. From the above rule and $\langle N(p), N(p)\rangle = 1$ for all $p \in M$, we immediately deduce

$$0 = D_v\langle N, N\rangle(p) = 2\langle D_v N(p), N(p)\rangle$$

This means that $D_v N(p) \in T_pM$ for all $p \in M$.

This remark justifies the following definition:

**Definition C.1.12.** Let $M \subset \mathbb{R}^n$ be a smooth orientable hypersurface with unit normal vector field $N \in \mathscr{C}^\infty(M,\mathbb{R}^n)$.
(i) The map

$$L_p : T_pM \to T_pM, v \mapsto -D_vN(p)$$

is called the *Weingarten map* of $M$ at $p$.
(ii) The map

$$II_p : T_pM \times T_pM \to \mathbb{R}, (v,w) \mapsto \langle L_p(v), w\rangle = I_p(L(v), w)$$

is called the *second fundamental form* of $M$ at $p$.

**Remark C.1.13.** (i) The negative sign in the definition of the Weingarten map is standard in the literature. It seems to be there for historical reasons.
(ii) The Weingarten map is linear and therefore the second fundamental form is a bilinear form on the tangent space.
(iii) There is also the notion of the $k$-th fundamental form of a hypersurface in $\mathbb{R}^n$ for $k > 2$ - it is defined at every $p \in M$ to be the map $(v,w) \mapsto I_p(L^k(v), w)$ for $v,w \in T_pM$.

**Lemma C.1.14.** *The Weingarten map on a smooth orientable hypersurface is self-adjoint (with respect to the first fundamental form), i.e.*

$$I_p(L(v), w) = \langle L(v), w\rangle = \langle v, L(w)\rangle = I_p(v, L(w))$$

*for all $p \in M$ and $v,w \in T_pM$. In particular, the second fundamental form is a symmetric bilinear form.*

PROOF. This is a computation in local coordinates, done e.g. in [Sch93], Chapter 2.5 (p. 105) (note that Schneider defines the second fundamental form without a minus sign which does not influence the argument at all). □

**Proposition C.1.15.** *Let $M \subset \mathbb{R}^n$ be a smooth orientable hypersurface and fix a unit normal vector field $N \in \mathscr{C}^\infty(M;\mathbb{R}^n)$. Let $p \in M$ be a point.*
*(i) Let $V \subset \mathbb{R}^n$ be a neighbourhood of $p$ such that $M \cap V = \{x \in V : g(x) = 0\}$ for a function $g \in \mathscr{C}^\infty(V,\mathbb{R})$. Choose the sign of $g$ in such a way that $\frac{1}{\|g'(p)\|} g'(p) = N(p)$. Then $L_p : T_pM \to T_pM$ is the map $v \mapsto -\frac{1}{\|g'(x)\|} g''(p)v$ for all $p \in M$, which means for the second fundamental form:*

$$II_p(v,w) = \frac{-1}{\|g'(p)\|} \langle g''(p)v, w\rangle$$

*for all $v,w \in T_pM$.*
*(ii) Let $V \subset \mathbb{R}^n$ be a neighbourhood such that $M \cap V = \mathrm{graph}(f)$ for a map $f \in \mathscr{C}^\infty(U;\mathbb{R})$ and $p = (u, f(u))$. Assume that $\frac{1}{\|(f'(u),1)\|}(-f'(u),1) = N(p)$. Then the Weingarten map is given by $v \mapsto \frac{1}{\|(f'(u),1)\|} f''(u)v$ which again means for the second fundamental form*

$$II_p(v,w) = \frac{1}{\|(f'(u),1)\|} \langle f''(u)v, w\rangle$$

*(If $N(p) \neq \frac{1}{\|(f'(u),1)\|}(-f'(u),1)$, then $N(p) = -\frac{1}{\|(f'(u),1)\|}(-f'(u),1)$ and the statement remains true, if the signs are adequately changed everywhere.)*

PROOF. In both cases, the proof is a straightforward computation in the given local coordinates. □

**Definition C.1.16.** We say that a smooth orientable hypersurface is *positively curved* at $p \in M$, if the second fundamental form $II_p$ at $p$ is positive definite. We say that it has non-negative curvature at $p$, if $II_p$ is positive semi-definite.

**Remark C.1.17.** (i) From Proposition C.1.15(i) we see that a smooth hypersurface $M$ is positively curved at $p \in M$ if and only if a function $g \in \mathscr{C}^\infty(V;\mathbb{R})$ such that $M \cap V = Z(g)$ and $-\frac{1}{\|g'(q)\|}g'(q) = N(q)$ for a neighbourhood $V \subset \mathbb{R}^n$ of $p$ and all $q \in M \cap V$ is strictly quasiconcave at $p$. If we write $M$ locally as the graph of a function $f \in \mathscr{C}^\infty(U;\mathbb{R})$ with $p = (u, f(u))$ and such that the vector $(-f'(v), 1)$ is a positive multiple of $N(v, f(v))$, then it is equivalent that $M$ is positively curved at $p$ and that the Hessian of $f$ is positive definite at $u$ (by Proposition C.1.15(ii)).
(ii) A positively curved hypersurface $M$ with unit normal vector field $N\colon M \to \mathbb{R}^n$ lies locally on one side of the tangent plane $T_pM$ for all $p \in M$, i.e. if $p \in M$, then there is a neighbourhood $U \subset M$ of $p$ such that $\langle q-p, N(p)\rangle \geq 0$ for all $q \in U$: Let $V \subset \mathbb{R}^n$ be a neighbourhood of $p$ such that there is a function $g \in \mathscr{C}^\infty(V;\mathbb{R})$ with $V \cap M = \{x \in V : g(x) = 0\}$ and $N(p) = \frac{1}{\|g'(p)\|}g'(p)$. Then $II_p(v,w) = \frac{-1}{\|g'(p)\|}\langle g''(p)v, w\rangle$ (by Proposition C.1.15(i)) which is positive definite, i.e. $v^t g''(p)v < 0$ for all $v \in T_pM$. We take the Taylor expansion of $g$ at $p$:

$$g(p+h) = g(p) + g'(p)h + h^t g''(p)h + \text{ higher order terms} = g'(p)h + h^t g''(p)h + \dots$$

Now, if $h \in T_pM$, then $g(p+h) < 0$ for all $h \in T_pM$ with $\|h\| < \varepsilon$ for some $\varepsilon > 0$. If $h = -\alpha N(p)$ for some $\alpha > 0$, then

$$g(p+h) = g'(p)\left(\frac{-\alpha}{\|g'(p)\|}g'(p)^t\right) + \dots = -\alpha\|g'(p)\| + \dots$$

is again strictly negative for all $\alpha < \varepsilon$ for some (possibly smaller) $\varepsilon > 0$. So we have seen $g(p+h) < 0$, if $h$ lies to the outside of the tangent hyperplane to $M$ at $p$ (and $\|h\| < \varepsilon$ for some possibly even smaller $\varepsilon > 0$). This shows that there is a neighbourhood $W \subset \mathbb{R}^n$ of $p$ such that

$$M \subset \{q \in W : \langle q-p, g'(p)\rangle \geq 0\}$$

and $\langle q-p, g'(p)\rangle > 0$ for all $q \in W$, $q \neq p$.

# C.2. The Boundary of Convex Sets, Support Functions and the Minkowski Functional

In this part of the appendix, we will study the boundary of convex sets and the connections between the boundary being a smooth hypersurface of $\mathbb{R}^n$ and differentiability properties of the support function and the Minkowski functional of the set. These

functions are, basically, only a different point of view on compact and convex sets with non-empty interior. That is because they can be defined by axiomatic properties and are in bijection to compact and convex sets with non-empty interior.
As we will restrict our attention to such convex sets, it is convenient to give this class of sets a separate name:

**Definition C.2.1.** A compact, convex set $K \subset \mathbb{R}^n$ with non-empty interior is called a *convex body*.

We will start with the Minkowski functional:

**Definition C.2.2.** Let $K \subset \mathbb{R}^n$ be a convex body and suppose that $0 \in \text{int}(K)$. The *Minkowski functional* to the set $K$ is defined to be the map

$$p_K : \mathbb{R}^n \to \mathbb{R},\ x \mapsto \begin{cases} \min\{\lambda > 0 : \frac{1}{\lambda}x \in K\} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

**Remark C.2.3.** (i) In the above definition, the existence of the infimum of the set $\{\lambda > 0 : \frac{1}{\lambda}x \in K\}$ is due to the compactness of $K$.
(ii) The Minkowski functional is positively homogeneous, i.e. for all $a \in \mathbb{R}_{\geq 0}$ and all $x \in \mathbb{R}^n$ we have

$$p_K(ax) = a p_K(x)$$

In fact, it is also convex: Take $x, y \in \mathbb{R}^n \setminus \{0\}$. The convexity of $K$ yields

$$K \ni \frac{p_K(x)}{p_K(x) + p_K(y)} \left( \frac{1}{p_K(x)} x \right) + \frac{p_K(y)}{p_K(x) + p_K(y)} \left( \frac{1}{p_K(y)} y \right) = \frac{1}{p_K(x) + p_K(y)} (x + y)$$

which in turn implies $p_K(x + y) \leq p_K(x) + p_K(y)$.
(iii) The Minkowski functional determines the set $K$, from which it is derived. In fact, it also determines its boundary and its interior in the following way:

$$\begin{aligned} K &= \{x \in \mathbb{R}^n : p_K(x) \leq 1\} \\ \partial K &= \{x \in \mathbb{R}^n : p_K(x) = 1\} \\ \text{int}(K) &= \{x \in \mathbb{R}^n : p_K(x) < 1\} \end{aligned}$$

The first equality is clear from the definition of the Minkowski functional. For the other two, we only have to see the equality for the boundary: This follows from the fact that $\{\lambda x : \lambda \in \mathbb{R}_{\geq 0}\} \cap K = [0, y]$ for a point $y \in \partial K$ and $[0, y) \subset \text{int}(K)$ since 0 is an interior point of $K$.
(iv) The assumption $0 \in \text{int}(K)$ is only a normalisation. If $0 \notin \text{int}(K)$, then fix an interior point $x_0 \in \text{int}(K)$ and define $p_K(x) = \min\{\lambda > 0 : \lambda^{-1}(x - x_0) \in K\}$ for $x \neq x_0$ and $p_K(x_0) = 0$. However, this function will not be positively homogeneous if $x_0 \neq 0$.

We now turn our attention to support functions.

**Definition C.2.4.** Let $K \subset \mathbb{R}^n$ be a convex body. The *support function* of $K$ is the following map

$$h_K : \mathbb{R}^n \to \mathbb{R},\ p \mapsto \max\{\langle x, p\rangle : x \in K\}$$

**Remark C.2.5.** (i) The existence of the supremum of the set $\{\langle x, p\rangle : x \in K\}$ is again due to the compactness of $K$.
(ii) The support function of a convex body is a positively homogeneous and convex function:

$$\begin{aligned} h_K(p+q) &= \max\{\langle x, p+q\rangle : x \in K\} \\ &\leq \max\{\langle x, p\rangle : x \in K\} + \max\{\langle x, q\rangle : x \in K\} = h_K(p) + h_K(q) \end{aligned}$$

(iii) A coordinate free definition of the support function is the following:
Let $K \subset V$ be a convex set in an $\mathbb{R}$-vector space $V$. Define

$$h : V^\vee \to \mathbb{R}, \ell \mapsto \sup\{\ell(x) : x \in K\}$$

This gives exactly the definition above, if we fix a basis on $V$ and identify $V^\vee$ with $V$ via the canonical scalar product. We stick to the above definition because we want to encourage the geometric connection between the support function and supporting hyperplanes to $K$ (cf. Remark C.2.12), i.e. the support function measures the distance of a supporting hyperplane to $K$ and the origin. Here the supporting hyperplanes are identified with points in $\mathbb{R}^n$ by taking outward (i.e. the set $K$ is contained in the opposite closed half space defined by the hyperplane) normal vectors.

On the other hand, given a positively homogeneous and convex function, there is a convex body with the given function as its support function:

**Theorem C.2.6** ( [Sch93], Theorem 1.7.1)**.** *If $f : \mathbb{R}^n \to \mathbb{R}$ is a positively homogeneous and convex function, then there is a unique convex body $K \subset \mathbb{R}^n$ with support function $f$.*
*In particular we see that there is a bijection between convex bodies and positively homogeneous and convex functions given by the support function.*

**Remark C.2.7.** (i) In the proof of the above theorem in [Sch93], it is shown that the set $K := \{x \in \mathbb{R}^n : \langle x, p\rangle \leq f(p)\ \textit{for all}\ p \in \mathbb{R}^n\}$ is the convex body whose existence is claimed, i.e. $f$ is the support function of this convex body $K$.
(ii) This theorem allows us to define a support function as a positively homogeneous and convex function. This definition then gives the same class of functions as the one given above.

The Minkowski functional and the support function of a convex body are closely related. To make this precise, we need the following notion:

**Definition C.2.8.** Let $K \subset \mathbb{R}^n$ be a set. The *polar* of $K$ is the set

$$K^o := \{x \in \mathbb{R}^n : \langle x, p\rangle \leq 1\ \textit{for all}\ p \in K\}$$

For a convex body, this notion is very well behaved for it is a duality of sets:

**Theorem C.2.9** (Bipolar Theorem, cf. [Bar02], Theorem IV.1.2)**.** *Let $K \subset \mathbb{R}^n$ be a convex body containing the origin. Then the bipolar of $K$ equals $K$: $(K^o)^o = K$.*

With the help of the polar, we can now precisely relate the support function and the Minkowski functional of a convex body.

**Theorem C.2.10** ( [Sch93], Theorem 1.7.6)**.** *Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior. Then the Minkowski functional of $K$ is the support function of the polar of $K$: $p_K(x) = h_{K^o}(x)$ for all $x \in \mathbb{R}^n$.*

Next, we want to relate the order of differentiability of the Minkowski functional, the support function and the boundary of the convex set as a hypersurface in $\mathbb{R}^n$:

**Lemma C.2.11.** *Let $K \subset \mathbb{R}^n$ be a convex body with $0 \in \mathrm{int}(K)$. The following two statements are equivalent:*
*(i) The boundary of $K$ is a smooth hypersurface in $\mathbb{R}^n$.*
*(ii) The Minkowski functional $p_K$ of $\mathbb{R}^n$ is smooth in all points except the origin and $p'_K(x) \neq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.*
*Furthermore, the following two statements are equivalent:*
*(iii) The boundary of $K$ is a smooth and positively curved hypersurface in $\mathbb{R}^n$.*
*(iv) The support function $h_K$ of $K$ is smooth in all points except the origin, $h'_K(p) \neq 0$ for all $p \in \mathbb{R}^n \setminus \{0\}$ and $h_K$ is strictly quasiconvex.*

PROOF. Statement (i) implies (ii) by the implicit function theorem applied to the fact that $p_K(x)$ is the non-negative real number $\lambda$ such that $\frac{1}{\lambda}x \in \partial K$, i.e. $g(\frac{1}{\lambda}x) = 0$ for a function $g\colon V \to \mathbb{R}$ defining $\partial K$ in a neighbourhood $V$ of $p = \frac{1}{p_K(x)}x$. The reverse implication is again an application of the implicit function theorem (combine Lemma C.1.1 and Remark C.2.3(iii)). The equivalence of (iii) and (iv) is [Gho04], Lemma 3.1. □

A similar statement can be found in [KP99] (cf. Corollary 6.3.13).

**Remark C.2.12.** Let $K \subset \mathbb{R}^n$ be a convex body and $x \in \partial K$. Assume that there is a neighbourhood $U \subset \mathbb{R}^n$ of $x$ such that $U \cap \partial K$ is a smooth hypersurface in $\mathbb{R}^n$. In this case, the unique supporting hyperplane at $x$ is the affine tangent space $\widehat{T}_x(U \cap \partial K)$ at $x$ to $U \cap \partial K$ and therefore we conclude for the support function

$$h(N(x)) = \langle y, N(x) \rangle$$

where $N$ is an outward unit normal vector field of $U \cap \partial K$ and $y$ some point in $K \cap \widehat{T}_x(U \cap \partial K)$, e.g. $x$. This means $h(p) = \langle y, p \rangle$ for any $y \in N^{-1}(p)$ (the value of the scalar product does not depend on the choice) and all $p \in N(U \cap \partial K) \subset S^{n-1}$.

**Corollary C.2.13.** *Let $K \subset \mathbb{R}^n$ be a convex body. The boundary of $K$ is a smooth and positively curved hypersurface in $\mathbb{R}^n$ if and only if the Minkowski functional $p_K$ of $K$ is smooth in all points except the origin, $p'_K(x) \neq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $p_K$ is strictly quasiconvex in all $x \in \partial K$.*

PROOF. The claim for the smoothness of the boundary and the Minkowski functional is Lemma C.2.11. We need to check the claim for the curvature: For a point $x \in \partial K$, the negative gradient of the Minkowski-functional $-p'_K(x)$ is an inward normal, i.e. $K \subset \{v \in \mathbb{R}^n : \langle v - x, -p'_K(x)\rangle \geq 0\}$, because $K = \{x \in \mathbb{R}^n : p_K(x) \leq 1\}$. This means that the second fundamental form at $x$ is given by

$$II_x(v,w) = \frac{1}{\|p'_K(x)\|}\langle p''_K(x)v, w\rangle$$

This shows the claimed equivalence of the statement that $\partial K$ is positively curved and the statement that $p_K$ is strictly quasiconvex in all points $x \in \partial K$. □

# Zusammenfassung auf Deutsch

In dieser Arbeit werden die wesentlichen Ergebnisse aus den Arbeiten [HN09] and [HN10] im Detail dargestellt. Es geht dabei im Wesentlichen um die Frage, wann eine konvexe, semi-algebraische Menge die Projektion eines Spektraeders ist - ein Spektraeder ist eine von einer linearen Matrixungleichung definierte Menge, d.h. eine Menge der Form

$$\{(x_1,\dots,x_n)\in\mathbb{R}^n : A_0 + x_1A_1 + \dots + x_nA_n \textit{ ist positiv semi-definit}\}$$

wo $A_0,\dots,A_n \in \mathrm{Sym}_{d\times d}(\mathbb{R})$ symmetrische Matrizen mit reellen Einträgen sind. Es ist recht einfach einzusehen, dass Spektraeder konvexe und basisch-abgeschlossene, semi-algebraische Mengen sind. Deshalb folgt sofort, dass Projektionen von Spektraedern ebenfalls konvex und semi-algebraisch sein müssen (nach dem Projektionssatz für semi-algebraische Mengen, s. [BCR98], Theorem 2.2.1). Weitere Eigenschaften von Spektraedern und ihren Projektionen werden im ersten Teil des zweiten Kapitels (nach der Einleitung) besprochen. In den verbleibenden beiden Abschnitten des ersten Kapitels geht es um Konstruktionen, die mit der konvexen Hülle von endlich vielen (Projektionen von) Spektraedern zusammenhängen. Das Hauptergebnis ist ein Lokal-Global-Prinzip, das im letzten Abschnitt des ersten Kapitels bewiesen wird.
Im dritten Kapitel wird die Lasserre-Relaxierung eingeführt. Dabei handelt es sich um eine konstruktive Methode, um aus den definierenden Polynomen einer basisch abgeschlossenen, semi-algebraischen Menge mit Hilfe von quadratischen Moduln (oder spezieller Präordnungen) eine Darstellung der Menge als Projektion eines Spektraeders zu erhalten. Im ersten Teil des Kapitels werden Grundlagen, die mit quadratischen Moduln und konvexen Kegeln zusammenhängen, zusammengefasst und im zweiten Teil dazu verwendet, den wichtigsten Satz über die Lasserre-Relaxierung zu zeigen. Dieser besagt, dass sie genau dann eine Darstellung als Projektion eines Spektraeders liefert (wir sagen "exakt wird"), wenn alle linearen Polynome, die nicht-negativ auf der gegebenen Menge sind, eine Darstellung im von den definierenden Polynomen erzeugten quadratischen Modul haben (mit Gradschranken für die benötigten Quadratsummen von Polynomen).
Im vierten Kapitel werden dann die Ergebnisse aus den beiden Arbeiten von Helton und Nie dargestellt. Diese Ergebnisse verwenden die im dritten Kapitel vorgestellte Lasserre-Relaxierung, um notwendige Bedingungen, die die Exaktheit der Lasserre-Relaxierung sicherstellen, an die definierenden Polynome einer basisch-abgeschlossenen semi-algebraischen Menge zu zeigen. Im ersten Teil des Kapitels wird eine Basisversion der folgenden Resultate gezeigt. In den beiden weiteren Abschnitten des Kapitels wird diese Basisversion unter steigendem technischen Aufwand immer weiter verfein-

ert. Eine der Hauptschwächen der Resultate ist wohl die Beschränkung auf basisch-abgeschlossene, semi-algebraische Mengen. Sie wird nur durch die Resultate über konvexe Hüllen von (Projektionen von) Spektraedern aus Kapitel 2 teilweise aufgewogen, weil die konvexe Hülle von basisch-abgeschlossenen, semi-algebraischen Mengen im allgemeinen nicht mehr basisch-abgeschlossen ist. Andererseits sind die notwendigen Bedingungen für basisch-abgeschlossene Mengen nicht weit von der hinreichenden Bedingung, die im zweiten Teil von Anhang A bewiesen wird, entfernt. Tatsächlich handelt es sich lediglich um die Frage, ob die Hesse-Matrix eines definierenden Polynoms negativ definit oder negativ semi-definit ist. Trotzdem ist diese Lücke groß genug und die Resultate decken nicht alle Fälle ab, in denen die Lasserre-Relaxierung exakt ist.
In den Anhängen werden verschiedene Informationen zusammengefasst, die nicht leicht in der Literatur zu finden sind und für die Arbeit relevant sind.

# Bibliography


[Bar02] A. Barvinok. *A course in convexity*, volume 54 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI (**2002**).

[BCR98] J. Bochnak, M. Coste, and M.-F. Roy. *Real algebraic geometry*, volume 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin (**1998**). Translated from the 1987 French original, Revised by the authors.

[Blo97] E. D. Bloch. *A first course in geometric topology and differential geometry*. Birkhäuser Boston Inc., Boston, MA (**1997**).

[Bou81] N. Bourbaki. *Espaces vectoriels topologiques. Chapitres 1 à 5*. Masson, Paris, new edition (**1981**). Éléments de mathématique. [Elements of mathematics].

[Brä10] P. Brändén. *Obstructions to determinantal representability* (**2010**). Preprint.
URL `http://arxiv.org/abs/1004.1382`

[BV04] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, Cambridge (**2004**).

[CLR80] M. D. Choi, T. Y. Lam, and B. Reznick. *Real zeros of positive semidefinite forms. I. Math. Z.*, 171(1):1–26 (**1980**).

[Eis95] D. Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York (**1995**). With a view toward algebraic geometry.

[Gho04] M. Ghomi. *Optimal smoothing for convex polytopes*. *Bull. London Math. Soc.*, 36(4):483–492 (**2004**).

[GN10] J. Gouveia and T. Netzer. *Positive polynomials and projections of spectrahedra* (**2010**). Preprint, work in progress.

[GPT08] J. Gouveia, P. A. Parrilo, and R. R. Thomas. *Theta bodies for polynomial ideals* (**2008**). Preprint.
URL `http://arxiv.org/abs/0809.3480`

[Gre67] M. J. Greenberg. *Lectures on algebraic topology*. W. A. Benjamin, Inc., New York-Amsterdam (**1967**).

[Hen09] D. Henrion. *Semidefinite representation of convex hulls of rational varieties* (**2009**). Preprint.
URL `http://arxiv.org/abs/0901.1821`

[HF] D. Hoffmann and W. Forst. *Optimization - Theory and Practice*. SUMAT. Springer-Verlag. To appear, a preview of Chapter 2 can be found on the given URL.
URL `http://www.math.uni-konstanz.de/~hoffmann/OPTIMIZATION/`

[Hic65] N. J. Hicks. *Notes on differential geometry*. Van Nostrand Mathematical Studies, No. 3. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London (**1965**).

[HN09] J. W. Helton and J. Nie. *Sufficient and necessary conditions for semidefinite representability of convex hulls and sets*. *SIAM J. Optim.*, 20(2):759–791 (**2009**).

[HN10] J. W. Helton and J. Nie. *Semidefinite representation of convex sets*. *Math. Program.*, 122(1, Ser. A):21–64 (**2010**).

[HV07] J. W. Helton and V. Vinnikov. *Linear matrix inequality representation of sets*. *Comm. Pure Appl. Math.*, 60(5):654–674 (**2007**).

[KMS05] S. Kuhlmann, M. Marshall, and N. Schwartz. *Positivity, sums of squares and the multi-dimensional moment problem. II*. *Adv. Geom.*, 5(4):583–606 (**2005**).

[Köt69] G. Köthe. *Topological vector spaces. I*. Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159. Springer-Verlag New York Inc., New York (**1969**).

[KP99] S. G. Krantz and H. R. Parks. *The geometry of domains in space*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston Inc., Boston, MA (**1999**).

[Las09] J. B. Lasserre. *Convex sets with semidefinite representation*. *Math. Program.*, 120(2, Ser. A):457–477 (**2009**).

[LPR05] A. S. Lewis, P. A. Parrilo, and M. V. Ramana. *The Lax conjecture is true*. *Proc. Amer. Math. Soc.*, 133(9):2495–2499 (electronic) (**2005**).

[Mar08] M. Marshall. *Positive polynomials and sums of squares*, volume 146 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI (**2008**).

[Nar68] R. Narasimhan. *Analysis on real and complex manifolds*. Advanced Studies in Pure Mathematics, Vol. 1. Masson & Cie, Éditeurs, Paris (**1968**).

[NPS10] T. Netzer, D. Plaumann, and M. Schweighofer. *Exposed faces of semidefinitely representable sets*. *SIAM J. Optim*, 20(4):1944–1955 (**2010**).

[NS07] J. Nie and M. Schweighofer. *On the complexity of Putinar's Positivstellensatz*. *J. Complexity*, 23(1):135–150 (**2007**).

[NS09] T. Netzer and R. Sinn. *A note on the convex hull of finitely many projections of spectrahedra* (**2009**).
URL `http://arxiv.org/abs/0908.3386`

[PD01] A. Prestel and C. N. Delzell. *Positive polynomials*. Springer Monographs in Mathematics. Springer-Verlag, Berlin (**2001**). From Hilbert's 17th problem to real algebra.

[PS01] V. Powers and C. Scheiderer. *The moment problem for non-compact semialgebraic sets*. *Adv. Geom.*, 1(1):71–88 (**2001**).

[RG95] M. Ramana and A. J. Goldman. *Some geometric results in semidefinite programming*. *J. Global Optim.*, 7(1):33–50 (**1995**).

[Roc70] R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J. (**1970**).

[Sch08] C. Scheiderer. *Reelle Algebraische Geometrie* (**2007/08**). Lecture Notes of his lecture 2007/2008 at Konstanz (in german).
URL `www.math.uni-konstanz.de/~scheider/lehre.html`

[Sch93] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge (**1993**).

[Sch05a] C. Scheiderer. *Distinguished representations of non-negative polynomials*. *J. Algebra*, 289(2):558–573 (**2005**).

[Sch05b] M. Schweighofer. *Optimization of polynomials on compact semialgebraic sets*. *SIAM J. Optim.*, 15(3):805–825 (electronic) (**2005**).

[Spi70] M. Spivak. *A comprehensive introduction to differential geometry. Vol. One*. Published by M. Spivak, Brandeis Univ., Waltham, Mass. (**1970**).

[SSS09] R. Sanyal, F. Sottile, and B. Sturmfels. *Orbitopes* (**2009**). Preprint.
URL `http://arxiv.org/abs/0911.5436`

[Ste96] G. Stengle. *Complexity estimates for the Schmüdgen Positivstellensatz*. *J. Complexity*, 12(2):167–174 (**1996**).

[VB96] L. Vandenberghe and S. Boyd. *Semidefinite programming*. *SIAM Rev.*, 38(1):49–95 (**1996**).