Asymptotic stability in a second-order symmetric hyperbolic system modeling the relativistic dynamics of viscous heat-conductive fluids with diffusion

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Abstract

This paper establishes nonlinear asymptotic stability of homogeneous reference states in dissipative relativistic fluid dynamics. The result is a counterpart for general non-barotropic fluids of one obtained by the author in a previous paper on barotropic fluids. Differently from that of this earlier finding, the proof here crucially relies on analyzing the corresponding linearized problem in Fourier space, with different scalings for small and large wave numbers.

1 Introduction

The present paper studies a second-order quasilinear system of partial differential equations,

$$A^{\alpha\beta g}(\psi)\frac{\partial\psi_g}{\partial x^\beta} = \frac{\partial}{\partial x^\beta}\left(B^{\alpha\beta g\delta}(\psi)\frac{\partial\psi_g}{\partial x^\delta}\right), \tag{1.1}$$

that was recently proposed in [4, 8] as a model for the relativistic dynamics of fluids in the presence of viscosity, heat conduction, and diffusion.[1] To motivate system (1.1), we start from the relativistic Euler equations

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta}\right) = 0, \tag{1.2}$$

$$\frac{\partial}{\partial x^\beta}\left(N^{\beta}\right) = 0, \tag{1.3}$$

in which

$$T^{\alpha\beta} = (\rho+p)u^\alpha u^\beta + \rho g^{\alpha\beta}, \quad N^\beta = nu^\beta$$

are the energy-momentum tensor and the matter current of an ideal fluid, of specific energy ρ, pressure p, and particle number density n, that moves at a 4-velocity u^α. The fluid is specified by an equation of state which identifies its specific internal energy $e = e(n,s)$ as a function of n and

[1] We use the Einstein summation convention, Greek indices run from 0 to 3 and are raised or lowered by contraction with $g^{\alpha\beta}, g_{\alpha\beta}$, where $g^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ is the standard Minkowski metric; cf., e.g., [20, Section 2.5]. Roman indices a, g, e run from 0 to 4 and are symbolically used like the Greek ones, including the summation convention.

specific entropy s. Expressions for other thermodynamic state variables such as ρ and p as well as the temperature θ and chemical potential υ derive from the equation of state as

$$\begin{aligned} &\rho = \rho(n,s) = ne(n,s), \quad &&p = p(n,s) = n^2 e_n(n,s) = \hat{p}(\rho, s), \\ &\theta = \theta(n,s) = e_s(n,s), \quad &&\upsilon = (\rho + p)/n - \theta s = \hat{\upsilon}(p, \theta). \end{aligned} \tag{1.4}$$

On the equation of state, we assume only the two fundamental properties characterizing the generic causal non-barotropic one-phase fluid:

$$0 < \frac{\partial \hat{p}}{\partial \rho}(\rho, s) < 1, \tag{1.5}$$

$$D^2 \hat{\upsilon}(p, \theta) < 0. \tag{1.6}$$

Assumption (1.5) means that the speed of sound is strictly positive and strictly smaller than the speed of light, while (1.6) says that system (1.2), (1.3) possesses a convex mathematical entropy (cf. [17]). This allows writing (1.2), (1.3) as a symmetric hyperbolic system

$$A^{a\beta g}(\psi) \frac{\partial \psi_g}{\partial x^\beta} = 0, \tag{1.7}$$

in the Godunov-Boillat variables $\psi^\alpha = u^\alpha/\theta$, $\psi^4 = \upsilon/\theta$ as new primary unknowns; in (1.7), the coefficients $A^{a\beta g}$ derive as

$$A^{a\beta g}(\psi) = \frac{\partial^2 \left(\hat{X}(\theta, \psi^4) \psi^\beta \right)}{\partial \psi_a \partial \psi_g}.$$

from a scalar function $\hat{X}(\theta, \psi^4)$ which is induced by the equation of state [17]. The left-hand side of (1.1) is thus the relativistic Euler operator for general non-barotropic ideal fluids in Godunov-Boillat form. Note that (1.6) excludes the barotropic fluids studied in [18].[2]

Now, compared to the equations (1.2), (1.3) of the ideal case, dissipative fluid dynamics is often formulated as

$$\frac{\partial}{\partial x^\beta} \left(T^{\alpha\beta} + \Delta T^{\alpha\beta} \right) = 0, \tag{1.8}$$

$$\frac{\partial}{\partial x^\beta} \left(N^\beta + \Delta N^\beta \right) = 0, \tag{1.9}$$

where $\Delta T^{\alpha\beta}$, ΔN^β augment the energy-momentum tensor and the matter current to describe the dissipation effects. System (1.1) expresses (1.8),

[2]For these, $\upsilon \equiv 0$ and equations (1.2) uncouple from (1.3); cf. [5].

(1.9) in the Godunov-Boillat variables. We assume that the corresponding coefficients $B^{\alpha\beta g\delta}$ on the right-hand side of (1.1) are given by [3]

$$\begin{aligned} B^{\alpha\beta\gamma\delta} = \chi\theta^2 u^\alpha u^\gamma g^{\beta\delta} - \sigma\theta u^\beta u^\delta \Pi^{\alpha\gamma} + \tilde{\zeta}\theta\Pi^{\alpha\beta}\Pi^{\gamma\delta} \\ + \eta\theta(\Pi^{\alpha\gamma}\Pi^{\beta\delta} + \Pi^{\alpha\delta}\Pi^{\beta\gamma} - \frac{2}{3}\Pi^{\alpha\beta}\Pi^{\gamma\delta}) \\ + \sigma\theta(u^\alpha u^\beta g^{\gamma\delta} - u^\alpha u^\delta g^{\beta\gamma}) + \chi\theta^2(u^\beta u^\gamma g^{\alpha\delta} - u^\gamma u^\delta g^{\alpha\beta}) \end{aligned} \quad (1.10)$$

$$B^{\alpha\beta 4\delta} = 0, \quad B^{4\beta\gamma\delta} = \sigma\theta\left(u^\beta g^{\gamma\delta} - u^\delta g^{\beta\gamma}\right), \quad (1.11)$$

$$B^{4\beta 4\delta} = \mu g^{\beta\delta}, \quad (1.12)$$

where $\eta, \tilde{\zeta}, \chi$ and μ denote the coefficients of shear viscosity, modified bulk viscosity, heat conduction, and diffusion, and $\sigma = (4/3)\eta + \tilde{\zeta}$. With this choice the right-hand side of (1.1) is the second-order symmetric hyperbolic description of dissipation given by Freistühler and Temple. Without entering into details of its derivation and justification we refer the reader to [4, 8] for these and limit ourselves here to the remark that this description has been provided as an alternative to previous formulations which are known to suffer from deficits such as a lack of causality (cf. references brought up in the introductions of [7, 8]).

The purpose of this paper is to show the following.

1.1 Theorem. *Let $s \geq 3$ and $\bar{\psi} = (\bar{\theta}^{-1}, 0, 0, 0, \bar{\upsilon}/\bar{\theta})$ with $\bar{\theta}$ and $\bar{\upsilon}$ being constant. Then there exist $\delta_0 > 0$, $C_0 = C_0(\delta_0) > 0$ such that the following holds: For all $({}^0\psi, {}^1\psi)$ with $({}^0\psi - \bar{\psi}, {}^1\psi) \in \left(H^{s+1} \times H^s\right) \cap \left(L^1 \times L^1\right)$ and $\|({}^0\psi - \bar{\psi}, {}^1\psi)\|^2_{s+1,s,1} < \delta_0$ there exists a unique solution ψ to* (1.1) *with $\psi(0) = {}^0\psi$, $\psi_t(0) = {}^1\psi$ and*

$$\psi - \bar{\psi} \in \bigcap_{j=0}^{s} C^j\left([0,\infty), H^{s+1-j}\right).$$

Furthermore

$$\begin{aligned} \|(\psi(t) - \bar{\psi}, \psi_t(t))\|^2_{s+1,s} + \int_0^t \|(\psi(\tau) - \bar{\psi}, \psi_t(\tau))\|^2_{s+1,s} d\tau \\ \leq C_0\|({}^0\psi - \bar{\psi}, {}^1\psi)\|^2_{s+1,s}, \end{aligned} \quad (1.13)$$

$$\|(\psi(t) - \bar{\psi}, \psi_t(t))\|_{s,s-1} \leq C_0(1+t)^{-\frac{3}{4}}\|({}^0\psi - \bar{\psi}, {}^1\psi)\|_{s,s-1,1} \quad (1.14)$$

for all $t \in [0, \infty)$.

[3] We use the standard projection $\Pi^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$.

This means nonlinear asymptotic stability of homogeneous reference states of (1.1). The result is thus a counterpart for general non-barotropic fluids of one obtained in [18] for barotropic fluids. While for barotropic fluids the conservation laws (1.8) for energy and momentum uncouple from the matter conservation law (1.9), the five equations (1.8), (1.9) now remain genuinely coupled. The treatment of the linearized system in [18], namely via energy methods in the spirit of Kawashima [12, 13], [2, 3], relies on the particular structure of the coefficient matrices, in a way that does not seem to permit carrying that treatment over to the present non-barotropic context. In this paper a finer analysis of the linearized problem in Fourier space is used to obtain decay estimates. Analogues of some of the key features of this method have earlier been proposed by Bianchini, Hanouzet, Natalini in [1] in the context of first-order nonlinear hyperbolic systems with partial dissipation.[4]

To show Theorem 1.1 we proceed as follows. In the central Section 2, we show the uniform definiteness of families of matrices which depend on a parameter that will later correspond to the wave number. Very similarly to parts of the argumentation in [1], in different regimes, the matrices are decomposed with respect to carefully chosen invariant subspaces and expanded in Taylor series. The desired definiteness properties follow from algebraic conditions, 2.1, 2.4, that we introduce so as to reflect dissipativity of the PDE system in the various regimes. In Section 3 we use the results of Section 2 to derive decay and energy estimates for the linearized system. Finally in Section 4 the full quasi-linear system is considered. Following the classical approach for dissipative wave equations in [16], that was also employed for hyperbolic-parabolic systems [12, 13] and first-order symmetric hyperbolic systems [1, 9, 14, 21, 22], we view the nonlinearities as perturbations of the linearized system and show that the estimates of Section 3 carry over to the nonlinear case.

It is an interesting question as to wether system (1.1) can be cast in the form of a symmetric hyperbolic system of first order, in which case methods and results of [1, 9, 14, 21, 22] might provide an alternative proof of Theorem 1.1. Note, however, that standard ways of introducing first-order derivatives as additional dependent variables do *not* readily achieve this. For connections between our dissipativity conditions 2.1, 2.4 with the celebrated Kawashima-Shizuta condition [13], cf. [6].

As a preparatory step, we now bring our problem into a convenient form from which we will start the considerations in Sections 3 and 4.

[4] I thank an anonymous referee for pointing out this important paper to me.

It was shown in [4] that (1.1) is symmetric hyperbolic in the sense of Hughes-Kato-Marsden [10]. Thus, using

$$B^{a\beta g\delta}(\psi)\frac{\partial\psi_\gamma}{\partial x^\beta \partial x^\delta} = \tilde{B}^{a\beta g\delta}(\psi)\frac{\partial\psi_\gamma}{\partial x^\beta \partial x^\delta},$$

with

$$\tilde{B}^{a\beta g\delta}(\psi) = \frac{1}{2}\left(B^{a\beta g\delta}(\psi) + B^{a\beta g\delta}(\psi)\right),$$

we can write (1.1) as

$$A(\psi)\psi_{tt} - B^{ij}(\psi)\psi_{x_i x_j} + D^j(\psi)\psi_{tx_j} + f(\psi, \psi_t, \partial_x\psi) = 0, \tag{1.15}$$

where

$$\begin{aligned} A &= (-\tilde{B}^{a0g0})_{0\le a,g\le 4}, \quad B^{ij} = (\tilde{B}^{aigj})_{0\le a,g\le 4},\\ D^j &= (-2\tilde{B}^{a0gj})_{0\le a,g\le 4} \end{aligned}$$

are symmetric 5×5 matrices, $A(\psi)$ is positive definite, $\xi_i B^{ij}(\psi)\xi_j$ is positive definite for arbitrary $\xi \in \mathbb{R}^3 \setminus \{0\}$, and

$$f^a = A^{a\beta g}\frac{\partial\psi_g}{\partial x^\beta} - \frac{\partial}{\partial x^\beta}\left(B^{a\beta g\delta}(\psi)\right)\frac{\partial\psi_g}{\partial x^\delta}, \;\; a = 0,1,2,3,4.$$

In the sequel, we will consider the Cauchy problem associated with (1.15):

$$A\psi_{tt} - B^{ij}\psi_{x_i x_j} + D^j\psi_{tx_j} + f = 0 \text{ on } (0,T]\times\mathbb{R}^3, \tag{1.16}$$

$$\psi(0) = {}^0\psi \text{ on } \mathbb{R}^3, \tag{1.17}$$

$$\psi_t(0) = {}^1\psi \text{ on } \mathbb{R}^3. \tag{1.18}$$

Notation. In the statement and in the sequel, the following notation is used. For $p \in [1,\infty]$ and some $n, m \in \mathbb{N}$ just write L^p for $L^p(\mathbb{R}^n, \mathbb{R}^m)$. For $s \in \mathbb{N}_0$ we denote by H^s the L^2-Sobolev-space of order s, namely

$$H^s := \{u \in L^2 : \forall\ \alpha \in \mathbb{N}_0^n\ (|\alpha| \le s) : \|\partial_x^\alpha u\|_{L^2} < \infty\}$$

with norm

$$\|u\|_s = \left(\sum_{0\le|\alpha|\le s} \|\partial_x^\alpha u\|_{L^2}\right)^{\frac{1}{2}}.$$

We just write $\|u\|$ instead of $\|u\|_0$. For $s, k \in \mathbb{N}_0$ and $U = (u_1, u_2) \in H^s \times H^k$ set

$$\|U\|_{s,k} = \left(\|u_1\|_s^2 + \|u_2\|_k^2\right)^{\frac{1}{2}}$$

and for $U \in (H^s \times H^k) \cap (L^p \times L^p)$ set

$$\|U\|_{s,k,p} = \|U\|_{s,k} + \|u_1\|_{L^p} + \|u_2\|_{L^p}.$$

For $u \in H^s$ and integers $0 \le k \le s$, $\partial_x^k u$ shall denote the vector whose entries are the partial derivatives of u of order k. Also we just write ψ instead of ψ_e for the state variable.

2 Perturbation theory for a class of linear second-order symmetric hyperbolic operators

The Fourier analysis of the linearization of (1.1) in Section 3 will require an understanding of the time asymptotics of certain families

$$W_t(\kappa, t) = M(\kappa)W(\kappa, t)$$

of systems of ordinary differential equations. The goal is to show that with respect to appropriate bases the self-adjoint part $M(\kappa) + M(\kappa)^*$ becomes definite, uniformly in κ in appropriate ranges.

We consider two situations that will later correspond to small wave numbers, (a), and large wave wave numbers, (b), and a connection between them, (c). In the following a, b, B are fixed symmetric $N \times N$ matrices with a, B positive definite.

(a) Small wave numbers

We first study the matrix family

$$M(\kappa) = \begin{pmatrix} 0 & I \\ -\kappa^2 B + i\kappa b & -a \end{pmatrix}, \quad \kappa \in \mathbb{C}. \tag{2.1}$$

We set

$$\tilde{B} = a^{-\frac{1}{2}} B a^{-\frac{1}{2}}, \quad \tilde{b} = a^{-\frac{1}{2}} b a^{-\frac{1}{2}}$$

and

$$\tilde{N}(\kappa) = -\kappa^2 \tilde{B} + i\kappa\tilde{b}.$$

Requiring definiteness of one operator on all eigenspaces of a second one, the following dissipativity condition 2.1, as well as condition 2.4 below, are similar in spirit to the Kawashima-Shizuta condition ([13], [1, 14, 15, 22]).

2.1 Condition. There exists a $C > 0$ such that for all $\lambda \in \sigma(\tilde{b})$, each $u \in \mathrm{Eig}_{\tilde{b}}(\lambda)$ satisfies

$$\langle (\tilde{B} - \tilde{b}a^{-1}\tilde{b})u, u\rangle \geq C|u|^2.$$

2.2 Proposition. *If Condition 2.1 holds, then there exist $r_0, c_0 > 0$ and a holomorphic family of invertible $2N \times 2N$ matrices*

$$\{S(\kappa) : |\kappa| \leq r_0\},$$

such that $M_*(\kappa) = (S(\kappa))^{-1}M(\kappa)S(\kappa)$ satisfies

$$\langle (M_*(\kappa) + (M_*(\kappa))^*)u, u\rangle \leq -c_0\kappa^2|u|^2,$$

for all $u \in \mathbb{C}^{2N}$ and $\kappa \in \mathbb{R}$, $|\kappa| \leq r_0$.

Proof. For $\kappa = 0$, $\mathbb{R}^{2N}$ splits as $\ker(M(0)) \oplus \mathrm{im}(M(0))$; the main idea is to continue this splitting regularly as κ is varied away from 0. (The same idea was used before in [1].)

First set

$$S_1 = \begin{pmatrix} a^{\frac{1}{2}} & a^{\frac{1}{2}} \\ 0 & -a^{\frac{3}{2}} \end{pmatrix}.$$

and

$$T(\kappa) := S_1^{-1}M(\kappa)S_1 = \begin{pmatrix} \tilde{N}(\kappa) & \tilde{N}(\kappa) \\ -\tilde{N}(\kappa) & -\tilde{N}(\kappa) - a \end{pmatrix}.$$

Then write $T(\kappa) = T^{(0)} + \kappa T^{(1)} + \kappa^2 T^{(2)}$, where

$$T^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}, \; T^{(1)} = i\begin{pmatrix} \tilde{b} & \tilde{b} \\ -\tilde{b} & -\tilde{b} \end{pmatrix}, \; T^{(2)} = \begin{pmatrix} -\tilde{B} & -\tilde{B} \\ \tilde{B} & \tilde{B} \end{pmatrix}.$$

For $\zeta \in \rho(T^{(0)})$ let $R(\zeta)$ denote the resolvent of $T^{(0)}$. Let $0 < \tau < \min \sigma(a)$ and set $\Gamma = \partial B_\tau(0) \subset \mathbb{C}$. Now choose $r > 0$ such that

$$\max_{\kappa \in \overline{B_r(0)}} |A(\kappa)| < \min_{\zeta \in \Gamma} |R(\zeta)|^{-1},$$

where $A(\kappa) := T(\kappa) - T^{(0)}$. Then for $\kappa \in \overline{B_r(0)}$ and $\zeta \in \Gamma$, the resolvent $R(\zeta, \kappa)$ of $T(\kappa)$ is well-defined and holomorphic on $\Gamma \times \overline{B_r(0)}$ (cf. [11, Chapter II.1]). Next define

$$P(\kappa) = -\frac{1}{2\pi i}\int_\Gamma R(\zeta, \kappa)d\zeta.$$

Since $\mu_0 = 0$ is the only eigenvalue of $T^{(0)}$ lying inside Γ, $P(\kappa)$ is the total projection for the μ_0-group of eigenvalues of $T(\kappa)$ (cf. [11, chapter II.2]), particularly

$$P := P(0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

We now use the so called reduction process (cf. ibd.) to expand the product $P(\kappa)T(\kappa)P(\kappa)$ in a Taylor series. To this end consider the operator

$$\tilde{T}(\kappa) = \frac{1}{\kappa} T(\kappa)P(\kappa), \ \tilde{T}(0) = PT^{(1)}P.$$

Due to $T(\kappa)R(\zeta,\kappa) = I + \zeta R(\zeta,\kappa)$ and $R(\zeta)$ having a pole of order one at $\zeta = 0$, $\tilde{T}$ is holomorphic. Furthermore

$$\tilde{T}(0) = \begin{pmatrix} i\tilde{b} & 0 \\ 0 & 0 \end{pmatrix}$$

and thus

$$\sigma(\tilde{T}(0)) = i\sigma(\tilde{b}) \cup \{0\}.$$

Let $\lambda_1, \ldots, \lambda_s$ be (the pairwise distinct) eigenvalues of $\tilde{b}$. At this point we may assume w.l.o.g $\lambda_i \neq 0$ for $1 \leq i \leq s$ (otherwise we could have considered $M + i\alpha\kappa I$ instead of M for a suitable $\alpha > 0$). Let

$$\gamma = \min\{|\lambda_j|, |\lambda_i - \lambda_j| : 1 \leq i,j \leq s, i \neq j\} > 0.$$

and $\Gamma_j = \partial B_{\gamma/2}(i\lambda_j)$ for $j = 1, \ldots, s$. Then the resolvent $\tilde{R}(\zeta)$ of $\tilde{T}(0)$ is well defined for $\zeta \in \Gamma_j$ and satisfies

$$|\tilde{R}(\zeta)|^{-1} = \mathrm{dist}(\zeta, \sigma(\tilde{T}(0)) \geq \frac{\gamma}{4}.$$

Now chose $0 < \tilde{r} < r$ such that

$$|\tilde{A}(\kappa)| < \frac{\gamma}{4}, \text{ for all } \kappa \in \overline{B_{\tilde{r}}(0)},$$

where $\tilde{A}(\kappa) = \tilde{T}(\kappa) - \tilde{T}(0)$. Then for $\zeta \in \Gamma_j$, $|\kappa| \leq \tilde{r}$, the resolvent $\tilde{R}(\zeta,\kappa)$ of $\tilde{T}(\kappa)$ is well-defined and holomorphic. Again, the total projection for the $i\lambda_j$-group of eigenvalues of $\tilde{T}(\kappa)$, i.e.

$$P_j(\kappa) = -\frac{1}{2\pi i} \int_{\Gamma_j} \tilde{R}(\zeta,\kappa) d\zeta, \qquad (2.2)$$

is itself holomorphic for $|\kappa| \leq \tilde{r}$ and

$$P_j := P_j(0) = \begin{pmatrix} p_j & 0 \\ 0 & 0 \end{pmatrix},$$

where p_j is the eigenprojection corresponding to the eigenvalue λ_j of $\tilde{b}$. Furthermore each eigenvalue of the μ_0-group belongs to some $i\kappa\lambda_j$-group and the total projection for this group is $P_j(\kappa)$. Now, the construction of the transformation matrices $S(\kappa)$ works as follows (cf. [11, Chapter II.4]): Set

$$P_0(\kappa) = I - \sum_{j=1}^{s} P_j(\kappa) \ (P_0 = P_0(0)),$$

$$Q(\kappa) = \sum_{j=0}^{s} (\partial_\kappa P_j(\kappa)) P_j(\kappa)$$

and consider the matrix-valued initial-value problem

$$\partial_\kappa U(\kappa) = Q(\kappa) U(\kappa), \quad U(0) = I. \tag{2.3}$$

The unique solution $S(\kappa)$ is invertible, holomorphic and has the property

$$S(\kappa) P_j (S(\kappa))^{-1} = P_j(\kappa), \ j = 0, \dots, s.$$

Define $M_*(\kappa) = (S(\kappa))^{-1} T(\kappa) S(\kappa)$. By definition each P_j is $M_*(\kappa)$-invariant. Since the P_j are orthogonal projections, $P_j P_k = \delta^{jk} P_k$, and $\sum_{j=0}^{s} P_j = I$, the assertion is shown if we can prove

$$\begin{aligned}\langle (M_*(\kappa) + (M_*(\kappa))^*) \, P_j u, P_j u \rangle & \\ \leq -c\kappa^2 |P_j u|^2, \ \kappa \in [0, \tilde{r}], \ u \in \mathbb{C}^{2N}, \ j = 0, \dots, s.&\end{aligned}$$

First consider $j = 0$. Since $S(0) = I$ and M_* is holomorphic, we have

$$P_0 M_*(\kappa) P_0 = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix} + \kappa P_0 H_1(\kappa) P_0,$$

where $H_1(\kappa)$ is holomorphic. As $a > 0$, this yields

$$\langle (M_*(\kappa) + (M_*(\kappa))^*) \, P_0 u, P_0 u \rangle \ \leq |P_0 u|^2 (-c + O(\kappa)), \ \kappa \in [0, \tilde{r}]. \tag{2.4}$$

Next let $j \in \{1, \dots, s\}$. As $P_j T^{(0)} = T^{(0)} P_j = 0$, we can write

$$P_j M_*(\kappa) P_j = \kappa M_*^{(1j)} + \kappa^2 M_*^{(2j)} + \kappa^3 P_j H_2(\kappa) P_j,$$

where $H_2(\kappa)$ is holomorphic

$$M_*^{(1j)} = P_j T^{(1)} P_j = \begin{pmatrix} i\lambda_j p_j & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$M_*^{(2j)} = P_j T^{(2)} P_j + Z^{(1)} T^{(1)} P_j + P_j T^{(1)} W^{(1)} + Z^{(1)} T^{(0)} W^{(1)},$$

whith $W^1 = \partial_\kappa S(0) P_j$ and $Z^1 = P_j \partial_\kappa((S(\kappa))^{-1})\,|_{\kappa=0}$. From (2.3) we get $W^{(1)} = P_j'(0) P_j$ and $Z^{(1)} = P_j P_j'(0)$. The general form of $P_j'(0)$ can be computed by inserting the Neumann-series for the resolvent $\tilde{R}$ of $\tilde{T}$ in (2.2) (which is done for the general case in [11, chapter II.2]). In our case some computations lead to

$$M_*^{(2j)} = \begin{pmatrix} -p_j(\tilde{B} - \lambda_j^2 a^{-1}) p_j & 0 \\ 0 & 0 \end{pmatrix}.$$

Since λ_j is a real number, condition 2.1 yields

$$\left\langle \left(M_*(\kappa)) + (M_*^{(j)}(\kappa)))^* \right) P_j u, P_j u \right\rangle \leq |P_j u|^2 (-c\kappa^2 + O(\kappa^3))$$

for $\kappa > 0$, which together with (2.4) proves the assertion. □

2.3 Corollary. *Assume Condition 2.1. Let $r_0 > 0$ be the constant in Proposition 2.2 and $\lambda_1, \dots, \lambda_s$ be the eigenvalues of $\tilde{b}$ with corresponding eigenprojections $p_1, \dots, p_s$. Furthermore for each $j = 1, \dots, s$, let $\beta_{j1}, \dots, \beta_{jn}$ be the eigenvalues of*

$$p_j(\tilde{B} - \lambda_j^2 a^{-1}) p_j|_{\mathrm{im}(p_j)}.$$

Then for $|\kappa| \leq r_0$ the spectrum of $M(\kappa)$ is given as

$$\sigma(M(\kappa)) = \{\gamma_{11}, \dots, \gamma_{1n}, \dots, \gamma_{s1}, \dots, \gamma_{sn}, \upsilon_1, \dots, \upsilon_m\}$$

with

$$\gamma_{jk} = i\kappa\lambda_j - \kappa^2 \beta_{jk} + o(\kappa^2), \tag{2.5}$$

and

$$\upsilon_k = -\alpha_k + o(1), \tag{2.6}$$

where $\alpha_1, \dots, \alpha_m$ are the eigenvalues of a. In particular there exists a $c > 0$ such that

$$\Re(\gamma) \leq -c\kappa^2$$

for all $\gamma \in \sigma(M(\kappa))$ and $\kappa \in \mathbb{R}$, $|\kappa| \leq r_0$.

Proof. Let $\kappa \in \mathbb{R}$, $|\kappa| \leq r_0$. Obviously, $M_*(\kappa)$ has the same spectrum as $M(\kappa)$. In the proof of Proposition 2.2 we have seen that for $j = 1, \ldots, s$,

$$P_j M_*(\kappa) P_j = \kappa M_*^{(1j)} + \kappa^2 M_*^{(2j)} + O(\kappa^3)$$

where

$$P_j = \begin{pmatrix} p_j & 0 \\ 0 & 0 \end{pmatrix},$$

$$M_*^{(1j)} = \begin{pmatrix} i\lambda_j p_j & 0 \\ 0 & 0 \end{pmatrix},$$

$$M_*^{(2j)} = \begin{pmatrix} -p_j(\tilde{B} - \lambda_j^2 a^{-1})p_j & 0 \\ 0 & 0 \end{pmatrix}.$$

By [11, Chapter II.2] the eigenvalues of $P_j M_*(\kappa) P_j$ are given by γ_{jk} ($k = 1, \ldots, n$) as defined in (2.5). Furthermore for

$$P_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

we have

$$P_0 M_*(\kappa) P_0 = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix} + O(\kappa), \quad |\kappa| \leq r_0,$$

and hence the eigenvalues of $P_0 M_*(\kappa) P_0$ are given by υ_k as defined by (2.6). Since $P_0, \ldots, P_s$ form a complete $M_*(\kappa)$-invariant set of projections the assertion is proven. $\square$

(b) Large wave numbers

Consider now the matrix family

$$K(\nu) = \begin{pmatrix} 0 & I \\ -B + i\nu b & -\nu a \end{pmatrix}, \quad \nu \in \mathbb{C}$$

which is "dual" to (2.1).

2.4 Condition. Let there exist a $C > 0$ such that for each $\mu \in \sigma(B)$, each $u \in \mathrm{Eig}_B(\mu)$ satisfies

$$\langle (a + B^{-\frac{1}{2}} b)u, u \rangle \geq C|u|^2,$$
$$\langle (a - B^{-\frac{1}{2}} b)u, u \rangle \geq C|u|^2.$$

2.5 Proposition. *Let Condition 2.4 hold. Then there exist* $r^\infty, c_\infty > 0$ *and an holomorphic, invertible family of* $2N \times 2N$ *matrices*

$$\{S(\nu) : |\nu| \le r^\infty\}$$

such that $K_*(\nu) = (S(\nu))^{-1}K(\nu)S(\nu)$ *satisfies*

$$\langle (K_*(\nu)) + (K_*(\nu))^*)u, u\rangle \le -c_\infty \nu |u|^2,$$

for all $u \in \mathbb{C}^{2N}$ *and* $\nu \in \mathbb{R}$, $|\nu| \le r^\infty$.

Proof. The crucial point is again the continuation of a natural splitting, at $\nu = 0$, into invariant subspaces.

First set

$$S_1 := \begin{pmatrix} B^{-\frac{1}{2}} & B^{-\frac{1}{2}} \\ iI & -iI. \end{pmatrix}$$

and then define

$$K_1(\nu) := S_1^{-1}K(\nu)S_1 = K_1^{(0)} + \nu K_1^{(1)},$$

where

$$K_1^{(0)} = \begin{pmatrix} iB^{\frac{1}{2}} & 0 \\ 0 & -iB^{\frac{1}{2}} \end{pmatrix},$$

$$K_1^{(1)} = \frac{1}{2}\begin{pmatrix} -a + bB^{\frac{1}{2}} & -a + bB^{\frac{1}{2}} \\ a - bB^{\frac{1}{2}} & -a - bB^{\frac{1}{2}}. \end{pmatrix}$$

Obviously

$$\sigma(K_1^{(0)}) = \{i\mu_1^{\frac{1}{2}}, \dots, i\mu_l^{\frac{1}{2}}, -i\mu_1^{\frac{1}{2}}, \dots, -i\mu_l^{\frac{1}{2}}\},$$

where μ_j $(1 \le j \le l)$ are the (strictly positive) eigenvalues of B. Due to Condition 2.4 it can be shown, as in the proof of Proposition 2.2, that there exists an $r > 0$ such that for each $j = 1, \dots, l$ the total projections $P_{+j}(\nu)$ and $P_{-j}(\nu)$ for the $i\mu_j^{\frac{1}{2}}$- and $-i\mu_j^{\frac{1}{2}}$-group of eigenvalues are well-defined and holomorphic on $\overline{B_r(0)}$. Furthermore,

$$P_{+j} := P_{+j}(0) = \begin{pmatrix} p_j & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{-j} = P_{-j}(0) = \begin{pmatrix} 0 & 0 \\ 0 & p_j \end{pmatrix},$$

where p_j is the eigenprojection for the eigenvalue μ_j of B. Then, again as in the proof of Proposition 2.2, there exists an invertible holomorphic family of $2N \times 2N$ matrices

$$\{S(\nu) : |\nu| \le r\},$$

such that

$$S(\nu)P_{\pm j}(S(\nu))^{-1} = P_{\pm j}(\nu),\ j = 0,\dots,l.$$

Now define $K_*(\nu) = (S(\nu))^{-1}K_1(\nu)S(\nu)$. Since $S(0) = I$ and

$$S'(0)K_1^{(0)}P_{\pm j} = P_{\pm j}K_1^{(0)}\frac{d}{d\nu}S(\nu)^{-1}\mid_{\nu=0} = 0$$

(cf. [11, Chapter II.4]), one gets

$$\begin{aligned} P_{+j}K_*(\nu)P_{+j} &= P_{+j}K_1^{(0)}P_{+j} + \nu P_{+j}K_1^{(1)}P_{+j} + \nu^2 P_{-j}H_+(\nu)P_{-j} \\ &= \begin{pmatrix} i\mu^{\frac{1}{2}} - \frac{\nu}{2}p_j\left(a - \mu_j^{-\frac{1}{2}}b\right)p_j & 0 \\ 0 & 0 \end{pmatrix} + \nu^2 P_{+j}H_+(\nu)P_{+j} \end{aligned}$$

and

$$P_{-j}K_*(\nu)P_{-j} = \begin{pmatrix} 0 & 0 \\ 0 & -i\mu^{\frac{1}{2}} - \frac{\nu}{2}p_j\left(a + \mu^{-\frac{1}{2}}b\right)p_j \end{pmatrix} + \nu^2 P_{-j}H_-(\nu)P_{-j},$$

where $H_\pm$ are holomorphic. Thus Condition 2.4 yields

$$\langle (K_*(\nu) + (K_*(\nu))^*)\, P_{\pm j}u, P_{\pm j}u\rangle \le |P_{\pm_j}u|^2(-c\nu + 0(\nu^2))$$

for $\nu \in [0,r]$, $u \in \mathbb{C}^{2N}$, $1 \le j \le l$. As $\{P_{+j}, P_{-j}\}_{1\le j\le l}$ form a complete set of K_*-invariant, orthogonal projections, this proves the assertion. $\square$

Next, define the matrix family

$$\tilde{M}(\kappa) = \kappa K(\kappa^{-1}) = \begin{pmatrix} 0 & \kappa I_N \\ -\kappa B + ib & -a \end{pmatrix}, \quad \kappa \in \mathbb{C}.$$

Then the following result is an immediate consequence of Proposition 2.5.

2.6 Corollary. *Let Condition 2.4 hold. Then there exist* $r_\infty, c_\infty > 0$ *and an invertible holomorphic family of* $2N \times 2N$ *matrices*

$$\{S(\kappa) : |\kappa| \ge r_\infty\}$$

such that $\kappa \mapsto S(\kappa)$, $\kappa \mapsto S^{-1}(\kappa)$ *are bounded on* $[r_\infty,\infty)$ *and* $\tilde{M}_*(\kappa) = (S(\kappa))^{-1}\tilde{M}(\kappa)S(\kappa)$ *satisfies*

$$\langle (\tilde{M}_*(\kappa)) + (\tilde{M}_*(\kappa))^*)u, u\rangle \le -c_\infty |u|^2,$$

for all $u \in \mathbb{C}^{2N}$ *and* $\kappa \in \mathbb{R}$, $|\kappa| \ge r^\infty$.

2.7 Corollary. *Assume Condition 2.4. Let $r_\infty > 0$ be the constant from Corollary 2.6, $\mu_1, \dots, \mu_l$ be the eigenvalues of B with corresponding eigenprojections $p_1, \dots, p_l$. Furthermore, for each $j = 1, \dots, l$ let $\eta^\pm_{j1}, \dots, \eta^\pm_{jn}$ be the of*

$$p_j \left(a \pm \mu_j^{-\frac{1}{2}} b \right) p_j$$

Then for $|\kappa| \geq r_\infty$ the spectrum of $M(\kappa)$ is given by

$$\sigma(M(\kappa)) = \{\omega^\pm_{11}, \dots, \omega^\pm_{1n}, \dots, \omega^\pm_{l1}, \dots, \omega^\pm_{ln}\}$$

with

$$\omega_{jk} = \mp i\kappa\mu_j - \frac{1}{2}\eta^\pm_{jk} + h(\kappa), \tag{2.7}$$

where

$$h(\kappa) \to 0, \quad \text{as } \kappa \to \infty$$

In particular there exists a $c > 0$ such that

$$\Re(\omega) \leq -c$$

for all $\omega \in \sigma(M(\kappa))$, $\kappa \in \mathbb{R}$, $|\kappa| \geq r^\infty$.

Proof. As

$$M(\kappa) = (L(\kappa))^{-1}\tilde{M}(\kappa)L(\kappa),$$

where

$$L(\kappa) = \begin{pmatrix} I & 0 \\ 0 & \kappa \end{pmatrix},$$

$M(\kappa)$ and $\tilde{M}(\kappa)$ have the same spectrum for $\kappa \neq 0$. Hence the assertion follows from the proof of Proposition 2.5 and Corollary 2.7 in the same way as Corollary 2.3 followed from the proof of Proposition 2.2. □

(c) A connection between Conditions 2.1 and 2.4

2.8 Proposition. *Let $B = I$. Then the following statements are equivalent:*

1. *Condition 2.1 holds.*
2. *Condition 2.4 holds.*
3. *The matrices $a + b$ and $a - b$ are positive definite.*

Futhermore if these conditions are satisfied, then each eigenvalue of $M(\kappa)$ has strictly positive real part for all $\kappa \in \mathbb{R}_+$.

Proof. We first prove the equivalence of the three statements. That Condition 2.4 is equivalent to $a+b>0$ and $a-b>0$ for $B=I$ is obvious. Furthermore with $B=I$, Condition 2.1 reads:

$$\langle a^{-1}(I-\lambda^2)u,u\rangle \geq C|u|^2 \tag{2.8}$$

for all $\lambda \in \sigma(\tilde{b})$, each $u \in \mathrm{Eig}_{\tilde{b}}(\lambda)$. Since

$$\langle a^{-1}(I-\lambda^2)u,u\rangle = (1-\lambda^2)|a^{-\frac{1}{2}}u|^2$$

and $a>0$, (2.8) is equivalent to $|\lambda|<1$. This holds true if and only if

$$I \pm \tilde{b} = I \pm a^{-\frac{1}{2}}ba^{-\frac{1}{2}} > 0.$$

It is obvious that this is the case if and only if $a+b>0$ and $a-b>0$.

Due to Corollaries 2.3 and 2.7, the second part of the assertion is shown if we can prove that no eigenvalue of $M(\kappa)$ is purely imaginary for $\kappa>0$. To this end suppose that there exist $\kappa>0$, $\beta\in\mathbb{R}$ and $U\in\mathbb{C}^{2N}\setminus\{0\}$ such that

$$(i\beta - M(\kappa))U = 0. \tag{2.9}$$

Write $U=(v,w)$ $(v,w\in\mathbb{C}^N)$. Then it follows from (2.9) with $B=I$ that $w=i\beta v$ (in particular $v\neq 0$) and

$$\left(\kappa^2 - i\kappa b + (i\beta)^2 + i\beta a\right)v = 0.$$

Taking the scalar product of this equation with v and using the symmetry of a and b gives

$$(\kappa^2-\beta^2)|v|^2 = 0,$$
$$\langle(-\kappa b+\beta a)v,v\rangle = 0.$$

Hence $\kappa=|\beta|$ and

$$\beta\langle(a\pm b)v,v\rangle = 0$$

Since $a\pm b>0$ and $|\beta|=\kappa>0$, this is a contradiction. □

3 Decay Estimates for the Linearized System

In this section we consider the linearization of (1.15) about an arbitrary fixed homogeneous reference state, in the fluid's rest frame $\bar{\psi}=(\bar{\psi}^\alpha,\bar{\psi}^4)=(1/\bar{\theta},0,0,0,\bar{\upsilon}/\bar{\theta})$, where $\bar{\theta}>0$ and $\bar{\upsilon}$ are the constant temperature and chemical potential at the reference state.

We start our considerations with a useful observation about the structure of the first-order terms of (1.15).

3.1 Remark. As stated in Section 1, the fluid's equation of state induces a scalar function $\hat{X}(\theta, \psi_4)$ such that for

$$X^\beta(\psi) = \hat{X}(\theta, \psi_4)\psi^\beta, \tag{3.1}$$

we have

$$A^{a\beta g}(\psi) = \frac{\partial^2 X^\beta(\psi)}{\partial \psi_g \partial \psi_a}.$$

Some computations then lead to

$$A^{\alpha\beta\gamma} = \theta^2 \frac{\partial \hat{X}}{\partial \theta} \left(u^\alpha g^{\beta\gamma} + u^\beta g^{\alpha\gamma} + u^\gamma g^{\alpha\beta} \right) + \frac{\partial}{\partial \theta} \left(\theta^3 \frac{\partial \hat{X}}{\partial \theta} \right) u^\alpha u^\beta u^\gamma, \tag{3.2}$$

$$A^{\alpha\beta 4} = \theta \frac{\partial^2 \hat{X}}{\partial \psi_4 \partial \theta} u^\alpha u^\beta + \frac{\partial \hat{X}}{\partial \psi_4} g^{\beta\gamma} = A^{4\beta\alpha}, \tag{3.3}$$

$$A^{4\beta 4} = \frac{\partial^2 \hat{X}}{\partial \psi_4^2} \psi^\beta. \tag{3.4}$$

Furthermore, due to (1.5),

$$A^{a0b}(\psi) - A^{aib}(\psi)\omega_i > 0 \text{ for all } \omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2. \tag{3.5}$$

By evaluating $\tilde{B}^{a\beta g\delta}$ and $A^{a\beta g}$ at $\bar{\psi}$, we find that the linearization at the reference state is given by

$$A_{(1)}\psi_{tt} - B^{ij}_{(1)}\psi_{x_i x_j} + a_{(1)}\psi_t + b^j_{(1)}\psi_{x_j} = 0, \tag{3.6}$$

$$A_{(1)} = \begin{pmatrix} \chi\bar{\theta}^2 & 0 & 0 \\ 0 & \sigma\bar{\theta} I_3 & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

$$B^{ij}_{(1)} = \begin{pmatrix} \chi\bar{\theta}^2\delta^{ij} & 0 & 0 \\ 0 & \eta\bar{\theta}\delta^{ij} I_3 + \frac{1}{2}\bar{\theta}(\frac{1}{3}\eta + \tilde{\zeta})(e^i \otimes e^j + e^j \otimes e^i) & 0 \\ 0 & 0 & \mu\delta^{ij} \end{pmatrix},$$

$$a_{(1)} = \begin{pmatrix} \bar{\theta}^3 \hat{X}_{\theta\theta} & 0 & \bar{\theta}\hat{X}_{\psi\theta} - \hat{X}_\psi \\ 0 & \bar{\theta}^2 \hat{X}_\theta I & 0 \\ \bar{\theta}\hat{X}_{\psi\theta} - \hat{X}_\psi & 0 & \bar{\theta}^{-1}\hat{X}_{\psi\psi} \end{pmatrix},$$

$$b^j_{(1)} = \bar{\theta}^2 \hat{X}_\theta (e^0 \otimes e^j + e^j \times e^0) + \hat{X}_\psi (e^5 \otimes e^j + e^j \otimes e^5).$$

Note that $\chi, \sigma, \eta, \tilde{\zeta}, \mu$ and the derivatives of X are all evaluated at the reference state and that no mixed derivative ψ_{tx_j} occurs here, as

$$\tilde{B}^{a0gj} = \tilde{B}^{ajg0} = 0$$

at the reference state. Next multiply (3.6) by $(A_{(1)})^{-\frac{1}{2}}$ and write it in variables $(A_{(1)})^{\frac{1}{2}}\psi$ to obtain

$$\psi_{tt} - \bar{B}^{ij} + a\psi_{tt} + b^j\psi_{x_j} = 0, \tag{3.7}$$

where

$$\bar{B}^{ij} = (A_{(1)})^{-\frac{1}{2}} B^{ij}_{(1)} (A_{(1)})^{-\frac{1}{2}},$$

$$a = (A_{(1)})^{-\frac{1}{2}} a_{(1)} A_{(1)}^{-\frac{1}{2}}, \quad b^j = (A_{(1)})^{-\frac{1}{2}} b^j_{(1)} (A_{(1)})^{-\frac{1}{2}}.$$

Our goal is to prove decay and energy estimates for the Cauchy problem associated with (3.6),

$$\psi_{tt} - \bar{B}^{ij}\psi_{x_i x_j} + a\psi_t + b^j\psi_{x_j} = 0, \tag{3.8}$$

$$\psi(0) = {}^0\psi, \tag{3.9}$$

$$\psi_t(0) = {}^1\psi, \tag{3.10}$$

which will be considered on a fixed time interval $[0, T]$. The proof relies on pointwise estimates for the Fourier transform of(3.8)-(3.10):

$$\hat{\psi}_{tt} + |\xi|^2 B(\check{\xi})\hat{\psi} + a\hat{\psi}_t - i|\xi| b(\check{\xi})\hat{\psi} = 0, \tag{3.11}$$

$$\hat{\psi}(0) = {}^0\hat{\psi}(\xi), \tag{3.12}$$

$$\hat{\psi}_t(0) = {}^1\hat{\psi}(\xi), \tag{3.13}$$

where $\check{\xi} = \xi/|\xi|$ and for $\omega \in \mathbb{S}^2$

$$B(\omega) = \omega_i \bar{B}^{ij}\omega_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma^{-1}\left(\eta I + \left(\tilde{\zeta} + \frac{1}{3}\eta\right)(\omega \otimes \omega)\right) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$b(\omega) = b^j\omega_j = \begin{pmatrix} 0 & (\chi\sigma\bar{\theta})^{-\frac{1}{2}}\hat{X}_\theta\omega^t & 0 \\ (\chi\sigma\bar{\theta})^{-\frac{1}{2}}\hat{X}_\theta\omega & 0 & (\mu\sigma\bar{\theta})^{-\frac{1}{2}}\hat{X}_\psi\omega \\ 0 & (\mu\sigma\bar{\theta})^{-\frac{1}{2}}\hat{X}_\psi\omega^t & 0 \end{pmatrix}.$$

The structure of (3.11) can be simplified due to its following property.

3.2 Remark. For each $\xi \in \mathbb{R}^3$, consider the orthogonal decomposition

$$\mathbb{C}^5 = (\mathbb{C} \times \xi\mathbb{C} \times \mathbb{C}) \oplus (\{0\} \times \{\xi\}^\perp \times \{0\}) := J_1(\xi) \oplus J_2(\xi).$$

Setting $\hat{\psi} = w + v$ with $w \in J_1(\xi)$ and $v \in J_2(\xi)$, (3.11) decomposes into the two uncoupled systems

$$w_{tt} + |\xi|^2 w + \tilde{a} w_t - i|\xi|\tilde{b} w = 0, \tag{3.14}$$

$$v_{tt} + \eta\sigma^{-1}|\xi|^2 v + \sigma^{-1}\bar{\theta} X_\theta v_t = 0, \tag{3.15}$$

where

$$\tilde{a} = \begin{pmatrix} \chi^{-1}(\hat{X}_\theta + \bar{\theta}\hat{X}_{\theta\theta}) & 0 & (\chi\mu)^{-\frac{1}{2}}(\hat{X}_{\psi\theta} - \bar{\theta}^{-1}\hat{X}_\psi) \\ 0 & \sigma^{-1}\bar{\theta}\hat{X}_\theta & 0 \\ (\chi\mu)^{-\frac{1}{2}}(\hat{X}_{\psi\theta} - \bar{\theta}^{-1}\hat{X}_\psi) & 0 & (\mu\bar{\theta})^{-1}\hat{X}_{\psi\psi} \end{pmatrix},$$

$$\tilde{b} = \begin{pmatrix} 0 & (\chi\sigma\bar{\theta})^{-\frac{1}{2}}\hat{X}_\theta & 0 \\ (\chi\sigma\bar{\theta})^{-\frac{1}{2}}\hat{X}_\theta & 0 & (\mu\sigma\bar{\theta})^{-\frac{1}{2}}\hat{X}_\psi \\ 0 & (\mu\sigma\bar{\theta})^{-\frac{1}{2}}\hat{X}_\psi & 0 \end{pmatrix}.$$

In particular, (3.5) yields $\tilde{a} \pm \tilde{b} > 0$.

We get the following pointwise decay estimate for solutions to (3.11)-(3.13).

3.3 Lemma. *For some $s \in \mathbb{N}_0$ let $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s)$ and $(\psi(t), \psi_t(t)) \in H^{s+1} \times H^s$ be a solution of* (3.8)-(3.10). *Then there exist $c, C > 0$ such that for each $t \in [0,T]$, $\xi \in \mathbb{R}^3$,*

$$(1+|\xi|^2)|\hat{\psi}(t,\xi)|^2 + |\hat{\psi}_t(t,\xi)|^2 \leq C\exp(-c\rho(\xi)t)\left((1+|\xi|^2)|{}^0\hat{\psi}(\xi)|^2 + |{}^1\hat{\psi}(\xi)|^2\right), \tag{3.16}$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$.

Proof. Let $\hat{\psi}$ be the Fourier transform of a solution ψ to (3.8)-(3.10). For fixed $\xi \in \mathbb{R}^3$ write $\hat{\psi} = w + v$ with $w \in J_1(\xi)$, $v \in J_2(\xi)$. Then w and v satisfiy (3.14) and (3.15), respectively (see Remark 3.2).

First, take the scalar product (in $\mathbb{C}^2$) of (3.15) with v_t. The real part of the resulting equation reads

$$\frac{1}{2}\frac{d}{dt}\left(|v_t|^2 + \eta\sigma^{-1}|\xi|^2|v|^2\right) + \sigma^{-1}\bar{\theta}X_\theta|v_t|^2 = 0 \tag{3.17}$$

The real part of the scalar product of (3.15) with v is given by

$$\frac{1}{2}\frac{d}{dt}\left(2\Re\langle v_t, v\rangle + \sigma^{-1}\bar{\theta}X_\theta|v|^2\right) - |v_t|^2 + \eta\sigma^{-1}|\xi|^2|v|^2 = 0. \tag{3.18}$$

Set $d = \sigma^{-1}\bar{\theta}X_\theta$. Then adding (3.17)+$d$(3.18) yields

$$\frac{1}{2}\frac{d}{dt}E^{(2)} + F^{(2)}, \tag{3.19}$$

where

$$E^{(2)} = |v_t|^2 + d^2/2|v|^2 + d\Re\langle v_t, v\rangle + \eta\sigma^{-1}|\xi|^2|v|^2 \tag{3.20}$$

and

$$F^{(2)} = d/2|v_t|^2 + d\eta\sigma^{-1}|\xi|^2|v|^2. \tag{3.21}$$

It is obvious that $E^{(2)}$ is uniformly equivalent to $E_0^{(2)} = |v_t|^2 + (1+|\xi|^2)|v|^2$ and $F^{(2)} \geq c\rho(\xi)E_0^{(2)}$ for some constant $c > 0$. Hence applying Gronwall's Lemma to (3.19) leads to

$$(1+|\xi|^2)|v|^2 + |v_t|^2 \leq C\exp(-c\rho(\xi)t)((1+|\xi|^2)|v(0)|^2 + |v_t(0)|^2). \tag{3.22}$$

Next we consider (3.14). Set $W = (w, w_t)$, then we can write (3.14) as

$$W_t = M(|\xi|)W, \quad \xi \in \mathbb{R}^3, \tag{3.23}$$

where

$$M(\kappa) = \begin{pmatrix} 0 & I_3 \\ -\kappa^2 I_3 + i\kappa\tilde{b} & -\tilde{a} \end{pmatrix}, \kappa \in \mathbb{C}. \tag{3.24}$$

By Proposition 2.2 there exist $r_0 > 0, c_0 > 0$ and a family of holomorphic and invertible 6×6 matrices

$$\{S(\kappa) : |\kappa| \leq r_0\},$$

such that $M_*(\kappa) = (S(\kappa))^{-1}M(\kappa)S(\kappa)$ satisfies

$$\langle (M_*(\kappa) + M_*(\kappa)^*)Z, Z\rangle \leq -c_0\kappa^2|Z|^2, \tag{3.25}$$

for $Z \in \mathbb{C}^6$, and $\kappa \in [0, r_0]$. Now, set $\tilde{W}(\xi) = S^{-1}(|\xi|)W(\xi)$. $\tilde{W}$ satisfies

$$\tilde{W}_t = M_*(|\xi|)\tilde{W}.$$

Taking the scalar product of this equation with $\tilde{W}$, considering the real part and using (3.25) gives

$$\frac{d}{dt}|\tilde{W}(\xi)|^2 = \langle (M_*(|\xi|) + M_*(|\xi|))^*)\tilde{W}(\xi), \tilde{W}(\xi)\rangle \leq -c_0|\xi|^2|\tilde{W}(\xi)|^2$$

for $|\xi| \in [0, r_0]$. Applying Gronwall's Lemma yields

$$|\tilde{W}(t,\xi)|^2 \leq \exp(-c_0|\xi|^2 t)|\tilde{W}(0,\xi)|^2, \ (t, |\xi|) \in [0,T]\times[0,r_0].$$

Since $S(|\xi|)$ is continuous on $[0, r_0]$, there exists a $C > 0$ (independent of ξ) such that

$$|W(t,\xi)|^2 \le C \exp(-c_0|\xi|^2 t)|W(0,\xi)|^2, \ (t,|\xi|) \in [0,T] \times [0,r_0]. \quad (3.26)$$

Due to

$$\rho(\xi) \le |\xi|^2 \le r_0^2, \ |\xi| \in [0, r_0],$$

(3.26) yields

$$\begin{aligned}(1+|\xi|^2)|w(t,\xi)|^2 + |w_t(t,\xi)|^2 \\ \le C \exp(-c_0\rho(\xi)t) \left((1+|\xi|^2)|w(0,\xi)|^2 + |w_t(0,\xi)|^2\right), \\ (t,|\xi|) \in [0,T] \times [0,r_0]. \quad (3.27)\end{aligned}$$

Next set $V = (|\xi| w, w_t)$ and write (3.14) as

$$V_t = N(|\xi|)V,$$

$$N(\kappa) = \begin{pmatrix} 0 & \kappa I_3 \\ -\kappa I_3 + i\tilde{b} & -\tilde{a} \end{pmatrix}, \ \kappa \in \mathbb{C}.$$

Due to Corollary 2.6 there exist $r_\infty, c_\infty > 0$ and a family of bounded and invertible 6×6 matrices

$$\{L(\kappa) : |\kappa| \ge r_\infty\},$$

where $\kappa \mapsto (L(\kappa)^{-1})$ is also bounded such that $N_*(\kappa) = L(\kappa)^{-1}N(\kappa)L(\kappa)$ satisfies

$$\langle (N_*(\kappa) + (N_*(\kappa))^*)Z, Z\rangle \le -c_\infty |Z|^2,$$

for all $Z \in \mathbb{C}^6$, $\kappa \in [r_\infty, \infty)$. It follows with the same arguments as in the first part of the proof that there exists a $C > 0$ such that

$$|V(t,\xi)|^2 \le C \exp(-c_\infty t)|V(0,\xi)|^2, \ (t,|\xi|) \in [0,T] \times [r_\infty, \infty). \quad (3.28)$$

Using $\rho(\xi) \le 1$, this gives

$$\begin{aligned}(1+|\xi|^2)|w(t,\xi)|^2 + |w_t(t,\xi)|^2 \\ \le C \exp(-c_\infty\rho(\xi)t) \left((1+|\xi|^2)|w(0,\xi)|^2 + |w_t(0,\xi)|^2\right), \\ (t,|\xi|) \in [0,T] \times [r_\infty,\infty). \quad (3.29)\end{aligned}$$

Lastly, let $\xi_0 \in \mathbb{R}^3$ with $r_0 \le |\xi_0| \le r_\infty$. Due to Lemma 2.8 each eigenvalue of $M(|\xi_0|)$ has negative real part. Therefore there exists an invertible matrix $Q_{\xi_0} \in \mathbb{C}^{6\times 6}$ and a $c > 0$ such that $M_*(\kappa) = Q_{\xi_0}^{-1} M(\kappa) Q_{\xi_0}$ satisfies

$$\langle (M_*(|\xi_0|) + (M_*(|\xi_0|))^*)\, Z, Z\rangle \le -c|Z|^2,\ Z \in \mathbb{C}^6.$$

Since M is continuous there also exist $\delta(\xi_0), c(\xi_0) > 0$ such that for all $\xi \in B_\delta(\xi_0)$

$$\langle (M_*(|\xi|) + (M_*(|\xi|))^*)\, Z, Z\rangle \le -c|Z|^2,\ Z \in \mathbb{C}^6.$$

Hence by Gronwall's Lemma there exists a $C > 0$ such that

$$|W(t,\xi)|^2 \le Ce^{-ct}|W(0,\xi)|^2,\ t \in [0,T], \tag{3.30}$$

for $\xi \in B_\delta(\xi_0)$. As $K = \{\xi \in \mathbb{R}^3 | r_0 \le |\xi| \le r_\infty\}$ is compact, there exist $c, C > 0$ (independent of ξ) such that (3.30) holds for all $\xi \in K$, and thus

$$\begin{aligned}(1+|\xi|^2)&|w(t,\xi)|^2 + |w_t(t,\xi)|^2\\ &\le C\exp(-c\rho(\xi)t)\left((1+|\xi|^2)|w(0,\xi)|^2 + |w_t(0,\xi)|^2\right),\\ &\qquad\qquad\qquad\qquad (t,|\xi|) \in [0,T]\times[r_0, r_\infty].\end{aligned}$$

This, together with (3.27) and (3.29), proves that

$$\begin{aligned}(1+|\xi|^2)&|w(t,\xi)|^2 + |w_t(t,\xi)|^2\\ &\le C\exp(-c\rho(\xi)t)\left((1+|\xi|^2)|w(0,\xi)|^2 + |w_t(0,\xi)|^2\right)\end{aligned} \tag{3.31}$$

holds for all $(t,\xi) \in [0,T]\times\mathbb{R}^3$. The assertion follows by adding (3.31) and (3.22). □

Based on Lemma 3.3 the proof of the following decay estimate goes as [2, Proof of Theorem 3.1].

3.4 Proposition. *For some $s \in \mathbb{N}_0$ let $({}^0\psi, {}^1\psi) \in (H^{s+1}\times H^s)\cap(L^1\times L^1)$ and $(\psi(t), \psi_t(t)) \in H^{s+1}\times H^s$ be a solution to (3.8)-(3.10). Then there exist $c, C > 0$ such that for all integers $0 \le k \le s$ and all $t \in [0,T]$*

$$\begin{aligned}\|\partial_x^k\psi(t)\|_1 + \|\partial_x^k\psi_t(t)\| \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}&\left(\|{}^0\psi\|_{L^1} + \|{}^1\psi\|_{L^1}\right)\\ &+ Ce^{-ct}\left(\|\partial_x^k({}^0\psi)\|_1 + \|\partial_x^k({}^1\psi)\|\right).\end{aligned} \tag{3.32}$$

Next, consider the inhomogeneous initial-value problem

$$\psi_{tt} - \bar{B}^{ij}\psi_{x_i x_j} + a\psi_t + b^j\psi_{x_j} = h, \tag{3.33}$$
$$\psi(0) = {}^0\psi, \tag{3.34}$$
$$\psi_t(0) = {}^1\psi. \tag{3.35}$$

for some $h : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^5$. Due to Duhamel's principle the following result is an immediate consequence of Proposition 3.4.

3.5 Proposition. *Let s be a non-negative integer, $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ and $h \in C([0,T], H^s \cap L^1)$. Then the solution ψ of* (3.33)-(3.35) *satisfies*

$$\begin{aligned}\|\partial_x^k \psi(t)\|_1 + \|\partial_x^k \psi_t(t)\| &\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|{}^0\psi\|_{L^1} + \|{}^1\psi\|_{L^1}) \\ &\quad + Ce^{-ct}(\|\partial_x^k({}^0\psi)\|_1 + \|\partial_x^k({}^1\psi)\|) \\ &\quad + C\int_0^t (1+t-\tau)^{-3/4-k/2}\|h(\tau)\|_{L^1} \\ &\qquad\qquad + Ce^{-c(t-\tau)}\|\partial_x^k h(\tau)\| d\tau, \end{aligned} \tag{3.36}$$

for all $t \in [0,T]$ and $0 \leq k \leq s$.

Furthermore we can prove the following energy estimate.

3.6 Proposition. *Let s be a non-negative integer. There exists a $C > 0$ such that for all $({}^0\psi, {}^1\psi) \in H^{s+1} \times H^s$ and $h \in C([0,T], H^s)$ the solution ψ of* (3.33)-(3.35) *satisfies*

$$\begin{aligned}&\|\partial_x^k \psi(t)\|_1^2 + \|\partial_x^k \psi_t(t)\|^2 + \int_0^t \|\partial_x^{k+1}\psi(\tau)\|^2 + \|\partial_x^k \psi_t(\tau)\|^2 d\tau \\ &\qquad \leq C\left(\|\partial_x^k({}^0\psi)\|_1^2 + \|\partial_x^k({}^1\psi)\|^2\right) + C\int_0^t \|\partial_x^k \psi(\tau)\|^2 \\ &\quad + C\left|\int_0^t \left(\partial_x^k h(\tau), \frac{a}{2}\partial_x^k \psi(\tau)\right)_{L^2} d\tau\right| + C\left|\int_0^t \left(\partial_x^k h(\tau), \partial_x^k \psi_t(\tau)\right)_{L^2} d\tau\right| \end{aligned} \tag{3.37}$$

for all $t \in [0,T]$ and integers $0 \leq k \leq s$.

Proof. Consider (3.33) in Fourier space, i.e.,

$$\hat{\psi}_{tt} + |\xi|^2 B(\check{\xi})\hat{\psi} + a\hat{\psi}_t - i|\xi|b(\check{\xi})\hat{\psi} = \hat{h}. \tag{3.38}$$

For fixed $\xi \in \mathbb{R}^3$ write (3.38) as

$$w_{tt} + |\xi|^2 w + \tilde{a} w_t - i|\xi|\tilde{b} w = \hat{h}_1, \tag{3.39}$$

$$v_{tt} + \eta\sigma^{-1}|\xi|^2 v + \sigma^{-1}\bar{\theta} X_\theta v_t = \hat{h}_2, \tag{3.40}$$

where $\hat{\psi} = w + v$, $\hat{h} = \hat{h}_1 + \hat{h}_2$ with $w, \hat{h}_1 \in J_1(\xi)$ and $v, \hat{h}_2 \in J_2(\xi)$ (see Remark 3.2). As in the proof of Lemma 3.3 set $d = \sigma^{-1}\bar{\theta} X_\theta$ and take the scalar product of (3.40) with $v_t + (d/2)v$. The real part reads

$$\frac{1}{2}\frac{d}{dt}E^{(2)} + F^{(2)} = \Re\left\langle \hat{h}_2, v_t + \frac{d}{2}v \right\rangle, \tag{3.41}$$

where $E^{(2)}$ and $F^{(2)}$ are defined by (3.20) and (3.21). Furthermore $E^{(2)}$ is uniformly equivalent to $(1 + |\xi|^2)|v|^2 + |v_t|^2$ and $F^2 \geq c(|v_t|^2 + |\xi|^2|v|^2)$. Thus integrating (3.41) leads to

$$\begin{aligned} C_1 \left(|v_t|^2 + (1 + |\xi|^2)|v|^2 \right) + C_1 \int_0^t |v_t|^2 + |\xi|^2|v|^2 d\tau \\ \leq C_2 \left(|v_t(0)|^2 + (1 + |\xi|^2)|v(0)|^2 \right) + \int_0^t \Re\langle \hat{h}_2, v_t + \frac{d}{2}v\rangle d\tau. \end{aligned} \tag{3.42}$$

Next, take the scalar product of (3.39) with $w_t + (\tilde{a}/2)w$. The real part reads

$$\frac{1}{2}\frac{d}{dt}E^{(1)} + F^{(1)} = \Re\langle \hat{h}_1, w_t + \frac{1}{2}\tilde{a}w\rangle, \tag{3.43}$$

where

$$E^{(1)} = |w_t|^2 + |\xi|^2|w|^2 + \frac{1}{2}|\tilde{a}w|^2 + \Re\langle \tilde{a}w_t, w\rangle,$$

and

$$F^{(1)} = \frac{1}{2}\langle \tilde{a}w_t, w_t\rangle + \Re\langle -i|\xi|\tilde{b}w, w_t\rangle + \frac{1}{2}|\xi|^2\langle \tilde{a}w, w\rangle - \frac{1}{2}\Re\langle i|\xi|\tilde{b}w, \tilde{a}w\rangle.$$

Using Young's inequality it is easy to see that $E^{(1)}$ is uniformly equivalent to $|w_t|^2 + (1 + |\xi|^2)|w|^2$. Furthermore

$$F^{(1)} = \frac{1}{2}\langle DY, Y\rangle_{\mathbb{C}^6} - \frac{1}{2}\Re\langle i|\xi|\tilde{b}w, \tilde{a}w\rangle,$$

where

$$D = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{a} \end{pmatrix}$$

and $Y = (w_t, -i|\xi|w)$. As

$$\sigma(D) = \sigma(\tilde{a} + \tilde{b}) \cup \sigma(\tilde{a} - \tilde{b}),$$

and $\tilde{a} \pm \tilde{b} > 0$ (see Remark 3.2), D is positive definite. Hence, there exist $c_1, c_2 > 0$ such that

$$F^{(1)} \geq c_1(|w_t|^2 + |\xi|^2|w|^2) - c_2|\xi||w||w| \geq \frac{c_1}{2}(|w_t|^2 + |\xi|^2|w|^2) - \frac{c_2^2}{2c_1}|w|^2.$$

Thus integrating (3.43) leads to

$$\begin{aligned} &C_1\left(|w_t|^2 + (1+|\xi|^2)|w|^2\right) + C_1\int_0^t |w_t|^2 + |\xi|^2|w|^2 d\tau \\ &\leq C_2\left(|w_t(0)|^2 + (1+|\xi|^2)|w(0)|^2\right) + \int_0^t C_2|w|^2 + \Re\langle \hat{h}_1, w_t + \frac{\tilde{a}}{2}w\rangle d\tau. \end{aligned} \tag{3.44}$$

Adding (3.42) and (3.44) gives

$$\begin{aligned} &|\hat{\psi}_t|^2 + (1+|\xi|^2)|\hat{\psi}|^2 + \int_0^t |\hat{\psi}_t|^2 + |\xi|^2|\hat{\psi}|^2 d\tau \\ &\leq C\left(|{}^1\hat{\psi}|^2 + (1+|\xi|^2)|{}^0\hat{\psi}|^2\right) + \int_0^t C|\hat{\psi}|^2 + c\Re\langle \hat{h}, \hat{\psi}_t + \frac{a}{2}\hat{\psi}\rangle d\tau. \end{aligned} \tag{3.45}$$

Finally the assertion follows by multiplying (3.45) with $\xi^{2\alpha}$ for $\alpha \in \mathbb{N}_0^n$, $0 \leq |\alpha| \leq s$, integrating with respect to ξ, and using Plancherel's identity. □

4 Global existence and asymptotic decay

To complete the proof of Theorem 1.1 we now combine the above findings with analogues of results obtained in [18].

Consider the Cauchy problem for a second order symmetric hyperbolic system of partial differential equations

$$A(\psi)\psi_{tt} - B^{ij}(\psi)\psi_{x_ix_j} + D^j(\psi)\psi_{tx_j} + f(\psi, \psi_t, \partial_x\psi) = 0 \text{ on } \mathbb{R}^n \times [0, T], \tag{4.1}$$

$$\psi(0) = {}^0\psi \text{ on } \mathbb{R}^n, \tag{4.2}$$

$$\psi_t(0) = {}^1\psi \text{ on } \mathbb{R}^n, \tag{4.3}$$

where $(\psi(t), \psi_t(t), \psi_x(t)) \in \Omega = \Omega_1 \times \Omega_2 \times \Omega_3$ for some simply connected domains $\Omega_1, \Omega_2 \in \mathbb{R}^N$, $\Omega_2 \in \mathbb{R}^{nN}$, $A(\psi), B^{ij}(\psi)$, $D^j(\psi) \in \mathbb{R}^{N\times N}$ symmetric matrices depending smoothly on $\psi \in \Omega_1$ with $A(\psi) > 0$, $B^{ij}(\psi)\omega_i\omega_j > 0$ for all $\omega \in \mathbb{S}^{n-1}$, and $f : \Omega \to \mathbb{R}^N$ a smooth function. Furthermore let there exist a constant state $\bar{\psi} \in \Omega_1$, satisfying $f(\bar{\psi}, 0, 0) = 0$ and

$$D^j(\bar{\psi}) = 0, \quad \partial_\psi f(\bar{\psi}, 0, 0) = 0.$$

Then the linearization of (4.1) is given as

$$\bar{A}\psi_{tt} - \bar{B}^{ij}\psi_{x_i x_j} + a\psi_t + b^j\psi_{x_j} = 0, \tag{4.4}$$

where $\bar{A} = A(\bar{\psi})$, $\bar{B}^{ij} = B^{ij}(\bar{\psi})$, $a = \partial_\eta f(\bar{\psi}, 0, 0)$, and $b^j = \partial_{\zeta_j} f(\bar{\psi}, 0, 0)$ (η and ζ_j denoting the ψ_t and $\partial_{x_j}\psi$ components, respectively). W.l.o.g we may suppose $\bar{\psi} = 0$ as otherwise we can consider the state variables $\psi - \bar{\psi}$ and coefficients $A(\cdot + \bar{\psi})$ and so on. Following [12, 16] we consider the functional

$$N_s(t)^2 = \sup_{\tau \in [0,T]} \|(\psi(\tau), \psi_t(\tau))\|^2_{s+1,1} + \int_0^t \|(\psi(\tau), \psi_t(\tau))\|^2_{s+1,s} d\tau,$$

with $s \in \mathbb{N}$, $t \in [0, T]$ and $(\psi, \psi_t) \in H^s \times H^{s+1}$ a solution to (4.1).

Now assume that solutions to the inhomogeneous version of (4.4) satisfy the decay estimate Proposition 3.5 - where the decay rate 3/4 which corresonds to 3 space dimensions is replaced by $n/4$ in the general case of n space dimensions - and the energy estimate Proposition 3.6. Then solutions to the nonlinear equations (4.1) satisfy the following decay and energy estimates (cf. [18]).

4.1 Proposition. *Let $n \geq 3$, $s \geq [n/2] + 2$. Then there exist constants a_1, $\delta_1 = \delta_1(a_1)$, $C_1 = C_1(a_1, \delta_1) > 0$ such that the following holds: If $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ with $\|({}^0\psi, {}^1\psi)\|_{s,s-1,1} \leq \delta_1$ and*

$$\psi \in \bigcap_{j=0}^{s} C^j\left([0, T], H^{s+1-j}\right)$$

is a solution to (4.1)-(4.3) satisfying $N_s(T) \leq a_1$, then the decay estimate

$$\|(\psi(t), \psi_t(t))\|_{s,s-1} \leq C_1(1 + t)^{-\frac{n}{4}} \|({}^0\psi, {}^1\psi)\|_{s,s-1,1} \tag{4.5}$$

holds for all $t \in [0, T]$.

4.2 Proposition. *Let $n \geq 3$, $s \geq [n/2] + 2$. Then there exist constants $a_2, C_2 = C_2(a_2), c_2 = c_2(a_2) > 0$, such that the following holds: If $({}^0\psi, {}^1\psi) \in H^{s+1} \times H^s$ and*

$$\psi \in \bigcap_{j=0}^{s} C^j\left([0, T], H^{s+1-j}\right)$$

is a solution to (4.1)-(4.3) *satisfying* $N_s(T) \le a_2$*, then the energy estimate*

$$\|(\psi(t), \psi_t(t))\|_{s+1,s}^2 + \int_0^t \|\partial_x \psi(\tau)\|_s^2 + \|\psi_t(\tau)\|_s^2 d\tau \\ - c_2 \int_0^t \|\psi(\tau)\|_s^2 d\tau \le C_2 \left(\|({}^0\psi, {}^1\psi)\|_{s+1,s}^2 + N_s(t)^3 \right) \quad (4.6)$$

holds for all $t \in [0, T]$.

Integrating (4.5) with respect to t and adding the resulting inequality to $(c_2/2)$(4.6) leads to the following result.

4.3 Proposition. *In the situation of Proposition 4.1 there exist constants* $a_3(\le \min\{a_2, a_1\})$*,* $C_3 = C_3(a_3, \delta_1) > 0$ *(*δ_1 *being the constant in Proposition 4.1), such that the the following holds: If* $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ *with* $\|({}^0\psi, {}^1\psi)\|_{s,s-1,1} \le \delta_1$ *and*

$$\psi \in \bigcap_{j=0}^{s} C^j \left([0, T], H^{s+1-j} \right)$$

is a solution to (4.1)-(4.3) *satisfying* $N_s(T) \le a_3$*, then the estimate*

$$N_s(t)^2 \le C_3^2 \|({}^0\psi, {}^1\psi)\|_{s+1,s,1}^2 \quad (4.7)$$

holds for all $t \in [0, T]$.

Global existence and asymptotic decay of solutions to (4.1)-(4.3) near $\bar{\psi}$ follow from Propositions 4.1 and 4.3 as in similar considerations in [18] by extending local solutions that exist due to the second-order symmetric hyperbolicity [10]. We get the following result.

4.4 Theorem. *Let* $n \ge 3$, $s \ge [n/2]+2$. *There exist* $\delta_0 > 0$, $C_0 = C_0(\delta_0) > 0$ *such that the following holds: Let* $({}^0\psi, {}^1\psi) \in \left(H^{s+1} \times H^s\right) \cap \left(L^1 \times L^1\right)$ *with* $\|({}^0\psi, {}^1\psi)\|_{s+1,s,1}^2 < \delta_0$*. Then there exists a unique solution*

$$\psi \in \bigcap_{j=0}^{s} C^j \left([0, \infty), H^{s+1-j} \right)$$

to (4.1)-(4.3)*. Furthermore*

$$\|(\psi(t), \psi_t(t))\|_{s+1,s}^2 + \int_0^t \|(\psi(\tau), \psi_t(\tau))\|_{s+1,s}^2 d\tau \le C_0 \|({}^0\psi, {}^1\psi)\|_{s+1,s}^2,$$

$$\|(\psi(t), \psi_t(t))\|_{s,s-1} \le C_0 (1+t)^{-\frac{n}{4}} \|({}^0\psi, {}^1\psi)\|_{s,s-1,1}$$

for all $t \in [0, \infty)$.

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