Asymptotic stability of homogeneous states in the relativistic dynamics of viscous, heat-conductive fluids

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Abstract

Global-in-time existence and asymptotic decay of small solutions to the Navier-Stokes-Fourier equations for a class of viscous, heat-conductive fluids are shown. As this second-order system is symmetric hyperbolic, existence and uniqueness on a short time interval follow from work of Hughes, Kato and Marsden. In this paper it is proven that solutions which are close to a homogeneous reference state can be extended globally and decay to the reference state. The proof combines decay results for the linearization with refined Kawashima-type estimates of the nonlinear terms.

1. Introduction

In relativistic fluid dynamics, stresses in perfect fluids are described by the inviscid energy-momentum tensor [1]

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + pg^{\alpha\beta}, \tag{1.1}$$

where ρ and p are the internal energy and the pressure of the fluid, u^α is its 4-velocity. In this paper we will exclusively consider causal barotropic fluids, a class defined by the property that there exists a one-to-one relation between ρ and p,

$$p = \hat{p}(\rho), \tag{1.2}$$

with a smooth function $\hat{p} : (0, \infty) \to (0, \infty)$ that satisfies $0 < \hat{p}' < 1$. One way to describe the dynamics of dissipative barotropic fluids is via a system[2]

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \Delta T^{\alpha\beta}\right) = 0, \;\; \alpha = 0, 1, 2, 3, \tag{1.3}$$

of partial differential equations - the conservation laws of energy and momentum -, in which the "dissipation tensor" $\Delta T^{\alpha\beta}$ is linear in the gradients of the state variables determined by coefficients η, ζ of viscosity and χ of heat conduction. Freistühler and Temple have recently proposed a particular new way of choosing $\Delta T^{\alpha\beta}$ such that basic requirements, notably of causality, are met; see [3] for this and also for a discussion of

[1] Greek indices run from 0 to 3 and are raised or lowered by contraction with $g^{\alpha\beta}, g_{\alpha\beta}$, where $g^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ is the standard Minkowski metric; cf., e.g., [7], Sec. 2.5.

[2] We use the Einstein summation convention.

the interesting history of the causality problem. According to [3], $\Delta T^{\alpha\beta}$ is given as

$$-\Delta T^{\alpha\beta} = B^{\alpha\beta\gamma\delta}(\psi)\frac{\partial\psi_\gamma}{\partial x^\delta},$$

where ψ denotes the so-called Godunov variables

$$\psi_\gamma = \frac{u_\gamma}{f}$$

with f the Lichnerowicz index of the fluid. The key property of Godunov variables is that in these, the first-order term of a system of conservation laws, here

$$\frac{\partial}{\partial x^\beta}T^{\alpha\beta},$$

becomes *symmetric* hyperbolic [4].[3] Now, the requirement that also

$$-\frac{\partial}{\partial x^\beta}\left(\Delta T^{\alpha\beta}\right)$$

should be symmetric hyberbolic when written in the same variables determines a set of coefficient fields $B^{\alpha\beta\gamma\delta}(\psi)$ which make (1.3) an element of a class of systems that was introduced by Hughes, Kato and Marsden and shown to be well-posed in Sobolev spaces [5]. As established in [3], the requirements of equivariance (isotropicity) and other physical necessities indeed make $B^{\alpha\beta\gamma\delta}(\psi)$ determined by the coefficients η, ζ, χ.

The purpose of this paper is to provide a global-in-time solution theory of these relativistic Navier-Stokes-Fourier equations (1.3). To this end, we analyze first the linearization of (1.3) at some homogeneous reference state and then the nonlinear problem as a pertubation of the linear one, both with techniques that were developed, or are simililar to techniques developed, by Kawashima and co-authors notably in [6], [1].

To have a clear setting, we carry out the whole argument under the additional assumption that the fluid is indeed thermobarotropic, which means, in addition to (1.2), its internal energy is a function of temperature alone

$$\rho = \hat{\rho}(\theta). \tag{1.4}$$

In this case, the Lichnerowicz index is identical with the temperature,

$$f = \theta, \tag{1.5}$$

[3] See [2] for details and the history of the use of such variables in relativistic fluid dynamics.

and actual heat conduction can be an integrated part of a four-field theory, see [2]. An important physical example of this is given by the case of the pure radiation fluid [7], whose internal energy as function of particle number, density and specific entropy is given by

$$\rho(n,s) = kn^{\frac{4}{3}}s^{\frac{4}{3}}.$$

The results of this paper extend to barotropic fluids that do not satisfy (1.4), (1.5) - one just has to replace θ by f everywhere -, but then the "χ-terms" attain the role of an "artificial heat conduction". We plan to later use this hyperbolic regularization for studying the "purely viscous" ($\chi = 0$) case via the limit $\chi \downarrow 0$.

2. Preliminaries and Main Result

We begin by introducing some notation. For $p \in [1,\infty]$ and some $m \in \mathbb{N}$ just write L^p for $L^p(\mathbb{R}^3, \mathbb{R}^m)$. For $s \in \mathbb{N}_0$ we denote by H^s the L^2-Sobolev-space of order s, namely

$$H^s := \{u \in L^2 : \forall\ \alpha \in \mathbb{N}_0^n\ (|\alpha| \le s) : \|\partial_x^\alpha u\|_{L^2} < \infty\}$$

with norm

$$\|u\|_s = \left(\sum_{0 \le |\alpha| \le s} \|\partial_x^\alpha u\|_{L^2} \right)^{\frac{1}{2}}.$$

We just write $\|u\|$ instead of $\|u\|_0$. For $s,k \in \mathbb{N}_0$ and $U = (u_1,u_2) \in H^s \times H^k$ set

$$\|U\|_{s,k} = \left(\|u_1\|_s^2 + \|u_2\|_k^2 \right)^{\frac{1}{2}}$$

and for $U \in (H^s \times H^k) \cap (L^p)^2$ set

$$\|U\|_{s,k,p} = \|U\|_{s,k} + \|U\|_{(L^p)^2}.$$

For $u \in H^s$ and integers $0 \le k \le s$, ∂_x^k shall denote the vector in $\mathbb{R}^N$, $N = m\#\{\alpha \in \mathbb{N}_0^n : |\alpha| = k\}$, whose entries are the partial derivatives of u of order k.

For $u \in H^s$, $v \in H^{l-1}$ $(0 \le l \le s)$ and $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le s$, set

$$[\partial_x^\alpha, u]v = \partial_x^\alpha(uv) - u\partial_x^\alpha v.$$

For $\delta > 0$ let ϕ_δ denote the Friedrichs mollifier and set

$$[\phi_\delta *, u]v = \phi_\delta * (uv) - u(\phi_\delta * v).$$

As stated in the introduction, the goal of this paper is to prove existence and asymptotic decay of global-in-time solutions of (1.3) near homogeneous reference states. First, writing (1.3) in Godunov variables gives

$$-B^{\alpha\beta\gamma\delta}(\psi)\frac{\partial\psi_\gamma}{\partial x^\beta\partial x^\delta}+\frac{\partial}{\partial x^\beta}T^{\alpha\beta}(\psi)-\frac{\partial}{\partial x^\beta}\left(B^{\alpha\beta\gamma\delta}(\psi)\right)\frac{\partial\psi_\gamma}{\partial x^\delta}=0,\qquad \alpha=0,1,2,3. \quad (2.1)$$

In our case of a thermobarotropic fluid the dissipation tensor and the inviscid energy-momentum tensor are given by [4]

$$\begin{aligned}B^{\alpha\beta\gamma\delta}(\psi)=\chi\theta^2u^\alpha u^\gamma g^{\beta\delta}&-\sigma\theta u^\beta u^\delta\Pi^{\alpha\gamma}+\tilde\zeta\theta\Pi^{\alpha\beta}\Pi^{\gamma\delta}\\&+\eta\theta(\Pi^{\alpha\gamma}\Pi^{\beta\delta}+\Pi^{\alpha\delta}\Pi^{\beta\gamma}-\frac{2}{3}\Pi^{\alpha\beta}\Pi^{\gamma\delta})\\&+\sigma\theta(u^\alpha u^\beta g^{\gamma\delta}-u^\alpha u^\delta g^{\gamma\delta})+\chi\theta^2(u^\beta u^\gamma g^{\gamma\delta}-u^\gamma u^\delta g^{\gamma\delta}),\end{aligned}$$

with $\sigma=(\frac{4}{3}\eta+\zeta)/(1-c_s^2)-c_s^2\chi\theta$, $\tilde\zeta=\zeta+c_s^2\sigma-c_s^2(1-c_s^2)\chi\theta$, where $c_s^2=\hat p'(\rho)$ is the speed of sound (cf. [3]), and

$$\frac{\partial}{\partial x^\beta}T^{\alpha\beta}=sn\theta^2\left[u^\alpha g^{\beta\gamma}+u^\beta g^{\alpha\gamma}+u^\gamma g^{\alpha\beta}+(3+c_s^{-2})u^\alpha u^\beta u^\gamma\right]\frac{\partial\psi_\gamma}{\partial x^\beta},$$

with particle number n and specific entropy s. It was shown in [3] that (2.1) is symmetric hyperbolic in the sense of Hughes-Kato-Marsden [5]. Thus, using

$$B^{\alpha\beta\gamma\delta}(\psi)\frac{\partial\psi_\gamma}{\partial x^\beta\partial x^\delta}=\tilde B^{\alpha\beta\gamma\delta}(\psi)\frac{\partial\psi_\gamma}{\partial x^\beta\partial x^\delta}$$

with

$$\begin{aligned}\tilde B^{\alpha\beta\gamma\delta}(\psi)&=\frac{1}{2}\left(B^{\alpha\beta\gamma\delta}(\psi)+B^{\alpha\delta\gamma\beta}(\psi)\right)\\&=\chi\theta^2u^\alpha u^\gamma g^{\beta\delta}-\sigma\theta u^\beta u^\delta\Pi^{\alpha\gamma}+\tilde\zeta\theta\Pi^{\alpha\beta\gamma\delta}+\eta\theta(\Pi^{\alpha\gamma}\Pi^{\beta\delta}+\frac{1}{3}\Pi^{\alpha\beta\gamma\delta}),\end{aligned}$$

where

$$\Pi^{\alpha\beta\gamma\delta}=\frac{1}{2}(\Pi^{\alpha\beta}\Pi^{\gamma\delta}+\Pi^{\alpha\delta}\Pi^{\beta\gamma}),$$

we can write (2.1) as

$$A(\psi)\psi_{tt}-\sum_{i,j=1}^{3}B_{ij}(\psi)\psi_{x_ix_j}+\sum_{j=1}^{3}D_j(\psi)\psi_{tx_j}+f(\psi,\psi_t,\partial_x\psi)=0, \quad (2.2)$$

[4] We use the standard projection $\Pi^{\alpha\beta}=g^{\alpha\beta}+u^\alpha u^\beta$.

where

$$A = (-\tilde{B}^{\alpha 0 \gamma 0})_{0\le\alpha,\gamma\le 3}, \quad B_{ij} = (\tilde{B}^{\alpha i \gamma j})_{0\le\alpha,\gamma\le 3},$$
$$D_j = (-\tilde{B}^{\alpha 0 \gamma j})_{0\le\alpha,\gamma\le 3}$$

are symmetric 4×4 matrices, $A(\psi)$ is positive definite, $\sum_{i,j=1}^{3} \xi_i B_{ij}(\psi)\xi_j$ is positive definite for arbitrary $\xi \in \mathbb{R}^3 \setminus \{0\}$, and

$$f^\alpha = \frac{\partial}{\partial x^\beta} T^{\alpha\beta}(\psi) - \frac{\partial}{\partial x^\beta}\left(B^{\alpha\beta\gamma\delta}(\psi)\right)\frac{\partial \psi_\gamma}{\partial x^\delta}, \;\; \alpha = 0,1,2,3.$$

Throughout the paper we will consider the Cauchy problem associated with (2.2):

$$A\psi_{tt} - \sum_{i,j=1}^{3} B_{ij}\psi_{x_i x_j} + \sum_{j=1}^{3} D_j \psi_{t x_j} + f = 0 \text{ on } (0,T]\times\mathbb{R}^3, \tag{2.3}$$
$$\psi(0) = {}^0\psi \text{ on } \mathbb{R}^3, \tag{2.4}$$
$$\psi_t(0) = {}^1\psi \text{ on } \mathbb{R}^3, \tag{2.5}$$

The main result is the following:

2.1 Theorem. *Let $s \ge 3$ and $\bar\psi = (\bar\theta^{-1}, 0, 0, 0,)^t$ with a constant temperature $\bar\theta > 0$. Then there exist $\delta_0 > 0$, $C_0 = C_0(\delta_0) > 0$ such that for all initial data $({}^0\psi, {}^1\psi_1) \in (H^{s+1}\times H^s)\cap(L^1\times L^1)$ satisfying $\|({}^0\psi - \bar\psi, {}^1\psi)\|^2_{s+1,s,1} < \delta_0$ there exists a unique solution ψ of the Cauchy problem* (2.3)-(2.5) *such that*

$$\psi - \bar\psi \in \bigcap_{j=1}^{s} C^j\left([0,\infty), H^{s+1-j}\right).$$

ψ satisfies the decay estimates

$$\|(\psi(t) - \bar\psi, \psi_t(t)\|^2_{s+1,s} + \int_0^t \|(\psi(\tau) - \bar\psi, \psi_t(\tau))\|^2_{s+1,s} d\tau \le C_0 \|({}^0\psi - \bar\psi, {}^1\psi)\|^2_{s+1,s}, \tag{2.6}$$

$$\|(\psi(t) - \bar\psi, \psi_t(t))\|_{s,s-1} \le C_0 (1+t)^{-\frac{3}{4}} \|({}^0\psi - \bar\psi, {}^1\psi)\|_{s,s-1,1} \tag{2.7}$$

for all $t \in [0,\infty)$.

3. Decay Estimates for the Linearized System

In this section we study the linearization of (2.2) about a quiescent, isothermal reference state $\bar{\psi} = u/\bar{\theta}$, $u = (1,0,0,0)^t$, $\bar{\theta} > 0$. The resulting equations read

$$A^{(1)}\psi_{tt} - \sum_{i,j=1}^{3} B_{ij}^{(1)}\psi_{x_i x_j} + a^{(1)}\psi_t + \sum_{j=1}^{3} b_j^{(1)}\psi_{x_j} = 0, \tag{3.1}$$

where

$$A^{(1)} = \begin{pmatrix} \chi\bar{\theta}^2 & 0 \\ 0 & \sigma\bar{\theta}I_3 \end{pmatrix},$$

$$B_{ij}^{(1)} = \begin{pmatrix} \chi\bar{\theta}^2\delta_{ij} & 0 \\ 0 & \bar{\theta}\eta I_3\delta_{ij} + \frac{1}{2}\bar{\theta}(\tilde{\zeta} + \frac{1}{3}\eta)(e_i \otimes e_j + e_j \otimes e_i) \end{pmatrix},$$

$$a^{(1)} = ns\bar{\theta}^2 \begin{pmatrix} c_s^{-2} & 0 \\ 0 & I_3 \end{pmatrix}, \quad b_j^{(1)} = ns\bar{\theta}^2(e_j \otimes e_0 + e_0 \otimes e_j),$$

where $n, s, \chi, c_s, \eta, \tilde{\zeta}$ are evaluated at the reference state. Note that no mixed derivative ψ_{tx_j} occurs here, as

$$\tilde{B}^{\alpha 0 \gamma j} = \tilde{B}^{\alpha j \gamma 0} = 0$$

at the reference state. Multiplying (3.1) by $(ns)^{-1}\bar{\theta}^{-2}$ and setting $\bar{\chi} = \chi(ns)^{-1}$, $\bar{\eta} = \eta(ns\bar{\theta})^{-1}$, $\bar{\zeta} = \tilde{\zeta}(ns\bar{\theta})^{-1}$, $\bar{\sigma} = \sigma(ns\bar{\theta})^{-1}$, we arrive at the equivalent system

$$A^{(2)}\psi_{tt} - \sum_{i,j=1}^{3} B_{ij}^{(2)}\psi_{x_i x_j} + a^{(2)}\psi_t + \sum_{j=1}^{3} b_j^{(2)}\psi_{x_j} = 0, \tag{3.2}$$

where

$$A^{(2)} = \begin{pmatrix} \bar{\chi} & 0 \\ 0 & \bar{\sigma}I_3 \end{pmatrix}, \quad B_{ij}^{(2)} = \begin{pmatrix} \bar{\chi}\delta_{ij} & 0 \\ 0 & \bar{\eta}I_3\delta_{ij} + \frac{1}{2}\left(\bar{\zeta} + \frac{1}{3}\bar{\eta}\right)(e_i \otimes e_j + e_j \otimes e_i) \end{pmatrix},$$

$$a^{(2)} = \begin{pmatrix} c_s^{-2} & 0 \\ 0 & I_3 \end{pmatrix}, \quad b_j^{(2)} = e_j \otimes e_0 + e_0 \otimes e_j.$$

Finally, multiplying (3.2) by $(A^{(2)})^{-\frac{1}{2}}$ and writing it in variables $(A^{(2)})^{\frac{1}{2}}\psi$ gives

$$\psi_{tt} - \sum_{i,j=1}^{3} \bar{B}_{ij}\psi_{x_i x_j} + a\psi_t + \sum_{j=1}^{3} b_j\psi_{x_j} = 0, \tag{3.3}$$

where

$$\bar{B}_{ij} = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \bar{\sigma}^{-1}\left(\bar{\eta} I_3 \delta_{ij} + \frac{1}{2}\left(\bar{\zeta} + \frac{1}{3}\bar{\eta}\right)(e_i \otimes e_j + e_j \otimes e_i)\right) \end{pmatrix},$$

$$a = \begin{pmatrix} c_s^{-2}\bar{\chi}^{-1} & 0 \\ 0 & \bar{\sigma}^{-1} I_3 \end{pmatrix}, \quad b_j = (\bar{\chi}\bar{\sigma})^{-\frac{1}{2}}(e_j \otimes e_0 + e_0 \otimes e_j).$$

The goal is to prove a decay estimate for the Cauchy problem associated with (3.3):

$$\psi_{tt} - \sum_{i,j=1}^{3} \bar{B}_{ij}\psi_{x_i x_j} + a\psi_t + \sum_{j=1}^{3} b_j \psi_{x_j} = 0 \text{ on } (0,T] \times \mathbb{R}^3, \tag{3.4}$$

$$\psi(0) = {}^0\psi \text{ on } \mathbb{R}^3, \tag{3.5}$$

$$\psi_t(0) = {}^1\psi \text{ on } \mathbb{R}^3. \tag{3.6}$$

3.1 Proposition. *For some $s \in \mathbb{N}_0$ let $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s) \cap (L^1)^2$ and $(\psi(t), \psi_t(t)) \in H^{s+1} \times H^s$ be a solution of (3.4)-(3.6). Then there exist $c, C > 0$ such that for all integers $0 \le k \le s$ and all $t \in [0,T]$*

$$\|\partial_x^k \psi(t)\|_1 + \|\partial_x^k \psi_t(t)\| \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\left(\|{}^0\psi\|_{L^1} + \|{}^1\psi\|_{L^1}\right) + Ce^{-ct}\left(\|\partial_x^k({}^0\psi)\|_1 + \|\partial_x^k({}^1\psi)\|\right). \tag{3.7}$$

To prove Proposition 3.1 we consider (3.4)-(3.6) in Fourier space, i.e.

$$\hat{\psi}_{tt} + |\xi|^2 B(\check{\xi})\hat{\psi} + a\hat{\psi}_t - i|\xi| b(\check{\xi})\hat{\psi} = 0 \text{ on } (0,T] \times \mathbb{R}^3, \tag{3.8}$$

$$\hat{\psi}(0) = {}^0\hat{\psi}(\xi) \text{ on } \mathbb{R}^3, \tag{3.9}$$

$$\hat{\psi}_t(0) = {}^1\hat{\psi}(\xi) \text{ on } \mathbb{R}^3, \tag{3.10}$$

where $\check{\xi} = \xi/|\xi|$,

$$B(\omega) = \sum_{i,j=1}^{3} \omega_i \bar{B}_{ij}\omega_j = \begin{pmatrix} 1 & 0 \\ 0 & \bar{\sigma}^{-1}\left(\bar{\eta} I_3 + \left(\bar{\zeta} + \frac{1}{3}\bar{\eta}\right)(\omega \otimes \omega)\right) \end{pmatrix},$$

$$b(\omega) = \sum_{j=1}^{3} b_j \omega_j = (\bar{\chi}\bar{\sigma})^{-\frac{1}{2}} \begin{pmatrix} 0 & \omega^t \\ \omega & 0 \end{pmatrix}, \ \omega \in \mathbb{S}^2.$$

We get the following pointwise decay estimate.

3.2 Lemma. *In the situation of Proposition 3.1 there exist $c, C > 0$ such that for $(t,\xi) \in [0,T] \times \mathbb{R}^n$*

$$
\begin{aligned}
&(1+|\xi|^2)|\hat{\psi}(t,\xi)|^2 + |\hat{\psi}_t(t,\xi)|^2 \\
&\qquad\qquad \le C\exp(-c\rho(\xi)t)\left((1+|\xi|^2)|^0\hat{\psi}(\xi)|^2 + |^1\hat{\psi}(\xi)|^2\right), \quad (3.11)
\end{aligned}
$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$.

Proof. Our goal is to arrive at an expression of the form

$$\frac{1}{2}\frac{d}{dt}E(t,\xi) + F(t,\xi) \le 0, \tag{3.12}$$

where $E(t,\xi)$ is uniformly equivalent to

$$E_0(t,\xi) = (1+|\xi|)^2|\hat{\psi}(t,\xi)|^2 + |\hat{\psi}_t(t,\xi)|^2$$

and $F \ge c\rho(\xi)E_0$. Then (3.11) follows by Gronwall's Lemma.

W.l.o.g. assume $\xi = (|\xi|, 0, 0)$ (otherwise rotate the coordinate system). Since $(4/3)\bar{\eta} + \bar{\zeta} = \bar{\sigma}$, (3.8) decomposes into the two uncoupled systems

$$w_{tt} + |\xi|^2 w + \tilde{a}w_t - i|\xi|\tilde{b}w = 0, \tag{3.13}$$

$$v_{tt} + \bar{\eta}\bar{\sigma}^{-1}|\xi|^2 v + \bar{\sigma}^{-1}v_t = 0, \tag{3.14}$$

where $w = (\hat{\psi}_0, \hat{\psi}_1)$, $v = (\hat{\psi}_2, \hat{\psi}_3)$,

$$\tilde{a} = \begin{pmatrix} \bar{\chi}^{-1}c_s^{-2} & 0 \\ 0 & \bar{\sigma}^{-1} \end{pmatrix}, \quad \tilde{b} = (\bar{\chi}\bar{\sigma})^{-\frac{1}{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.15}$$

Obviously, this allows us to prove estimate (3.11) for w and v independently.

First, consider (3.14), where the estimate is fairly easy to obtain. Take the scalar product (in $\mathbb{C}^2$) of this equations with $v_t + 1/(2\bar{\sigma})v$. The real part reads

$$\frac{1}{2}\frac{d}{dt}E^{(2)} + F^{(2)} = 0,$$

where

$$E^{(2)} = |v_t|^2 + \frac{\bar{\eta}}{\bar{\sigma}}|\xi|^2|v|^2 + \frac{1}{2\bar{\sigma}^2}|v|^2 + \frac{1}{\bar{\sigma}}\Re\langle v_t, v\rangle, \tag{3.16}$$

and

$$F^{(2)} = \frac{1}{2\bar{\sigma}}|v_t|^2 + \frac{\bar{\eta}}{2\bar{\sigma}^2}|\xi|^2|v|^2. \tag{3.17}$$

Since

$$|\bar{\sigma}^{-1}\Re\langle v_t, v\rangle| \le \frac{1}{3\bar{\sigma}^2}|v|^2 + \frac{3}{4}|v_t|^2,$$

$E^{(2)}$ is uniformly equivalent to $E_0^{(2)} = |v_t|^2 + (1+|\xi|^2)|v|^2$ and as

$$|\xi|^2 \geq \frac{1}{2}\rho(\xi)(1+|\xi|^2),$$

we have $F^{(2)} \geq c_1\rho(\xi)E_0^{(2)}$ for some $c_1 > 0$.

Next, we study system (3.13). For notational purposes set $a_1 = \bar{\chi}^{-1}c_s^{-1}$, $a_2 = \bar{\sigma}^{-2}$ and $b_1 = (\bar{\chi}\bar{\sigma})^{-\frac{1}{2}}$. Now, take the scalar product of (3.13) with $\tilde{a}w_t$. The real part of the resulting equation reads

$$\frac{1}{2}\frac{d}{dt}\Big(\langle\tilde{a}w_t, w_t\rangle + |\xi|^2\langle\tilde{a}w, w\rangle\Big) + |\tilde{a}w_t|^2 + \Re\langle -i|\xi|\tilde{b}w, \tilde{a}w_t\rangle = 0. \quad (3.18)$$

Taking the scalar product of (3.13) with $-i|\xi|\tilde{b}w$ and considering the real part gives

$$\frac{d}{dt}\Big(\Re\langle w_t, -i|\xi|\tilde{b}w\rangle\Big) + \Re\langle\tilde{a}w_t, -i|\xi|\tilde{b}w\rangle + |\xi|^2|\tilde{b}w|^2 = 0. \quad (3.19)$$

Then we take the scalar product of (3.13) with w. The real part is

$$\frac{1}{2}\frac{d}{dt}\left(\langle\tilde{a}w, w\rangle + 2\Re\langle w_t, w\rangle\right) - |w_t|^2 + |\xi|^2|w|^2 = 0. \quad (3.20)$$

Set

$$S = \frac{1}{2b_1}\begin{pmatrix} 0 & a_1 - a_2 \\ a_2 - a_1 & 0 \end{pmatrix}.$$

Since iS is Hermitian,

$$\Re\langle iSw, w_t\rangle = \frac{1}{2}\frac{d}{dt}\langle iSw, w\rangle$$

holds and we can write (3.20) as

$$\frac{1}{2}\frac{d}{dt}\left(\langle\tilde{a}w, w\rangle + 2\Re\langle w_t, w\rangle + 2|\xi|\langle iSw, w\rangle\right) \\ - |w_t|^2 + |\xi|^2|w|^2 - 2\Re(|\xi|\langle iSw, w_t\rangle) = 0. \quad (3.21)$$

Now, add (3.18)+(3.19)+α(3.21) (for some $\alpha > 0$ to be determined later) to obtain

$$\frac{1}{2}\frac{d}{dt}E^{(1)} + F^{(1)} = 0, \quad (3.22)$$

where

$$E^{(1)} = \langle\tilde{a}w_t, w_t\rangle + |\xi|^2\langle\tilde{a}w, w\rangle + 2\Re(\langle w_t, -i|\xi|\tilde{b}w\rangle) \\ + \alpha\left(\langle\tilde{a}w, w\rangle + 2\Re\langle w_t, w\rangle + 2|\xi|\langle iSw, w\rangle\right)$$

and

$$F^{(1)} = |\tilde{a}w_t|^2 - \alpha|w_t|^2 - 2\Re(i|\xi|\langle(\tilde{a}\tilde{b} - S)w, w_t\rangle) + |\xi|^2|\tilde{b}w|^2 + \alpha|\xi|^2|w|^2.$$

for Proposition 3.1 First, show that $E^{(1)}$ is uniformly equivalent to $E_0^{(1)} = (1+|\xi|^2)|w|^2 + |w_t|^2$. Obviously, there exists $C_1 > 0$ such that

$$E^{(1)} \le C_1 E_0^{(1)}.$$

For

$$M = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{a} \end{pmatrix}$$

and $W = (w_t, -i|\xi|w)$,

$$\langle \tilde{a}w_t, w_t\rangle + |\xi|^2\langle \tilde{a}w, w\rangle + 2\Re(\langle w_t, -i|\xi|\tilde{b}w\rangle) = \langle MW, W\rangle_{\mathbb{C}^4}.$$

It is easy to show that $\sigma(M) = \sigma(\tilde{a}+\tilde{b}) \cup \sigma(\tilde{a}-\tilde{b})$. Furthermore $c_s \in (0,1)$ yields $\tilde{a}+\tilde{b} > 0$, $\tilde{a}-\tilde{b} > 0$. Thus M is positive definite, i.e.

$$\langle \tilde{a}w_t, w_t\rangle + |\xi|^2\langle \tilde{a}w, w\rangle + 2\Re(\langle w_t, -i|\xi|\tilde{b}w\rangle) \ge C_2(|w_t|^2 + |\xi|^2|w|^2)$$

for a $C_2 > 0$. Furthermore, by Young's inequality there exists $C_3 > 0$ such that

$$|2\Re\langle w_t, w\rangle + 2i|\xi|\langle Sw, w\rangle| \le \frac{d}{2}|w|^2 + C_3(|\xi|^2|w|^2 + |w_t|^2),$$

where $d = \min\{a_1, a_2\}$. In conclusion

$$E^{(1)} \ge C_2(|w_t|^2 + |\xi|^2|w|^2) - \alpha C_3(|\xi|^2|w|^2 + |w_t|^2) + \alpha\frac{d}{2}|w|^2.$$

Hence, for α sufficiently small there exists $C_4 > 0$ such that

$$E^{(1)} \ge C_4 E_0^{(1)}.$$

Finally show $F^{(1)} \ge c\rho(\xi)E_0^{(1)}$ for α sufficiently small. To this end write $F^{(1)} = F_1^{(1)} + F_2^{(1)}$, where

$$\begin{aligned} F_1^{(1)} &= (a_1^2 - \alpha)|w_t^1|^2 + (b_1^2 + \alpha)|\xi|^2|w^2|^2 \\ &\quad - 2\Re\left(i|\xi|\left(a_1 b_1 + \alpha\frac{a_1 - a_2}{2b_1}\right)w^2\bar{w}_t^1\right), \\ F_2^{(1)} &= (a_2^2 - \alpha)|w_t^2|^2 + (b_1^2 + \alpha)|\xi|^2|w^1|^2 \\ &\quad - 2\Re\left(i|\xi|\left(a_2 b_1 + \alpha\frac{a_2 - a_1}{2b_1}\right)w^1\bar{w}_t^2\right). \end{aligned}$$

Since

$$(a_1^2 - \alpha)(b_1^2 + \alpha) - \left(a_1 b_1 + \alpha \frac{a_1 - a_2}{2b_1}\right)^2 = \alpha(a_1 a_2 - b_1^2) + O(\alpha^2)$$

and $a_1 a_2 > b_1^2$ there exist $c_2 > 0$ such that

$$F_1^{(1)} \geq \alpha c_2 (|w_t^1|^2 + |\xi|^2 |w^2|^2)$$

for α sufficiently small. In the same way we get

$$F_2^{(1)} \geq \alpha c_2 (|w_t^2|^2 + |\xi|^2 |w^1|^2).$$

Therefore

$$F^{(1)} \geq \alpha c_2 (|w_t|^2 + |\xi|^2 |w|^2) \geq \alpha \frac{c_1}{2} \rho(\xi) E_0^{(1)},$$

which finishes the proof. □

Based on Lemma 3.2 the proof for Proposition 3.1 goes as [1, Proof of Theorem 3.1].

Next consider the inhomogeneous initial-value problem

$$\psi_{tt} - \sum_{i,j=1}^{3} \bar{B}_{ij} \psi_{x_i x_j} + a\psi_t + \sum_{j=1}^{n} b_j \psi_{x_j} = h, \text{ on } (0,T] \times \mathbb{R}^3, \tag{3.23}$$

$$\psi(0) = {}^0\psi, \text{ on } \mathbb{R}^3, \tag{3.24}$$

$$\psi_t(0) = {}^1\psi, \text{on } \mathbb{R}^3. \tag{3.25}$$

for some $h : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^4$. We get the following results:

3.3 Proposition. *Let s be a non-negative integer, $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s) \cap (L^1)^2$ and $h \in C([0,T], H^s \cap L^1)$. Then the solution ψ of* (3.23)-(3.25) *satisfies*

$$\begin{aligned} \|\partial_x^k \psi(t)\|_1 + \|\partial_x^k \psi_t(t)\| \leq{}& C(1+t)^{-\frac{3}{4}-\frac{k}{2}} (\|{}^0\psi\|_{L^1} + \|{}^1\psi\|_{L^1}) \\ &+ Ce^{-ct}(\|\partial_x^k({}^0\psi)\|_1 + \|\partial_x^k({}^1\psi)\|) \\ &+ C\int_0^t (1+t-\tau)^{-3/4-k/2} \|h(\tau)\|_{L^1} \\ &\qquad + C\exp(-c(t-\tau))\|\partial_x^k h(\tau)\| d\tau \end{aligned} \tag{3.26}$$

for all $t \in [0,T]$ and $0 \leq k \leq s$.

Proof. For $t \in [0,T]$ let $T(t)$ be the linear operator which maps $({}^0\psi, {}^1\psi)$ to the solution $(\psi(t)), \psi_t(t))$ of the homogeneous IVP (3.4)-(3.6) at time t. By Duhamel's principle the solution of (3.23)-(3.25) is given by

$$(\psi(t), \psi_t(t)) = T(t)({}^0\psi, {}^1\psi) + \int_0^t T(t-\tau)(0, h(\tau))d\tau.$$

Hence the assertion is an immediate consequence of Proposition 3.1. □

3.4 Proposition. *Let s be a non-negative integer. There exist $C_1, C_2 > 0$ such that for all $({}^0\psi, {}^1\psi) \in H^{s+1} \times H^s$ and $h \in C([0,T], H^s)$ the solution ψ of* (3.23)-(3.25) *satisfies*

$$\begin{aligned} C_1 \left(\|\partial_x^\alpha \psi(t)\|_1^2 + \|\partial_x^\alpha \psi_t(t)\|^2 \right) + C_1 \int_0^t \|\partial_x^\alpha \partial_x \psi(\tau)\|^2 + \|\partial_x^\alpha \psi_t(\tau)\|^2 d\tau \\ \leq C_2 \left(\|\partial_x^\alpha({}^0\psi)\|_1^2 + \|\partial_x^\alpha({}^1\psi)\|^2 \right) \\ + \int_0^t C_2 \|\partial_x^\alpha \psi(\tau)\|^2 + \left(\partial_x^\alpha h(\tau), \frac{a}{2}\partial_x^\alpha \psi(\tau) + \partial_x^\alpha \psi_t(\tau) \right)_{L^2} d\tau \end{aligned} \tag{3.27}$$

for all $t \in [0,T]$ and $\alpha \in \mathbb{N}_0^3$, $|\alpha| = s$

Proof. Consider (3.23) in Fourier space, i.e.

$$\hat{\psi}_{tt} + |\xi|^2 B(\check{\xi})\hat{\psi} + a\hat{\psi}_t - i|\xi| b(\check{\xi})\hat{\psi} = \hat{h}$$

We proceed similarly as in the proof of Lemma 3.2. Again w.l.o.g. assume $\xi = (|\xi|, 0, 0)$, then (3.23) reads

$$w_{tt} + |\xi|^2 w + \tilde{a} w_t - i|\xi|\tilde{b} w = (\hat{h}^0, \hat{h}^1)^t, \tag{3.28}$$

$$v_{tt} + \bar{\eta}\bar{\sigma}^{-1}|\xi|^2 v + \bar{\sigma}^{-1} v_t = (\hat{h}^2, \hat{h}^3)^t, \tag{3.29}$$

where $w = (\hat{\psi}_0, \hat{\psi}_1)$, $v = (\hat{\psi}_2, \hat{\psi}_3)$, $\tilde{a}, \tilde{b}$ are given by (3.15). First, take the scalar product of (3.29) with $v_t + 1/(2\bar{\sigma})v$ and consider the real part

$$\frac{1}{2}\frac{d}{dt}E^{(2)} + F^{(2)} = \Re \left\langle (\hat{h}^2, \hat{h}^3)^t, v_t + \frac{1}{2\bar{\sigma}} v \right\rangle \tag{3.30}$$

where $E^{(2)}, F^{(2)}$ are given by (3.16), (3.17). Since $E^{(2)}$ is uniformly equivalent to $|v_t|^2 + (1 + |\xi|^2)|v|^2$ and $F^2 \geq c(|v_t|^2 + |\xi|^2|v|^2)$, integrating (3.30) leads to

$$\begin{aligned} C_1 \left(|v_t|^2 + (1 + |\xi|^2)|v|^2 \right) + C_1 \int_0^t |v_t|^2 + |\xi|^2 |v|^2 d\tau \\ \leq C_2 \left(|v_t(0)|^2 + (1 + |\xi|^2)|v(0)|^2 \right) + \int_0^t \Re \left\langle (\hat{h}^2, \hat{h}^3)^t, v_t + \frac{1}{2\bar{\sigma}} v \right\rangle d\tau. \end{aligned} \tag{3.31}$$

Next, take the scalar product of (3.28) with $w_t + (\tilde{a}/2)w$. The real part reads

$$\frac{1}{2}\frac{d}{dt}E^{(1)} + F^{(1)} = \Re\langle(\hat{h}^0, \hat{h}^1)^t, w_t + \frac{1}{2}\tilde{a}w\rangle, \tag{3.32}$$

where

$$E^{(1)} = |w_t|^2 + |\xi|^2|w|^2 + \frac{1}{2}|\tilde{a}w|^2 + \Re\langle\tilde{a}w_t, w\rangle$$

and

$$F^{(1)} = \frac{1}{2}\langle\tilde{a}w_t, w_t\rangle + \Re\langle -i|\xi|\tilde{b}w, w_t\rangle + \frac{1}{2}|\xi|^2\langle\tilde{a}w, w\rangle - \frac{1}{2}\Re\langle i|\xi|\tilde{b}w, \tilde{a}w\rangle.$$

Using Young's inequality it is easy to see that $E^{(1)}$ is uniformly equivalent to $|w_t|^2 + (1 + |\xi|^2)|w|^2$. Furthermore

$$F^{(1)} = \frac{1}{2}\langle MW, W\rangle_{\mathbb{C}^4} - \frac{1}{2}\Re\langle i|\xi|\tilde{b}w, \tilde{a}w\rangle,$$

where

$$M = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{a} \end{pmatrix}$$

and $W = (w_t, -i|\xi|w)$. As M is positive definite (see proof of Lemma 3.2) there exists $c_1, c_2 > 0$ such that

$$F^{(1)} \geq c_1(|w_t|^2 + |\xi|^2|w|^2) - c_2|\xi||w||w| \geq \frac{c_1}{2}(|w_t|^2 + |\xi|^2|w|^2) - \frac{c_2^2}{2c_1}|w|^2.$$

Thus integrating (3.32) leads to

$$\begin{aligned} &C_1\left(|w_t|^2 + (1 + |\xi|^2)|w|^2\right) + C_1\int_0^t |w_t|^2 + |\xi|^2|w|^2 d\tau \\ &\leq C_2\left(|w_t(0)|^2 + (1 + |\xi|^2)|w(0)|^2\right) + \int_0^t C_2|w|^2 + \Re\langle(\hat{h}^0, \hat{h}^1)^t, w_t + \frac{\tilde{a}}{2}w\rangle d\tau. \end{aligned} \tag{3.33}$$

Adding (3.31) and (3.33) gives

$$\begin{aligned} &C_1\left(|\hat{\psi}_t|^2 + (1 + |\xi|^2)|\hat{\psi}|^2\right) + C_1\int_0^t |\hat{\psi}_t|^2 + |\xi|^2|\hat{\psi}|^2 d\tau \\ &\quad \leq C_2\left(|{}^1\hat{\psi}|^2 + (1 + |\xi|^2)|{}^0\hat{\psi}|^2\right) + \int_0^t C_2|\hat{\psi}|^2 + \Re\langle\hat{h}, \hat{\psi}_t + \frac{a}{2}\hat{\psi}\rangle d\tau. \end{aligned} \tag{3.34}$$

Finally the assertion follows by multiplying (3.34) with $\xi^{2\alpha}$ for $\alpha \in \mathbb{N}_0^n$, $|\alpha| = s$, integrating with respect to ξ, and using Plancherel's identity. $\square$

4. Global Existence and Asymptotic Decay of Small Solutions

The goal of this section is to prove Theorem 2.1. We will proceed as follows: First we show a decay estimate for all but the highest order derivatives of a solution, Proposition 4.1, and then an energy estimate for the derivatives of highest order, Proposition 4.3. Then Theorem 2.1 follows from combining the two, Proposition 4.4.

As in Section 3 fix $\bar{\theta} > 0$, multiply (2.2) by $(n(\bar{\theta})s(\bar{\theta}))^{-1}\bar{\theta}^{-2}(A^{(2)})^{-\frac{1}{2}}$ and change the variables to $(A^{(2)})^{\frac{1}{2}}\psi$ such that the linearization at $(\bar{\theta}^{-1}, 0, 0, 0)$ is given by (3.3). In addition, consider $\psi - \bar{\psi}$ with $\bar{\psi} = (\bar{\theta}^{-1}, 0, 0, 0)$ instead of ψ, ${}^0\psi - \bar{\psi}$ instead of ${}^0\psi$, $A(\cdot + \bar{\psi})$ instead of $A(\cdot)$ and so on, such that the rest state is shifted from $(\bar{\theta}^{-1}, 0, 0, 0)$ to $(0, 0, 0, 0)$. In the following, when (2.2) or (2.3)-(2.5) are mentioned, we actually mean these modified equations.

Write $U = (\psi, \psi_t)$ and $U_0 = ({}^0\psi, {}^1\psi)$ for a solution to (2.3)-(2.5) and the initial values, respectively. Let $s \geq s_0 + 1$ ($s_0 = [3/2] + 1$), $T > 0$, $U_0 \in H^{s+1} \times H^s$, and ψ satisfy

$$\psi \in \bigcap_{j=0}^{s} C^j\left([0, T], H^{s+1-j}\right). \tag{4.1}$$

For $0 \leq t \leq t_1 \leq T$ define

$$N_s(t, t_1)^2 = \sup_{\tau \in [t, t_1]} \|U(\tau)\|_{s+1,s}^2 + \int_t^{t_1} \|U(\tau)\|_{s+1,s}^2 d\tau.$$

We write $N_s(t)$ instead of $N_s(0, t)$. Furthermore assume that $N_s(T) \leq a_0$ for an $a_0 > 0$. Since $s \geq s_0$, $H^s \hookrightarrow L^\infty$ is a continuous embedding. Hence $N_s(T) \leq a_0$ implies that $(\psi, \psi_t, \partial_x \psi)$ takes values in a closed ball $\overline{B(0, r)} \subset \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^{12}$ for some $r > 0$.

First we prove the decay estimate. To this end it is convenient to rewrite (2.3) as - cf. (3.3) -

$$\psi_{tt} - \sum_{i,j=1}^{3} \bar{B}_{ij}\psi_{x_i x_j} + a\psi_t + \sum_{j=1}^{3} b_j \psi_{x_j} = h(\psi, \psi_t, \partial_x \psi, \partial_x^2 \psi, \partial_x \psi_t), \tag{4.2}$$

where

$$h(\psi, \psi_t, \partial_x\psi, \partial_x^2\psi, \partial_x\psi_t) = \sum_{i,j=1}^{3} \left(A(\psi)^{-1} B_{ij}(\psi) - \bar{B}_{ij} \right) \psi_{x_i x_j}$$
$$- \sum_{j=1}^{3} A(\psi)^{-1} D_j(\psi) \psi_{t x_j}$$
$$- A(\psi)^{-1} f(\psi, \psi_t, \partial_x\psi) + a\psi_t + \sum_{j=1}^{3} b_j \psi_{x_j}. \quad (4.3)$$

4.1 Proposition. *There exist constants* $a_1(\leq a_0)$, $\delta_1 = \delta_1(a_1)$, $C_1 = C_1(a_1, \delta_1) > 0$ *such that the following holds: If* $\|U_0\|_{s,s-1,1}^2 \leq \delta_1$ *and* $N_s(T)^2 \leq a_1$ *for a solution* ψ *of* (2.3)-(2.5) *satisfying* (4.1), *then*

$$\|U(t)\|_{s,s-1} \leq C_1(1+t)^{-\frac{3}{4}} \|U_0\|_{s,s-1,1} \quad (t \in [0,T]). \quad (4.4)$$

Proof. Let $t \in [0,T]$ and ψ be a solution to (2.3)-(2.5). Since $B_{ij}(0) = \bar{B}_{ij}$, $D_j(0) = 0$ and

$$a\psi_t + \sum_{j=1}^{3} b_j \psi_{x_j} = Df(0)(\psi, \psi_t, \partial_x\psi),$$

Lemmas A.1, A.2 show that there exist $C, c > 0$ ($c \leq a_0$) such that $h(t) \in H^{s-1} \cap L^1$ and

$$\|h(t)\|_{s-1} \leq C\|\psi(t)\|_{s-1} \left(\|\partial_x^2\psi(t)\|_{s-1} + \|\partial_x\psi_t(t)\|_{s-1} \right)$$
$$+ C\|(\psi(t), \psi_t(t), \partial_x\psi(t))\|_{s-1}^2$$
$$\leq C\|U(t)\|_{s+1,s}\|U(t)\|_{s,s-1},$$
$$\|h(t)\|_{L^1} \leq C\|U(t)\|_{2,1}^2,$$

if $N_s(T) \leq c$, which we will assume throughout this proof. Proposition 3.3 yields

$$\|U(t)\|_{s,s-1} \leq C(1+t)^{-\frac{3}{4}} \|U_0\|_{s,s-1,1}$$
$$+ C \int_0^t \exp(-c(t-\tau))\|h(\tau)\|_{s-1} + (1+t-\tau)^{-\frac{3}{4}} \|h(\tau)\|_{L^1} d\tau,$$

which leads to

$$\|U(t)\|_{s-1,s} \leq C(1+t)^{-\frac{3}{4}}\|U_0\|_{s,s-1,1}$$
$$+ C \sup_{\tau\in[0,t]} \|U(\tau)\|_{s+1,s} \int_0^t \exp(-c(t-\tau))\|U(\tau)\|_{s,s-1} d\tau$$
$$+ C \int_0^t (1+t-\tau)^{-\frac{3}{4}}\|U(\tau)\|^2_{s,s-1} d\tau.$$

Multiplying with $(1+t)^{\frac{3}{4}}$ gives

$$(1+t)^{\frac{3}{4}}\|U(t)\|_{s,s-1} \leq C\|U_0\|_{s,s-1,1}$$
$$+ CN_s(t)\mu_1(t) \sup_{\tau\in[0,t]} (1+\tau)^{\frac{3}{4}}\|U(\tau)\|_{s,s-1}$$
$$+ C\mu_2(t) \sup_{\tau\in[0,t]} (1+\tau)^{\frac{3}{2}}\|U(\tau)\|^2_{s,s-1},$$

where

$$\mu_1(t) = (1+t)^{\frac{3}{4}} \int_0^t \exp(-c(t-\tau))(1+\tau)^{-\frac{3}{4}} d\tau$$
$$\mu_2(t) = (1+t)^{\frac{3}{4}} \int_0^t (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{3}{2}} d\tau.$$

Since μ_1, μ_2 are bounded functions on $[0,\infty)$, we get

$$\sup_{\tau\in[0,t]} (1+\tau)^{\frac{3}{4}}\|U(\tau)\|_{s,s-1} \leq C\|U_0\|_{s,s-1,1}$$
$$+ CN_s(t) \sup_{\tau\in[0,t]} (1+\tau)^{\frac{3}{4}}\|U(\tau)\|_{s,s-1}$$
$$+ C \sup_{\tau\in[0,t]} (1+\tau)^{\frac{3}{2}}\|U(\tau)\|^2_{s,s-1}.$$

We can deduce from this equation that there in fact exists $a_1 > 0$ ($a_1 \leq c$), $\delta_1 > 0$ and $C_1 > 0$, such that

$$\sup_{\tau\in[0,t]} (1+\tau)^{\frac{3}{4}}\|U(\tau)\|_{s,s-1} \leq C_1\|U_0\|_{s,s-1,1},$$

whenever $N_s(T)^2 \leq a_1$ and $\|U_0\|^2_{s,s-1,1} \leq \delta_1$. □

4.2 Corollary. *In the situation of Proposition 4.1 there exists a* $C_2 = C_2(a_1, \delta_1) > 0$ *such that*

$$N_{s-1}(T)^2 \leq C_2\|U_0\|^2_{s,s-1,1} \tag{4.5}$$

whenever $N_s(T)^2 \leq a_1$ *and* $\|U_0\|^2_{s,s-1,1} \leq \delta_1$.

Proof. The function $t \mapsto (1+t)^{-\frac{3}{4}}$ is square-integrable on $[0, \infty)$. Therefore the assertion is a direct consequence of Proposition 4.1. □

Now it is convenient to write (2.3) as

$$\psi_{tt} - \sum_{i,j=1}^{3} \bar{B}_{ij}\psi_{x_i x_j} + a\psi_t + \sum_{j=1}^{3} b_j \psi_{x_j} = L(\psi)\psi + h_2(\psi, \psi_t, \partial_x \psi), \tag{4.6}$$

where

$$L(\psi)\psi = (I - A(\psi))\psi_{tt} - \sum_{i,j=1}^{3} (\bar{B}_{ij} - B_{ij}(\psi))\psi_{x_i x_j} - \sum_{j=1}^{3} D_j(\psi)\psi_{tx_j},$$

$$h_2(\psi, \psi_t, \partial_x \psi) = a\psi_t + \sum_{j=1}^{3} b_j \psi_{x_j} - f(\psi, \psi_t, \partial_x \psi).$$

4.3 Proposition. *There exist constants $a_2 (\leq a_0)$ and $c_3, C_3 = C_3(a_2) > 0$ such that the following holds: If $N_s(T)^2 \leq a_2$ for a solution ψ of* (2.3)-(2.5) *satisfying* (4.1)*, then*

$$\begin{aligned} \|\partial_x^s \psi(t)\|_1^2 + \|\partial_x^s \psi_t(t)\|^2 + \int_0^t \|\partial_x^{s+1}\psi(\tau)\|^2 + \|\partial_x^s \psi_t(\tau)\|^2 d\tau \\ - c_3 \int_0^t \|\partial_x^s \psi(\tau)\|^2 d\tau \leq C_3 \left(\|U_0\|_{s,s+1}^2 + N_s(t)^3 \right) \quad (t \in [0,T]). \end{aligned} \tag{4.7}$$

Proof. We prove the result in two steps.

Step 1: Let $U_0 = ({}^0\psi, {}^1\psi) \in H^{s+1} \times H^s$ and

$$\psi \in \bigcap_{j=0}^{s} C^j \left([0,T], H^{s+2-j} \right) \tag{4.8}$$

be a solution to (2.3)-(2.5). By Lemma A.2 there exists a $c > 0$ such that $I - A(\psi), \bar{B}_{ij} - B_{ij}(\psi), D_j(\psi) \in H^{s+1}$ provided $N_s(T) \leq c$. We will assume this throughout the proof. Then due to (4.8) and [6, Lemma 2.3] $L(\psi)\psi \in H^s$. Lemma A.2 yields $h_2 \in H^s$. Thus we can conclude by Proposition 3.4 that

$$\begin{aligned} C_1 \left(\|\partial_x^\alpha \psi(t)\|_1^2 + \|\partial_x^\alpha \psi_t(t)\|^2 \right) + C_1 \int_0^t \|\partial_x^\alpha \partial_x \psi(\tau)\|^2 + \|\partial_x^\alpha \psi_t(\tau)\|^2 d\tau \\ \leq C_2 \left(\|\partial_x^\alpha ({}^0\psi)\|_1^2 + \|\partial_x^\alpha ({}^1\psi)\|^2 \right) \\ + C_2 \int_0^t \|\partial_x^\alpha \psi(\tau)\|^2 d\tau \\ + \int_0^t \left(\partial_x^\alpha (L(\psi(\tau))\psi(\tau) + h_2(\tau)), \partial_x^\alpha \psi_t(\tau) + \frac{a}{2} \partial_x^\alpha \psi(\tau) \right)_{L^2} d\tau \end{aligned} \tag{4.9}$$

for all $\alpha \in \mathbb{N}_0^3$, $|\alpha| = s$. First obviously

$$\left| \left(\partial_x^\alpha h_2, \partial_x^\alpha \psi_t + \frac{a}{2} \partial_x^\alpha \psi \right)_{L^2} \right| \le C \|h_2\|_s \|U\|_s \tag{4.10}$$

and integrating by parts gives

$$\begin{aligned} \left| \left(\partial_x^\alpha (L(\psi)\psi), \frac{a}{2} \partial_x^\alpha \psi \right)_{L^2} \right| &\le C \|L(\psi)\psi\|_{s-1} \|\psi\|_{s+1} \\ &\le C \|I - A(\psi)\|_s \|\psi_{tt}\|_{s-1} \|\psi\|_{s+1} \\ &+ C \sum_{i,j=1}^3 \|\bar{B}_{ij} - B_{ij}(\psi)\|_s \|\partial_x^2 \psi\|_{s-1} \|\psi\|_{s+1} \\ &+ C \sum_{j=1}^3 \|D_j(\psi)\|_s \|\partial_x \psi_t\|_{s-1} \|\psi\|_{s+1}. \end{aligned} \tag{4.11}$$

Next write

$$\begin{aligned} \partial_x^\alpha \left(L(\psi)\psi \right) = L(\psi) \partial_x^\alpha \psi + [\partial_x^\alpha, (I - A(\psi))] \psi_{tt} \\ - \sum_{i,j=1}^3 [\partial_x^\alpha, (\bar{B}_{ij} - B_{ij}(\psi))] \psi_{x_i x_j} - \sum_{j=1}^3 [\partial_x^\alpha, D_j(\psi)] \psi_{t x_j}. \end{aligned}$$

Since $I - A(\psi), \bar{B}_{ij} - B_{ij}(\psi), D_j(\psi) \in H^s$, [6, Lemma 2.5(i)] yields

$$\begin{aligned} \|[\partial_x^\alpha, (I - A(\psi))] \psi_{tt}\| &\le C \|\partial_x A(\psi)\|_{s-1} \|\psi_{tt}\|_{s-1} \\ \|[\partial_x^\alpha, (\bar{B}_{ij} - B_{ij}(\psi))] \psi_{x_i x_j}\| &\le C \|\partial_x B_{ij}(\psi)\|_{s-1} \|\psi_{x_i x_j}\|_{s-1} \\ \|[\partial_x^\alpha, D_j(\psi)] \psi_{t x_j}\| &\le C \|\partial_x D_j(\psi)\|_{s-1} \|\psi_{t x_j}\|_{s-1}. \end{aligned} \tag{4.12}$$

Furthermore integration by parts and the symmetry of A, B_{ij} and D_j give

$$\begin{aligned} &\int_0^t (L(\psi) \partial_x^\alpha \psi, \partial_x^\alpha \psi_t)_{L^2} \, d\tau \\ &\qquad \le C \int_0^t \|\partial_t A\|_{L^\infty} \|\partial_x^\alpha (\partial_x \psi, \psi_t)\|^2 d\tau \\ &+ \left(\sum_{i,j=1}^3 \|\partial_t B_{ij}\|_{L^\infty} + \|\partial_x B_{ij}\|_{L^\infty} + \sum_{j=1}^3 \|\partial_x D_j\|_{L^\infty} \right) \|\partial_x^\alpha (\partial_x \psi, \psi_t)\|^2 d\tau \\ &\qquad + C \left(\|I - A\|_{L^\infty} + \sum_{i,j=1}^3 \|\bar{B}_{ij} - B_{ij}\|_{L^\infty} \right) \|\partial_x^\alpha (\partial_x \psi, \psi_t)\|^2 \\ &\qquad\qquad + C \|\partial_x^\alpha ({\partial_x}^0 \psi, {}^1 \psi)\|^2. \end{aligned} \tag{4.13}$$

In conclusion, (4.9) and the estimates (4.10), (4.11), (4.12) (4.13) lead to

$$\begin{aligned}
\|\partial_x^\alpha\psi(t)\|_1^2 + \|\partial_x^\alpha\psi_t(t)\|^2 + \int_0^t \|\partial_x^\alpha\partial_x\psi(\tau)\|^2 + \|\partial_x^\alpha\psi_t(\tau)\|^2 d\tau \\
- c\int_0^t \|\partial_x^\alpha\psi(\tau)\|^2 d\tau \\
\leq C\|U_0\|_{s+1,s}^2 + C\int_0^t \|h_2(\psi)\|_s\|U\|_{s+1,s} + R_1(\psi)\|U\|_{s+1,s}^2 d\tau \\
+ C\int_0^t \|I - A(\psi)\|_s\|\psi_{tt}\|_{s-1}\|U\|_{s+1,s} d\tau \\
+ CR_2(\psi)\|U(t)\|_{s+1,s}^2, \quad (4.14)
\end{aligned}$$

where

$$R_1(\psi) = \|\partial_t A(\psi)\|_s + \|I - A(\psi)\|_s + \sum_{i,j=1}^3 \|\partial_t B_{ij}(\psi)\|_s + \|\bar{B}_{ij} - B_{ij}(\psi)\|_s + \sum_{j=1}^3 \|D_j(\psi)\|_s$$

and

$$R_2(\psi) = \|I - A(\psi)\|_s + \sum_{i,j=1}^3 \|\bar{B}_{ij} - B_{ij}(\psi)\|_s.$$

Step 2: Now let ψ be a solution to (2.3)-(2.5) satisfying (4.1). For $\delta > 0$ set $\psi^\delta = \phi_\delta * \psi$. Applying $\phi_\delta *$ to (4.6) yields

$$\psi_{tt}^\delta - \sum_{i,j=1}^3 \bar{B}_{ij}\psi_{x_ix_j}^\delta + a\psi_t^\delta + \sum_{j=1}^3 b_j\psi_{x_j}^\delta = L(\psi)\psi^\delta + R^\delta(\psi) + h_2^\delta,$$

where $h^\delta = \phi_\delta * h_2$ and

$$R^\delta(\psi) = [\phi_\delta *, (I - A(\psi))]\psi_{tt} - \sum_{i,j=1}^n [\phi_\delta *, \bar{B}_{ij} - B_{ij}(\psi)]\psi_{x_ix_j} - \sum_{j=1}^3 [\phi_\delta *, D_j(\psi)]\psi_{tx_j}.$$

Due to [6, Lemma 2.5 (ii)] $R^\delta(\psi) \in H^s$. Hence $L(\psi)\psi^\delta + R^\delta(\psi) + h_2^\delta \in H^s$. Thus proceeding as in step 1 yields

$$\begin{aligned}
\|\partial_x^\alpha \psi^\delta(t)\|_1^2 + \|\partial_x^\alpha \psi_t^\delta(t)\|^2 + \int_0^t \|\partial_x^\alpha \partial_x \psi^\delta(\tau)\|^2 + \|\partial_x^\alpha \psi_t^\delta(\tau)\|^2 d\tau \\
- c\int_0^t \|\partial_x^\alpha \psi^\delta(\tau)\|^2 d\tau \\
\leq C\|U_0^\delta\|_{s+1,s}^2
\end{aligned}$$

$$+C\int_0^t \|h_2^\delta\|_s\|U^\delta\|_{s+1,s} + R_1(\psi)\|U^\delta\|_{s+1,s}^2 + \|I-A(\psi)\|_s\|\psi_{tt}^\delta\|_{s-1}\|U^\delta\|_{s+1,s}d\tau$$

$$+ C\int_0^t \|R^\delta(\psi)\|_s\|U^\delta\|_{s+1,s}d\tau + CR_2(\psi)\|U^\delta(t)\|_{s+1,s}^2.$$

It is easy to see that $U^\delta \to U$ and $h_2^\delta \to h_2$ in $L^\infty\left([0,T], H^{s+1}\times H^s\right)$ and in $L^2([0,T],H^s)$, respectively, as $\delta \to 0$. Furthermore $R^\delta(\psi) \to 0$ in $L^2([0,T],H^s)$ as $\delta \to 0$ due to [6, Lemma 2.5(ii)]. Hence we get (4.14) for ψ satisfying (4.1).

Furthermore by Lemma A.1 and

$$\|h_2\|_s \leq C\|U\|_{s+1,s}^2.$$

and by Lemma A.2

$$R_1(\psi) + R_2(\psi) \leq C\|U\|_{s+1,s}$$

for $N_s(T)$ sufficiently small. Finally, since ψ satisfies (2.3),

$$\|\psi_{tt}\|_{s-1} \leq C(\|\partial_x^2\psi\|_{s-1} + \|\partial_x\psi_t\|_{s-1} + \|f(\psi,\psi_t,\partial_x\psi)\|_{s-1}) \leq C\|U\|_{s+1,s}$$

holds for $N_s(T)$ sufficiently small. Therefore we can deduce from (4.14) that

$$\begin{aligned}
\|\partial_x^\alpha \psi(t)\|_1^2 + \|\partial_x^\alpha \psi_t(t)\|^2 + \int_0^t \|\partial_x^\alpha \partial_x \psi(\tau)\|^2 + \|\partial_x^\alpha \psi_t(\tau)\|^2 d\tau \\
- c\int_0^t \|\partial_x^\alpha \psi(\tau)\|^2 d\tau \\
\leq C\|U_0\|_{s+1,s}^2 + C\|U(t)\|_{s+1,s}^3 + C\int_0^t \|U(\tau)\|_{s+1,s}^3 d\tau.
\end{aligned}$$

The assertion is an immediate consequence of this inequality. □

4.4 Proposition. *In the situation of Proposition 4.1 there exist constants* $a_3(\leq \min\{a_2, a_1\})$*,* $C_4 = C_4(a_3, \delta_1) > 0$ *(*δ_1 *being the constant in Proposition 4.1) such that the the following holds: If* $\|U_0\|^2_{s,s-1,1} \leq \delta_1$ *and* $N_s(T)^2 \leq a_3$ *for a solution* ψ *of* (2.3)-(2.5) *satisfying* (4.1)*, then*

$$N_s(t)^2 \leq C_4^2 \|U_0\|^2_{s+1,s,1} \quad (t \in [0, T]). \tag{4.15}$$

Proof. This follows directly by adding (4.5)+ε(4.7) for ε sufficiently small. □

Finally we turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $T_1 > 0, \delta_2 > 0$ such that for all $U_0 = (\psi_0, \psi_1) \in H^{s+1} \times H^s$, where $\|U_0\|^2_{s+1,s} < \delta_2$, there exists a solution $U = (\psi, \psi_t)$ of the Cauchy problem (2.3)-(2.5) with

$$\psi \in \bigcap_{j=1}^{s} C^j \left([0, T_1], H^{s+1-j} \right).$$

This is possible due to [5, Theorem III]. Furthermore let a_3, δ_1 and C_4 be the constants in Proposition 4.4. Choose $0 < \varepsilon < a_3/(2(1+T_1))$. Due to [5, Ibid.] there exists $\delta_3 > 0, (\delta_3 \leq \delta_2)$ such that for all $U_0 = (\psi_0, \psi_1) \in H^{s+1} \times H^s$, where $\|U_0\|^2_{s+1,s} < \delta_3$, the solution U of (2.3)-(2.5) satisfies

$$\sup_{t \in [0,T_1]} \|U(t)\|^2_{s+1,s} < \varepsilon.$$

Now set $\delta_0 = \min\{\delta_1, \delta_3, \delta_3/C_4, a_3/(2C_4)\}$ and choose any $U_0 \in \left(H^{s+1} \times H^s\right) \cap \left(L^1 \times L^1\right)$ for which $\|U_0\|^2_{s+1,s,1} < \delta_0$. Since $\delta_0 \leq \delta_3$, we have

$$N_s(T_1)^2 < \varepsilon + T_1 \varepsilon < \frac{a_3}{2}.$$

Hence by Proposition 4.4 and $\|U_0\|^2_{s+1,s,1} < \delta_1$

$$N_s(T_1)^2 \leq C_4 \|U_0\|^2_{s+1,s} < C_4 \delta_0 \leq \delta_3. \tag{4.16}$$

Furthermore due to Proposition 4.1, (2.7) holds for all $t \in [0, T_1]$. In particular (4.16) yields

$$\|U(T_1)\|^2_{s+1,s} < \delta_3. \tag{4.17}$$

Thus we can solve (2.3) on $[T_1, 2T_1]$ with initial values $(\psi(T_1), \psi_t(T_1))$ and get

$$N_s(T_1, 2T_1)^2 \leq \varepsilon + T_1 \varepsilon < \frac{a_3}{2}.$$

Now extend the solution (ψ, ψ_t) continuously on $[0, 2T_1]$. We can conclude

$$N_s(2T_1)^2 \leq N_s(T_1)^2 + N_s(T_1, 2T_1)^2 < \frac{a_3}{2} + \frac{a_3}{2} = a_3.$$

Since we have already assumed $\|U_0\|_{s+1,s,1}^2 < \delta_1$, Propositions 4.4 and 4.1 yield

$$N_s(2T_1) \leq C_4 \delta_0 \tag{4.18}$$

and (2.7) holds for all $t \in [0, 2T_1]$. Due to (4.18) we can repeat the former argument to obtain a solution on $[0, 3T_1]$ and further repetition proves the assertion. □

A. Appendix

A.1 Lemma. *Let $n, N \in \mathbb{N}$, $s \geq s_0 := [\frac{n}{2}]+1$ and $F \in C^\infty(\mathbb{R}^N)$, $F(0) = 0$. Then there exist $\delta > 0$, $C = C(\delta) > 0$ such that for all $u \in H^s$ with $\|u\|_s \leq \delta$, $F(u) - \partial_u F(0) \in H^s$ and*

$$\|F(u) - \partial_u F(0)u\|_s \leq C\|u\|_s^2.$$

Proof. Since $s \geq s_0$, there exists a $C_1 > 0$ such that

$$\|u\|_{L^\infty} \leq C_1 \|u\|_s$$

for all $u \in H^s$. Furthermore due to $F(0) = 0$ there exist $\delta_1 > 0$, $C_2 = C_2(\delta_1) > 0$ such that

$$|F(y) - \partial_y F(0)y| \leq C_2 |y|^2.$$

for all $y \in \mathbb{R}^N$ with $|y| \leq \delta_1$. Now let $u \in H^s$ such that $\|u\|_s \leq \delta_1/C_1$ (i.e. $\|u\|_{L^\infty} \leq \delta_1$). Then

$$\|F(u) - \partial_u F(0)u\| \leq C_2 \|u\|_{L^\infty} \|u\| \leq C_1 C_2 \|u\|_s^2. \tag{A.1}$$

Furthermore for $\alpha \in \mathbb{N}_0^n$ with $1 \leq |\alpha| = j \leq s$ we get

$$\partial_x^\alpha F(u) = \partial_u F(u) \partial_x^\alpha u + R,$$

where

$$R = \sum_{1 \leq |\beta| < j} \binom{\alpha}{\beta} \partial_x^\beta u \ \partial_x^{\alpha-\beta} F(u).$$

Since $\partial_x u \in H^{s-1}$ and $\|u\|_{L^\infty} \le \delta_1$, we get $\partial_x F(u) \in H^{s-1}$ and

$$\|\partial_x F(u)\|_{s-1} C_3 \|\partial_x u\|_{s-1}$$

for a $C_3 = C_3(\delta_2) > 0$ by [6, Lemma 2.4]. Therefore [6, Lemma 2.3] yields

$$\|R\| \le C_4 \|\partial_x u\|_{s-1} \|\partial_x F(u)\|_{s-1} \le C_3 C_4 \|\partial_x u\|_{s-1}^2$$

for a $C_4 > 0$. On the other hand there exist $\delta_2 > 0$, $C_5 = C_5(\delta_2) > 0$, such that

$$|\partial_y F(y) - \partial_y F(0)| \le C_5 |y|$$

for all $y \in \mathbb{R}^N$ with $|y| \le \delta_2$. Assuming $\|u\|_s \le \delta_2 / C_1$ entails

$$\begin{aligned}
\|\partial_x^\alpha (F(u) - \partial_u F(0))\| &\le \|(\partial_u F(u) - \partial_u F(0)) \partial_x^\alpha u\| + \|R\| \\
&\le \|\partial_u F(u) - \partial_u F(0)\|_{L^\infty} \|u\|_s + C_3 C_4 \|\partial_x u\|_{s-1} \\
&\le \max\{C_3 C_4, C_5\} \|u\|_s^2.
\end{aligned}$$

Since α was arbitrary, this estimate together with (A.1) yield the assertion for $\delta = \min\{\delta_1, \delta_2\}/C_1$. □

A.2 Lemma. *Let $n, N \in \mathbb{N}$, $s \ge s_0$ and $F \in C^\infty(\mathbb{R}^N, \mathbb{R}^{N \times N})$. Then there exist $\delta > 0$, $C = C(\delta) > 0$ such that for all $u \in H^s(\mathbb{R}^n, \mathbb{R}^N)$ with $\|u\|_s \le \delta$, $(F(u) - F(0))u \in H^s$ and*

$$\|(F(u) - F(0))u\|_s \le C \|u\|_s^2.$$

Proof. First note that there exist $\delta_1 > 0$, $C_1 = C_1(\delta_1) > 0$ such that

$$|F(y) - F(0)| \le C_1 |y|$$

for all $y \in \mathbb{R}^N$, $|y| \le \delta_1$ as well as $C_2 > 0$ such that

$$\|v\|_{L^\infty} \le C_2 \|v\|_s$$

for all $v \in H^s$. Now let $u \in H^s$, $\|u\|_s \le \delta_1 / C_2$. Then

$$\|F(u) - F(0)\| \le C_1 \|u\|_s$$

holds. On the other hand by [6, Lemma 2.4] $\partial_x F(u) \in H^{s-1}$ and

$$\|\partial_x F(u)\|_{s-1} \le C_3 \|\partial_x u\|_{s-1}$$

for a $C_3 = C_3(\delta_1) > 0$. Hence $F(u) - F(0) \in H^s$ and

$$\|F(u) - F(0)\|_s \le C_4 \|u\|_s$$

for $\|u\|_s \le \delta = \delta_1 / C_2$. Now the assertion follows from [6, Lemma 2.4]. □

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References

[1] P. M. Dharmawardane, J. E. Muñoz Rivera, and S. Kawashima. Decay property for second order hyperbolic systems of viscoelastic materials. *J. Math. Anal. and Appl.*, 366(2):621–635, 2010.

[2] H. Freistühler. Godunov variables in relativistic fluid dynamics. arXiv:1706.06673.

[3] H. Freistühler and B. Temple. Causal dissipation in the relativistic dynamics of barotropic fluids. *J. Math. Phys.*, to appear.

[4] S. K. Godunov. An interesting class of quasilinear systems. *Dokl. Akad. Nauk SSSR*, 139:521–523, 1961.

[5] T. J. R. Hughes, T. Kato, and J. E. Marsden. Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Rational Mech. and Anal.*, 63(3):273–294, 1977.

[6] S. Kawashima. *Systems of a Hyperbolic-Parabolic Composite Type, with Applications of Magnetohydrodynamics.* PhD thesis, Kyoto University, 1983.

[7] S. Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* John Wiley & Sons, New York, 1972.