# Non-forking formulas in Distal NIP theories

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- Definable sets: B.C. of polynomial equalities.
- strongly minimal definable sets in one variable finite or cofinite

# The Universe

O-min

$\cdot(\mathbb{R}, +, \times, 0, 1, \exp)$

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Strongly Minimal

· Vector spaces

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# Stable theories

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**Stable**

· Modules
· Sep. closed fields

**Strongly Minimal**

·Vector spaces

$\cdot(\mathbb{C}, +, \times, 0, 1)$

**Definition**

*A formula $\phi(x, y)$ is stable if there do not exist $(a_i)_{i\in\omega}$, $(b_i)_{i\in\omega}$ such that*

$$M \models \phi(a_i, b_j) \text{ if and only if } i < j$$

# Indiscernible Sequences

## Definition

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**Example**

In $(\mathbb{Q}, <)$, see blackboard.

# Dividing formulas

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*A formula $\phi(x, b)$ divides over $A$ if there is an indiscernible sequence $(b_i)_{i \in \omega}$ with $b_o = b$ such that $\{\phi(x, b_i) : i \in \omega\}$ is inconsitent.*

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## Remark

*In $(\mathbb{C}, +, \times, 0, 1)$ diving formulas give rise to a notion of rank that corresponds to transcendence degree.*

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**NIP**

· ACVF

**Stable**

· Modules
· Sep. closed fields

**Strongly Minimal**

·Vector spaces

$\cdot(\mathbb{C}, +, \times, 0, 1)$

**O-min**

$\cdot(\mathbb{R}, +, \times, 0, 1, \exp)$

$\cdot(\mathbb{R}, +, \times, 0, 1)$

## Definition

*A formula $\phi(x, y)$ is NIP if there is no infinite set $A$ of $|x|$-tuples such that:*

$$(IP)_{\phi,A} \text{ for all } A_0 \subseteq A \ \exists b_{A_0} \text{ such that } \phi(A, b_{A_0}) = A_0.$$

# NIP theories

**Fact**

*Let $T$ be an NIP theory, $M \models T$, $\phi(x, y)$ formula then if $\mathcal{C} = \{\phi(x, b) : b \in M^{|y|}\}$, the set system $(M^{|x|}, \mathcal{C})$ has finite (Vapnik – Chervonenkis) VC-dimension.*

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**Theorem ( The $(p,q)$-theorem *(Alon-Kleitman, Matousek)*)**

*Let $p \geq q$ be integers. Then there is an $N \in \mathbb{Z}$ such that the following holds:*
*Let $(X, \mathcal{S})$ be set system where every $S \in \mathcal{S}$ is non-empty.*
*Assume:*

- $VC^*(S) < q$;
- *For every $p$ sets of $\mathcal{S}$, some $q$ have non-empty intersection.*

*Then there is a subset of $X$ of size $N$ which intersects every element of $S$.*

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**Corollary (Chernikov-Simon)**

*$T$ NIP. Suppose $\phi(x, b)$ does not divide over $M$ then we can find a formula $\psi(y) \in tp(b/M)$ and a finite partition $\{W_j\}_{j=1}^{n}$ of the set $\psi(M)$ such that $\{\phi(x, b_i) : b_i \in W_i\}$ is consistent.*

**Conjecture (Definable $(p, q)$ - conjecture)**

*$T$ NIP. The finite partition is not needed,*
*i.e. Suppose $\phi(x, b) \in tp(b/M)$ does not divide over $M$ then we can find a formula $\psi(y)$ such that $\{\phi(x, b) : b \models \psi(y)\}$ is consistent.*

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**Fact**

*Definable $(p, q)$ - conjecture holds in Stable theories.*

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· ACVF

**Distal**

- $\mathbb{Q}_p$
- Transerries

**O-min**

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**Stable**

- Modules
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**Strongly Minimal**

- Vector spaces
- $(\mathbb{C}, +, \times, 0, 1)$

**Definition**

*A theory $T$ is distal if, for any small indiscernible sequence of the form $I + \{b\} + J$ in $M$, and any small $A \subseteq M$, if $I + J$ is indiscernible over $A$ then $I + \{b\} + J$ is indiscernible over $A$.*

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*Graphs definable in a distal structure have the strong Erdös-Hajnal property.*

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*A pair of subsets $A \subseteq M^m$, $B \subseteq M^n$ is called $R$-homogeneous if either $A \times B \subset R$ or $A \times B \cap R = \emptyset$.*
*We say a relation $R$ satisifies the strong Erdös -Hajnal property if there is a constant $\delta = \delta(R) > 0$ such that for any finite subsets $A, B$ there are $A_0 \subset A$ and $B_0 \subset B$ with $|A_0| > \delta|A|$ $|B_0| > \delta|B|$ such that $(A_0, B_0)$ is $R$ homogeneous.*

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**Question**

*Does the Definable $(p, q)$ - conjecture holds in NIP theories?*

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*Distal theories are not closed under reducts (e.g. $(M, =)$ is not distal).*

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*Let $M \models T$ we define the Shelah expansion of $M$, $M^{Sh}$ to be the expansion to the language $L_{Sh(M)}$ containing for each externally definable (i.e. definable in an elementary extension of $M$) $D \subset M^k$ a $k$-ary predicate $R_D(x)$.*

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## Theorem (Boxall-K)

*If $M^{Sh}$ is distal if and only if $M$ is distal (and thus any inbetween expansions is distal).*

# Thank you