Heights of pre-special points of Shimura varieties

Christopher Daw[1] Martin Orr[2]

[1]Institut des Hautes Études Scientifiques

[2]University College London

Workshop: O-Minimality and Applications, Konstanz 2015

Outline

Shimura Varieties

- Let G be an algebraic group over $\mathbb{Q}$ (semisimple, adjoint)
- Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ i.e. $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$
- Let $h : \mathbb{S} \to G_\mathbb{R}$ (satisfying three properties)

- Let X denote the conjugacy class of h under $G(\mathbb{R})^+$
- Let Γ be a congruence subgroup of $G(\mathbb{Q})^+$

- X is a complex manifold (hermitian, symmetric)
- $\Gamma\backslash X$ is a **Shimura variety** (quasi-projective, algebraic)

Special Subvarieties

- Let $x \in X$ and $H := \mathrm{MT}(x)$ the Mumford-Tate group
- i.e. $H \subseteq G$ is the smallest $\mathbb{Q}$-group such that $x(\mathbb{S}) \subseteq H_{\mathbb{R}}$
- Let X_H denote the conjugacy class of x under $H(\mathbb{R})^+$
- Let Γ_H be a congruence subgroup of $H(\mathbb{Q})^+$ contained in Γ
- $\Gamma_H \backslash X_H$ is a Shimura variety
- The morphism $\Gamma_H \backslash X_H \to \Gamma \backslash X$ is algebraic
- The image of $\Gamma_H \backslash X_H$ is called a **special subvariety**
- It is a **point** if and only if H is a torus (commutative)

André-Oort

Conjecture (André-Oort)

- *Let S be a Shimura variety*
- *Let Σ be a set of special points contained in S*
- *Let $\overline{\Sigma}$ denote the Zariski closure of Σ in S*
- *Let Z denote an irreducible component of $\overline{\Sigma}$*

Then Z is a special subvariety.

- Original proof under GRH by Klingler-Ullmo-Yafaev (2014)
- Unconditional proof for $\mathcal{A}_g$ by Pila, Tsimerman et al.
- Proof follows the so-called Pila-Zannier strategy

Definability

Theorem (Peterzil-Starchenko, Klingler-Ullmo-Yafaev)

- *Let π denote the uniformising map $X \to \Gamma\backslash X$*
- *Let $\mathcal{F}$ be a semi-algebraic fundamental set in X for Γ*

Then $\pi_{|\mathcal{F}}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$.

- Case of $\mathcal{A}_g$ due to Peterzil-Starchenko (2010)
- General case due to Klingler-Ullmo-Yafaev

Pila-Wilkie

Denote by $\mathcal{Z}$ the definable set $\pi^{-1}(Z) \cap \mathcal{F}$

Theorem (Pila-Wilkie)

- *Let $A \subseteq \mathbb{R}^n$ be a set definable in an o-minimal structure*
- *Let A^{alg} denote the union of the connected positive-dimensional semi-algebraic subsets of A*
- *Let $k \in \mathbb{N}$ and $\epsilon > 0$*

For all $T \geq 1$

$$|\{x \in A \setminus A^{\mathrm{alg}} : [\mathbb{Q}(x) : \mathbb{Q}] \leq k,\ H(x) \leq T\}| \ll_\epsilon T^\epsilon.$$

Ax-Lindemann-Weierstrass

The question is: *What is $\mathcal{Z}^{\mathrm{alg}}$?*

Theorem (Ax-Lindemann-Weierstrass)

Let Θ denote the set of positive-dimensional (weakly) special subvarieties contained in Z. Then

$$\mathcal{Z}^{\mathrm{alg}} = \bigcup_{V \in \Theta} \pi^{-1}(V) \cap \mathcal{F}.$$

- The compact case due to Ullmo-Yafaev
- Then the case of $\mathcal{A}_g$ due to Pila-Tsimerman (2014)
- General case due to Klingler-Ullmo-Yafaev
- All use o-minimality (in $G(\mathbb{Q})$ rather than X)

Theorem of Ullmo

Theorem (Ullmo)

- *Let S be a Shimura variety*
- *Let Z be a (Hodge-generic)* **proper** *subvariety of S*
- *If $S = S_1 \times S_2$ assume that Z is not of the form $S_1 \times Z_2$*

Then the set of positive-dimensional (weakly) special subvarieties contained in Z is not Zariski dense in Z.

- **Using Pila-Wilkie show that $\pi^{-1}(\Sigma) \cap (\mathcal{Z} \setminus \mathcal{Z}^{\mathrm{alg}})$ is finite**
- Ax-Lindemann-Weierstrass $\implies$ all but finitely many points in Σ belong to a positive-dimensional special subvariety contained in Z
- Ullmo $\implies$ Z is equal to S

Hodge Structures

- Choose a faithful representation $G \to \mathrm{GL}(V)$
- Choose a lattice $V_{\mathbb{Z}}$ in V
- For each $x \in X$ we obtain a $\mathbb{Z}$-Hodge structure V_x on $V_{\mathbb{Z}}$
- $\mathrm{End}_{\mathbb{Z}-\mathrm{HS}}(V_x) := \mathrm{End}_{\mathbb{Z}}(V_{\mathbb{Z}})^{\mathrm{MT}(x)}$
- $R_x := Z(\mathrm{End}_{\mathbb{Z}-\mathrm{HS}}(V_x))$
- $D_x := |\mathrm{disc}(R_x)|$

If V_x corresponds to an Abelian variety A_x then

$$\mathrm{End}_{\mathbb{Z}-\mathrm{HS}}(V_x) = \mathrm{End}(A_x).$$

Main Result

Theorem (D-Orr, Pila-Tsimerman)

Let S be a Shimura variety with the preceding notations. There exist positive constants C_1 and C_2 and an integer k such that for any pre-image $x \in \mathcal{F}$ of a special point, x has algebraic co-ordinates of degree at most k and

$$H(x) \leq C_1 D_x^{C_2}.$$

- Case of $\mathcal{A}_g$ due to Pila-Tsimerman (2013)

Galois Orbits

- S has a canonical model over a number field E
- Special subvarieties are defined over finite extensions

Conjecture (Edixhoven)

Let S be a Shimura variety with the preceding notations. There exists a positive constant C_3 such that for any special point $s \in S$,

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/E) \cdot s| \gg D_x^{C_3}.$$

- Known under the GRH by Ullmo-Yafaev (2015)
- Case of $\mathcal{A}_g$ recently announced by Tsimerman follows from Masser-Wüstholz and the averaged Colmez formula due to Andreatta-Goren-Howard-Madapusi Pera and Yuan-Zhang

Combing Heights and Galois Orbits for Finiteness

- Choose $x_0 \in \pi^{-1}(\Sigma) \cap (\mathcal{Z} \setminus \mathcal{Z}^{\mathrm{alg}})$
- Fix $\epsilon > 0$ and apply Pila-Wilkie to $\mathcal{Z}$ with $T = C_1 D_{x_0}^{C_2}$
- $A := |\{x \in \pi^{-1}(\Sigma) \cap (\mathcal{Z} \setminus \mathcal{Z}^{\mathrm{alg}}) : H(x) \leq C_1 D_{x_0}^{C_2}\}| \ll D_{x_0}^{C_2 \epsilon}$
- However, for all Galois conjugates x of x_0, $D_x = D_{x_0}$
- $\implies A \gg D_{x_0}^{C_3}$
- $\implies D_{x_0}$ is bounded on $\pi^{-1}(\Sigma) \cap (\mathcal{Z} \setminus \mathcal{Z}^{\mathrm{alg}})$
- $\implies \pi^{-1}(\Sigma) \cap (\mathcal{Z} \setminus \mathcal{Z}^{\mathrm{alg}})$ is finite