Normal distribution in the subanalytic setting

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2. Statement of the problem

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Motivation

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Parameterized integrals in the subanalytic setting:

- Comte, Lion, Rolin: $\int f(x,y)dy$
- Cluckers, Miller: $\int f(x,y)(log(g(x,y)))^n dy$
- Cluckers, Comte, Miller, Rolin, Servi: $\int e^{i\,g(x,y)} f(x,y)dy$
- Now: $\int e^{\frac{-y^2}{2t}} f(x,y)dy$

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Definition (Brownian Motion in $\mathbb{R}$)

An one dimensional stochastic process $(B_t)_{t\geq 0}$ is called Brownian Motion in $\mathbb{R}$ with start value z if it is characterised by the following facts:

- $B_0 = z$
- *Let $0 \leq s < t$. Then $B_t - B_s$ is normally distributed with expected value 0 and variance $t - s$.*
- *Let $n \geq 1$, $0 \leq t_0 < t_1 < \ldots < t_n$. Then $B_{t_0}, B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent random variables.*
- *Every path is almost surely continuous.*

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Definition (Brownian Motion in $\mathbb{R}^n$)

An n-dimensional stochastic process $(B_t = (B_t^1, \ldots, B_t^n))_{t \geq 0}$ is called Brownian Motion in $\mathbb{R}^n$ with start value $z \in \mathbb{R}^n$ if every stochastic process $(B_t^i)_{t \geq 0}$ is a Brownian Motion in $\mathbb{R}$ for $i \in \{1, \ldots, n\}$, the stochastic processes $B_t^1, \ldots, B_t^n$ are independent for every $t \geq 0$ and $B_0 = z$.

Let $A \subset \mathbb{R}^n$ be a borel set and let $z \in \mathbb{R}^n$ be the start value. The probability for $B_t \in A$ at time t is given by

$$P(B_t \in A) = \begin{cases} \delta_z(A), & t = 0, \\ \frac{1}{(2\pi t)^{\frac{n}{2}}} \int\limits_A e^{-\frac{|x-z|^2}{2t}} \, dx, & t > 0. \end{cases}$$

where

$$\delta_z(A) = \begin{cases} 1, & z \in A, \\ 0, & z \notin A. \end{cases}$$

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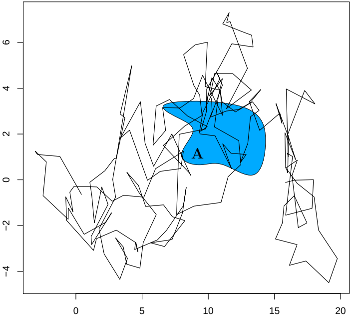
Motivation



- microscopic: wild
- macroscopic: tame if A is tame ?!

Statement of the problem

Let $A \subset \mathbb{R}^n \times \mathbb{R}^m$ be a semialgebraic set. Let

$$
\begin{aligned}
f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} &\longrightarrow [0,1] \\
(a, z, t) &\mapsto \begin{cases} \delta_z(A_a), & t = 0, \\ \frac{1}{(2\pi t)^{\frac{n}{2}}} \int\limits_{A_a} e^{\frac{-|x-z|^2}{2t}} dx, & t > 0. \end{cases}
\end{aligned}
$$

Question:

- Definability?
- Asymptotics?

Results for $\mathbb{R}$

Let $A \subset \mathbb{R}^n \times \mathbb{R}$ be definable in an o-minimal structure $\mathcal{M}$. The function

$$
\begin{aligned}
f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_{\geq 0} \quad &\longrightarrow \quad [0,1] \\
(a,z,t) \quad &\mapsto \quad P_z(B_t \in A_a) = \begin{cases} \delta_z(A_a), & t = 0 \\ \frac{1}{\sqrt{2\pi t}} \int\limits_{A_a} e^{\frac{-(x-z)^2}{2t}} dx, & t > 0 \end{cases}
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is definable in an expansion of the o-minimal structure $\mathcal{M}$.

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is definable in an expansion of the o-minimal structure $\mathcal{M}$.

$$\int_{C_a} e^{\frac{-(x-z)^2}{2t}} dx = \int_{\alpha(a)}^{\beta(a)} e^{\frac{-(x-z)^2}{2t}} dx$$

$$= \sqrt{2t}\frac{\sqrt{\pi}}{2}\left(\operatorname{erf}\left(\frac{\beta(a)-z}{\sqrt{2t}}\right)-\operatorname{erf}\left(\frac{\alpha(a)-z}{\sqrt{2t}}\right)\right)$$

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Theorem (Speissegger)

Suppose that $I \subseteq \mathbb{R}$ is an open interval, $a \in I$ and $g : I \to \mathbb{R}$ is definable in the Pfaffian closure $\mathcal{P}(\mathcal{M})$ and continuous. Then its antiderivative $F : I \to \mathbb{R}$ given by $F(x) := \int_a^x g(t)dt$ is also definable in $\mathcal{P}(\mathcal{M})$.

$$\int_{C_a} e^{\frac{-(x-z)^2}{2t}}\,dx = \int_{\alpha(a)}^{\beta(a)} e^{\frac{-(x-z)^2}{2t}}\,dx$$

$$= \sqrt{2t}\frac{\sqrt{\pi}}{2}\left(\operatorname{erf}\left(\frac{\beta(a)-z}{\sqrt{2t}}\right)-\operatorname{erf}\left(\frac{\alpha(a)-z}{\sqrt{2t}}\right)\right)$$

By Speissegger $f(a,z,t)$ is definable in $\mathcal{P}(\mathcal{M})$.

Results for higher dimensions

Let $A \subset \mathbb{R}^n \times \mathbb{R}^2$ be a globally subanalytic set. We assume that A_a is uniformly bounded and the start value z is 0. Then the function

$$\begin{aligned} f : \mathbb{R}^n \times \mathbb{R}_{\geq 0} &\longrightarrow [0,1] \\ (a,t) &\mapsto P_0(B_t \in A_a) = \frac{1}{2\pi t} \int_{A_a} e^{\frac{-|x|^2}{2t}} dx, t > 0 \end{aligned}$$

a) is definable in $\mathbb{R}_{\mathrm{an}}$ for $t \to \infty$ (by Comte, Lion, Rolin).

b) For $t \to 0$ we can establish an asymptotic expansion

$$f \sim \sum_{n=0}^{\infty} d_n(a) t^{\frac{n}{2q}},$$

where $d_n(a)$ is globally subanalytic for all $n \in \mathbb{N}_0$.

Sketch of the proof in the case without parameters:
With polar coordinate transformation and cell decomposition

$$\frac{1}{2\pi t}\int_C e^{-\frac{r^2}{2t}}\, r\; d(r,\varphi) \;=\; \frac{1}{2\pi t}\int_{r=\alpha}^{\beta}\int_{\varphi=\eta(r)}^{\psi(r)} e^{-\frac{r^2}{2t}}\, r\; d\varphi\; dr$$

$$= \frac{1}{2\pi t}\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}}\, r\; \psi(r)\; dr$$

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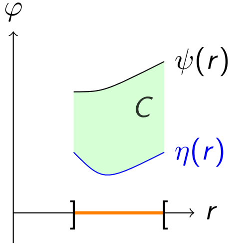

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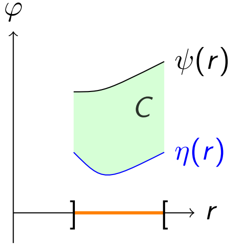

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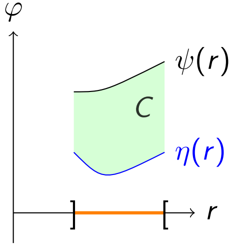

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$$\frac{1}{2\pi t}\sum_{n=0}^{\infty} c_n \int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}}\, r\, r^{\frac{n}{q}}\, dr$$

- for $t \to 0$: $f(t) \sim \sum_{n=0}^{\infty} d_n t^{\frac{n}{2q}}$.

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Summary

- for $\mathbb{R}$: $f(a, z, t)$ is definable in the Pfaffian closure $\mathcal{P}(\mathcal{M})$
- for higher dimensions with A_a uniformly bounded and start value $z = 0$:
  - $f(a, t)$ is definable in $\mathbb{R}_{\mathrm{an}}$ for $t \to \infty$
  - $f(a, t) \sim \sum_{n=0}^{\infty} d_n(a) t^{\frac{n}{2q}}$ for $t \to 0$