Real minimal Hahn fields have the independence property

Universität Konstanz

Lasse Vogel
joint work in progress with
Lothar Sebastian Krapp, Salma Kuhlmann

Abstract Shelah's conjecture for NIP fields suggests that every infinite NIP field should be separably closed, real closed or admit a non-trivial henselian valuation. Notably that would mean that NIP fields are 'very close' to being algebraically closed. A reasonable approach in the pursuit of this conjecture is therefore to show that fields that are 'far' from being algebraically closed necessarily have the independence property. In an earlier work, I could prove that all purely transcendental extensions of real fields have the independence property.This class of fields consists exactly of all fields that can be expressed as a rational function field over some real field K. The aim of this work is now to generalize this result to real minimal Hahn fields $K(G)$, where G is some arbitrary ordered abelian group. Note that rational function fields can be viewed as the minimal Hahn fields where $G = \mathbb{Z}$.

Independence on real rational function fields

Proposition 1. *Let K be a real field. Then $K(X)$ has the independence property.*

The crucial ingredients in my original proof were on the one hand that every positive-semidefinite rational function over $\mathbb{Q}$ can be expressed as a sum of at most five squares, on the other hand the simple fact that a sum of squares can never be negative in any ordering on a real field. Then the formula

$$\varphi(x; y) := \exists z, u_1, u_2, u_3, u_4, u_5 \colon \left(xz = y \wedge \sum_{i=1}^{5} u_i^2 = 1 - z^2 \right)$$

was shown to witness the independence property through the use of $p_n := \frac{X-n}{(n+1)(X^2+1)}$ for an $n \in \mathbb{N}$ and $b_I := \prod_{n\in I} b_n$ for any finite $I \subsetneq \mathbb{N}$. Note that, since $\mathbb{Q}$ embeds into any real field K, we can consider the p_n, b_I as element of any real rational function field $K(X)$.

For all real fields K it is $K(X) \models \varphi(p_n; b_I) \Leftrightarrow n \in I$ because if $n \in I$, then z must be $b_{I\setminus\{n\}}$, which is bounded by one and thus $1 - z^2$ is positive semidefinite. Hence we can find the required u_i already in $\mathbb{Q}(X)$ and the existential formula is inherited. On the other hand, if $n \notin I$, then $z = \frac{b_I}{p_n}$ is unbounded on infinitely many points near the singularity n. Hence $1 - z^2$ is negative at infinitely many points and any potential u_i would all be defined at one of these points. Then evaluating in this point would yield a negative element of K expressed as a sum of five squares, which leads to a contradiction.

Real minimal Hahn fields

Let K be a real field and G an ordered abelian group. We define the ring $K[G]$ as the subring of the formal power series field $K((G))$ which consists of all power series with finite support. That means any $s \in K[G]$ can be described as the formal expression

$$s = \sum_{\gamma\in\Gamma_s} a_\gamma X^\gamma$$

for Γ_s : $\mathrm{supp}(s) \subsetneq G$ finite. The **minimal Hahn field** $K(G)$ is now defined as the fraction field of $K[G]$. Note that every element of $K(G)$ can be written as the fraction of two expressions as above. Furthermore, for $G = \mathbb{Z}$ this is exactly the field of rational functions over K. Since $\mathbb{Z}$ as group can be embedded into any ordered abelian group G, we can always consider $K(X)$ as a subfield of $K(G)$. This allows us to consider the p_n and b_I as elements of $K(G)$ and we immediately obtain $n \in I \Rightarrow K(G) \models \varphi(p_n; b_I)$.

Interpreting $K(G)$ as functions

It only remains to show that for $n \in \mathbb{N} \setminus I$ it follows $K(G) \not\models \varphi(p_n; b_I)$. The crucial ingredient used in the case of the rational function field $K(X)$ is that its elements can be easily interpreted as functions over K. Furthermore, they are compatible with addition and multiplication as pointwise operation wherever they are defined and only have finitely many singularities.

In this more general setting now even interpreting the elements of $K(G)$ as functions may not be possible. Consider the example of $\mathbb{Q}(\mathbb{Q})$, this field contains $X^{\frac{1}{2}}$. But if we now naturally let X denote the identity on K, then $X^{\frac{1}{2}}$ would map every element $a \in \mathbb{Q}$ to a square root of a. This is obviously impossible, on one hand for example 2 has no square root in $\mathbb{Q}$, on the other hand -1 can not have a square root in any real extension of $\mathbb{Q}$.

In this general setting, we must therefore put in a lot more work to obtain the required argument. Since we can never avoid the issue of -1 not being a valid input to evaluate $X^{\frac{1}{2}}$, we will only try to consider elements of $K(G)$ as functions defined on the positive elements of K with under a fixed ordering. We will first develop the main idea in a restricted setting where we assume that both K and G admit an archimedean ordering.

The archimedean case

We now consider an archimedean ordered group G and a real field K that admits an archimedean ordering. Then both K and G can (independently) be embedded into $\mathbb{R}$ as ordered abelian group or real field respectively. We can now make use of those two embeddings to consider $K(G)$ as a subfield of $\mathbb{R}(\mathbb{R})$. We obtain $\mathbb{Q}(\mathbb{Z}) \subseteq K(G) \subseteq \mathbb{R}(\mathbb{R})$.

Proposition 2. *Every field L with $\mathbb{Q}(\mathbb{Z}) \subseteq L \subseteq \mathbb{R}(\mathbb{R})$ has the independence property.*

We define an embedding from $\mathbb{R}[\mathbb{R}]$ into the ring of maps from $\mathbb{R}_+$ to $\mathbb{R}$ with pointwise addition and multiplication. Consider an $s \in \mathbb{R}[\mathbb{R}]$ with the form

$$s = \sum_{r\in\mathrm{supp}\ s} a_r X^r$$

for some $r \in \mathbb{R}$, $a_r \in \mathbb{R}^\times$. We define the function

$$f_s \colon x \mapsto \sum_{r\in\mathrm{supp}\ s} a_r \exp(r \log(x)).$$

One can now easily confirm that the map $s \mapsto f_s$ is indeed a ring embedding. To extend this map from $\mathbb{R}[\mathbb{R}]$ to $\mathbb{R}(\mathbb{R})$ we now only need to define $f_{1/s}$ for an s as above. We can easily define $f_{1/s}(x) := \frac{1}{f_s(x)}$ for all $x \in \mathbb{R}_+$ where $f_s(x) \neq 0$, but must leave it undefined for all roots of f_s. If no confusion is likely to arise, we will write $s(x)$ for $f_s(x)$.

The last ingredient we now need is that every f_s has only finitely many roots. They are exactly the objects which are called **generalized polynomials** in [4], hence we obtain from [4, Corollary 3.2] that for all $s \in \mathbb{R}[\mathbb{R}]$ it is

$$|\{x \in \mathbb{R}_+ \mid f_s(x) = 0\}| < |\mathrm{supp}\ s|.$$

In fact, this result is strong enough that we can express it as a first order axiom scheme in the structure $(\mathbb{R}, +, \cdot, \exp)$. This will be very useful when generalizing to non-archimedean fields and groups. For now, we have all we need to see that for any finite $I \subsetneq \mathbb{N}$ and $n \in \mathbb{N} \setminus I$ we obtain $\mathbb{R}(\mathbb{R}) \not\models \varphi(p_n; b_I)$. Otherwise $1 - (\frac{b_I}{p_n})^2$ is negative at infinitely many points $x \in \mathbb{R}_+$, but can be expressed as a sum of squares $\sum_{i=1}^{5} u_i^2$. Then, since all u_i are defined at all but finitely many points in $\mathbb{R}_+$, we find at least one $x_0 \in \mathbb{R}_+$ where $0 > (1 - (\frac{b_I}{p_n})^2)(x_0) = \sum_{i=1}^{5}(u_i(x_0))^2 \geq 0$. Contradiction.

The general case

To now handle the general case in which we do not assume the field and group to be archimedean we require a larger structure to embed them in. Notably that larger structure must accommodate an exponential function to allow the embedding into the ring of functions as above. To this end we will consider the surreal numbers **No**. Notably they admit an addition, multiplication and ordering under which they act as an ordered field. Furthermore, every ordered field K and ordered abelian group G can be embedded into **No**,since by [2, Theorem 19, Theorem 9] their real closure or divisible closure respectively is isomorphic to a substructure of **No**. By [3, Chapter 10] they even admit an exponential function. Even better, by [1, Corollary 2.2] the structure $(\mathbf{No}, +, \cdot, \exp)$ is elementarily equivalent to $(\mathbb{R}, +, \cdot, \exp)$. This allows us to transfer the result of [4, Corollary 3.2] to generalized polynomials with surreal coefficients and exponents, since as mentioned earlier this result can be expressed as a first order axiom scheme.

With this said, it seems all conditions for the argument that $K(G) \not\models \varphi(p_n; b_I)$ if $n \in \mathbb{N} \setminus I$ are in place. There is however one caveat: The surreal numbers form a proper class and not a set. We can with a little work however choose an ordering on K and construct an ordered field extensions L/K such that we can interpret all elements of $K(G)$ as functions from K_+ to L.

We consider the images under all f_s for an $s \in K[G]$ of any $x \in K_+$. Those are at most $(|K|^2|G|)$-many. We obtain L by adjoining all these points to K. We now made sure that every element of $K(G)$ can be interpreted as a function from all but finitely many points of K_+ to L and the addition and multiplication on $K(G)$ are compatible with pointwise addition and multiplication outside of those singularities. Furthermore every $1 - (\frac{b_I}{p_n})^2$ is negative at infinitely many points of K_+ for some ordering on L whenever $n \in \mathbb{N} \setminus I$, as we can already find infinitely many such points in $\mathbb{Q}$. Thus if we now assume that $K(G) \models \varphi(p_n; b_I)$, then we find $u_1, \ldots, u_5$ and $x_0 \in K_+$ such that all u_i are defined at x_0 and

$$0 > (1 - (\frac{b_I}{p_n})^2)(x_0) = \sum_{i=1}^{5}(u_i(x_0))^2 \geq 0.$$

This is a contradiction and thus implies $K(G) \not\models \varphi(p_n; b_I)$. We therefore proved:

Theorem. *Let K be a real field and G an ordered abelian group. Then $K(G)$ has the independence property.*

References

[1] L. v.d.Dries and P. Ehrlich, 'Fields of surreal numbers and exponentiation', *Fund. Math.* 167 (2001) 173–188.

[2] P. Ehrlich, 'Number Systems with Simplicity Hierarchies: A Generalization of Conway's Theory of Surreal Numbers', *J. Symb. Log.* 66 (2001) 1231–1258.

[3] H. Gonshor, *An Introduction to the Theory of Surreal Numbers*, London Math. Soc. Lecture Note Ser. (Cambridge Univ. Press, Cambridge, 1986).

[4] G.J.O. Jameson, 'Counting zeros of generalized polynomials: Descartes' rule of signs and Laguerre's extensions', *Math. Gaz.* 90 (2006) 223–234.

Contact

Fachbereich Mathematik und Statistik
Universität Konstanz
78457 Konstanz, Germany
lasse.vogel@uni-konstanz.de

Acknowledgement

This work is part of my doctoral research project, which started in 2022 and is supervised by Professor Salma Kuhlmann and PD Dr. Lothar Sebastian Krapp at Universität Konstanz.