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Sommersemester 2022

Tutorials for 'Real Closed Fields and Integer Parts'

Exercise Sheet 1: Structures, Terms and Formulas

General Note: All statements must always be proven. The bonus exercise is voluntary and will be awarded extra points.

Exercise 1.1 (Structures; cf. [Lecture Notes, Exercise 2.1.7])
Consider the set $B = \{0, 1\}$ and assume that $0 \neq 1$.

(i) For each language $\mathcal{L}$ in Definition 2.1.6, i.e. for each

$$\mathcal{L} \in \{\mathcal{L}_<, \mathcal{L}_{\mathrm{admon}}, \mathcal{L}_{\mathrm{mon}}, \mathcal{L}_{\mathrm{g}}, \mathcal{L}_{\mathrm{mgp}}, \mathcal{L}_{\mathrm{semr}}, \mathcal{L}_{\mathrm{r}}, \mathcal{L}_{\mathrm{og}}, \mathcal{L}_{\mathrm{or}}, \mathcal{L}_{\mathrm{exp}}\},$$

how many different interpretations on B are there to obtain an $\mathcal{L}$-structure having B as domain? Justify your answer.

(ii) Find all $\mathcal{L}_<$-structures with domain B that are strict linear orders (see Definition E.1.2 (1)). How many of them are there up to isomorphism (which means counting isomorphic structures as the same)? Justify your answer.

(iii) Find all $\mathcal{L}_{\mathrm{semr}}$-structures with domain B that are semirings (see Definition E.1.1).

(iv) Find all $\mathcal{L}_{\mathrm{r}}$-structures with domain B that are fields.

(v) Is there an $\mathcal{L}_{\mathrm{or}}$-structure with domain B that is an ordered field (see Definition E.1.2)? Justify your answer.

Definition E.1.1. A **monoid** is an $\mathcal{L}_{\mathrm{mon}}$-structure $(M, \cdot, 1)$ fulfilling the axioms

- $\forall x, y, z \ (x \cdot y) \cdot z = x \cdot (y \cdot z)$ (associativity of $\cdot$),
- $\forall x \ (x \cdot 1 = x \wedge 1 \cdot x = x)$ (1 is a neutral element of $\cdot$).

A monoid $(M, \cdot, 1)$ is called **commutative** if it additionally fulfills the axiom

$$\forall x, y \ x \cdot y = y \cdot x \quad \text{(commutativity of } \cdot\text{)}.$$

A **semiring** is an $\mathcal{L}_{\mathrm{semr}}$-structure $(S, +, \cdot, 0, 1)$ such that

(1) its additive part $(S, +, 0)$ is a commutative monoid;

(2) its multiplicative part $(S, \cdot, 1)$ is a monoid;

(3) multiplication distributes over addition from both sides;

(4) 0 annihilates S, i.e. $0 \cdot s = s \cdot 0 = 0$ for any $s \in S$.

Definition E.1.2. An **ordered field** is an $\mathcal{L}_{\mathrm{or}}$-structure $(K, +, -, \cdot, 0, 1, <)$ such that

(1) the $\mathcal{L}_<$-structure $(K, <)$ is a **strict linear order**, i.e. $(K, <)$ fulfills the axioms

- $\forall x\ \neg x < x$ (irreflexivity or anti-reflexivity),
- $\forall x, y, z\ ((x < y \wedge y < z) \rightarrow x < z)$ (transitivity),
- $\forall x, y, z\ (x < y \vee x = y \vee x > y)$ (trichotomy);

(2) the $\mathcal{L}_{\mathrm{r}}$-structure $(K, +, -, \cdot, 0, 1)$ is a field;

(3) the ordering $<$ is compatible with addition and multiplication, i.e. $(K, +, -, \cdot, 0, 1, <)$ fulfills the axioms

- $\forall x, y, z\ (x < y \rightarrow x + z < y + z)$ (compatibility of $<$ and $+$),
- $\forall x, y\ ((0 < x \wedge 0 < y) \rightarrow 0 < x \cdot y)$ (compatibility of $<$ and $\cdot$).

Exercise 1.2 (Homomorphisms and Expansion; cf. [Lecture Notes, Exercise 2.1.12])
Consider the $\mathcal{L}_{\mathrm{r}}$-structure $\mathcal{M} = (M, +^{\mathcal{M}}, -^{\mathcal{M}}, \cdot^{\mathcal{M}}, 0^{\mathcal{M}}, 1^{\mathcal{M}})$ defined as follows:

- $M := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\} \subseteq \mathbb{R}^{2\times 2}$.
- $+^{\mathcal{M}}$, $-^{\mathcal{M}}$ and $\cdot^{\mathcal{M}}$ are standard addition, subtraction and multiplication of matrices.
- $0^{\mathcal{M}} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $1^{\mathcal{M}} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Show that $\mathcal{M} \cong \mathbb{C}_{\mathrm{r}}$ by finding a suitable $\mathcal{L}_{\mathrm{r}}$-isomorphism, and deduce that $\mathcal{M}$ is a field. Can the $\mathcal{L}_{\mathrm{r}}$-structure $\mathcal{M}$ be expanded to an $\mathcal{L}_{\mathrm{or}}$-structure that is an ordered field (see Definition E.1.2)? Justify your answer.

Exercise 1.3 (Terms & Formulas)

(a) Find three different $\mathcal{L}_{\mathrm{r}}$-terms t with interpretation $t^{\mathbb{Z}} \colon \mathbb{Z} \rightarrow \mathbb{Z}, n \mapsto 2n$ (cf. [Lecture Notes, Exercise 2.2.7]).

(b) Proceed by induction on the construction of terms to show that the interpretation $t^{\mathbb{R}}$ of any $\mathcal{L}_{\mathrm{r}}$-term t is a polynomial function on $\mathbb{R}$, i.e. a map of the form

$$\mathbb{R}^n \rightarrow \mathbb{R},\ \underline{a} \mapsto p(\underline{a})$$

for some $n \in \mathbb{N}$ and some polynomial $p \in \mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \ldots, X_n]$. Is the converse also true, i.e. can every polynomial function on $\mathbb{R}$ be expressed as interpretation $t^{\mathbb{R}}$ of some $\mathcal{L}_{\mathrm{r}}$-term t? Justify your answer.
(Hint: Apply Proposition 2.2.8 to answer the question.)

(c) Show that $\mathbb{Z}_< \models \forall x \exists y, z\ (y < x \wedge x < z)$ (cf. [Lecture Notes, Exercise 2.2.19]).
(Note: This $\mathcal{L}_<$-sentence says that the standard linear ordering on $\mathbb{Z}$ has no endpoints.)

(d) Consider the structure ω_{semr} and specify* a $\mathcal{L}_{\mathrm{semr}}$-formula

(i) $\beta(x)$ with one free variable such that for any $n \in \omega$,

$$\omega_{\mathrm{semr}} \models \beta(n) \text{ if and only if } n = 2;$$

(ii) $\kappa(x, y)$ with two free variables such that for any $n, m \in \omega$,

$$\omega_{\mathrm{semr}} \models \kappa(n, m) \text{ if and only if } n < m;$$

(iii) $\delta(x, y)$ with two free variables such that for any $n, m \in \omega$,

$$\omega_{\mathrm{semr}} \models \delta(n, m) \text{ if and only if } n \mid m;$$

(iv) $\psi(x)$ with one free variable such that for any $n \in \omega$,

$$\omega_{\mathrm{semr}} \models \psi(n) \text{ if and only if } n \text{ is a prime number};$$

(v) φ with no free variables (i.e. a $\mathcal{L}_{\mathrm{semr}}$-sentence) expressing that there are infinitely many pairs of twin primes (i.e. pairs of prime numbers that differ by 2).

*) It is not necessary to prove that the specified formula fulfills the required condition. Abbreviations may be used if their definition is provided.

Bonus Exercise

Consider the $\mathcal{L}_{\mathrm{r}}$-formula $\varphi(x)$ given by $\exists y \ (y \cdot y) + 1 = x$.

(i) For which values of a do we have $\mathbb{R}_{\mathrm{r}} \models \varphi(a)$?

(ii) For which values of a do we have $\mathbb{Z}_{\mathrm{r}} \models \varphi(a)$?

Please hand in your solutions by **Thursday, 21 April 2022, 11:45 (postbox 18 in F4).**