Prof. Dr. Salma Kuhlmann,
Dr. Lothar Sebastian Krapp,
Laura Wirth, Moritz Schick

Universität Konstanz
Fachbereich Mathematik und Statistik
Sommersemester 2022

Tutorials for 'Real Closed Fields and Integer Parts'

Exercise Sheet 2: Definability and Preservation

General Note: All statements must always be proven. The bonus exercise is voluntary and will be awarded extra points.

Exercise 2.1 ($\mathcal{L}_{\mathrm{mag}}$-Definability)

Consider the **language of magmas** $\mathcal{L}_{\mathrm{mag}} = \{\circ\}$, where $\circ$ denotes a binary function symbol.

(a) Let $\varphi(x, y, z)$ be the $\mathcal{L}_{\mathrm{mag}}$-formula $x \circ x = y \circ z$ and let $\psi(y, z)$ be the $\mathcal{L}_{\mathrm{mag}}$-formula $\exists x\ x \circ x = y \circ z$.

  (i) Determine the set that is defined by $\varphi(a, b, z)$ in the $\mathcal{L}_{\mathrm{mag}}$-structure $(\mathbb{Z}, +)$, where $a, b \in \mathbb{Z}$.

  (ii) Determine the set that is defined by $\varphi(a, b, z)$ in the $\mathcal{L}_{\mathrm{mag}}$-structure $(\mathbb{R}, \cdot)$, where $a, b \in \mathbb{R}$.

  (iii) Determine the subset of $\mathbb{R}^2$ that is defined by $\psi(y, z)$ in the $\mathcal{L}_{\mathrm{mag}}$-structure $(\mathbb{R}, \cdot)$.

  In some cases, you need to make a case distinctions depending on the parameters a and b.

(b) Show that the set of (positive and negative) primes is $\emptyset$-$\mathcal{L}_{\mathrm{mag}}$-definable in $(\mathbb{Z}, \cdot)$.

(c) Show that, given two primes $p, q \in \mathbb{Z}$, there exists an $\mathcal{L}_{\mathrm{mag}}$-automorphism $\sigma_{p,q}$ on $(\mathbb{Z}, \cdot)$ that interchanges p and q.
(Hint: Prime factorization!)

(d) Show that the set of odd integers is not $\emptyset$-$\mathcal{L}_{\mathrm{mag}}$-definable in $(\mathbb{Z}, \cdot)$.

(e) Show that, given $z \in \mathbb{Z}$, the singleton $\{z\}$ is $\emptyset$-$\mathcal{L}_{\mathrm{mag}}$-definable in $(\mathbb{Z}, \cdot)$ if and only if $z \in \{-1, 0, 1\}$.

(f) Show that the set $\{z \in \mathbb{Z} \mid z \geq 0\}$ is not $\mathcal{L}_{\mathrm{mag}}$-definable in $(\mathbb{Z}, \cdot)$.
(Hint: First prove that for any finite subset A of $\mathbb{Z}$ there exist primes $p, q \in \mathbb{Z}$ such that $p < 0 < q$ and $p \nmid a$ as well as $q \nmid a$ for any $a \in A$, and apply Proposition 2.2.21.)

Exercise 2.2 (Definable Operations and Orderings on Numbers)

(a) Show that the standard addition and multiplication on $\mathbb{R}$ are not $\emptyset$-$\mathcal{L}_{<}$-definable in $\mathbb{R}_{<}$.

(b) Show that the standard ordering $<$ on $\mathbb{Z}$ is not $\emptyset$-$\mathcal{L}_{\mathrm{g}}$-definable in $\mathbb{Z}_{\mathrm{g}}$, but the standard ordering $<$ on ω is $\emptyset$-$\mathcal{L}_{\mathrm{admon}}$-definable in ω_{admon}.

(c) Show that subtraction, i.e. the binary operation $-: (x, y) \mapsto x - y$, is an $\mathcal{L}_{\text{admon}}$-definable operation in $\mathbb{Z}_{\text{admon}}$ and $\mathbb{R}_{\text{admon}}$, but that standard multiplication is not $\emptyset$-$\mathcal{L}_{\text{admon}}$-definable in either structure.

(d) Show that the standard ordering $<$ on $\mathbb{R}$ is $\emptyset$-$\mathcal{L}_{\text{r}}$-definable in $\mathbb{R}_{\text{r}}$. Deduce that every subset of $\mathbb{R}^n$, where $n \in \mathbb{N}$, which is $\mathcal{L}_{\text{or}}$-definable in $\mathbb{R}_{\text{or}}$ is also $\mathcal{L}_{\text{r}}$-definable in $\mathbb{R}_{\text{r}}$. What about the converse?

Exercise 2.3 (Quantifier-Free Formulas & Preservation Laws)

(a) Write down a recursive definition of quantifier-free formulas in the style of Definition 2.2.9.

(b) Show that for any quantifier-free $\mathcal{L}_{\text{or}}$-formula $\varphi(\underline{x})$ and any $\underline{q} \in \mathbb{Q}$, we have

$$\mathbb{Q}_{\text{or}} \models \varphi(\underline{q}) \text{ if and only if } \mathbb{R}_{\text{or}} \models \varphi(\underline{q}).$$

(c) Find an $\mathcal{L}_{\text{or}}$-formula $\psi(\underline{x})$ and a $\underline{q} \in \mathbb{Q}$ such that $\mathbb{Q}_{\text{or}} \models \psi(\underline{q})$ but $\mathbb{R}_{\text{or}} \not\models \psi(\underline{q})$.

Bonus Exercise

(a) [Lecture Notes, Exercise 2.3.11]:

  (i) Let $\psi: \mathbb{R}_{\text{or}} \cong \mathbb{R}_{\text{or}}$. Show that $\psi = \text{id}_{\mathbb{R}}$.

  (ii) Show that there are only countably many distinct $\emptyset$-$\mathcal{L}_{\text{or}}$-definable subsets of $\mathbb{R}$.

  (iii) Deduce that there is a subset $A \subseteq \mathbb{R}$ that is not $\emptyset$-$\mathcal{L}_{\text{or}}$-definable but still preserved by all $\mathcal{L}_{\text{or}}$-automorphisms on $\mathbb{R}$.

(b) Show that the set of real numbers is not definable in the field of complex numbers, i.e. show that $\mathbb{R}$ is not $\mathcal{L}_{\text{r}}$-definable in $\mathbb{C}$.

Please hand in your solutions by **Thursday, 28 April 2022, 11:45 (postbox 18 in F4).**