Prof. Dr. Salma Kuhlmann,
Dr. Lothar Sebastian Krapp,
Laura Wirth, Moritz Schick

Universität Konstanz
Fachbereich Mathematik und Statistik
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Tutorials for 'Real Closed Fields and Integer Parts'
Exercise Sheet 3: Theories, Axiomatisations and Quantifier-Elimination

General Note: All statements must always be proven. The bonus exercise is voluntary and will be awarded extra points.

Exercise 3.1 (Theories and Models)

(a) Given a language $\mathcal{L}$ and a set Σ of $\mathcal{L}$-sentences, let $\mathrm{Mod}(\Sigma)$ be the class of $\mathcal{L}$-structures axiomatised by Σ. Show that

$$\mathrm{Th}(\mathrm{Mod}(\Sigma)) := \bigcap_{\mathcal{A}\in\mathrm{Mod}(\Sigma)} \mathrm{Th}(\mathcal{A})$$

is the deductive closure of Σ.

(b) Let $\mathcal{L}$ be a language and let $\mathcal{M}$ be an $\mathcal{L}$-structure. Show that $\mathrm{Th}(\mathcal{M})$ is a complete $\mathcal{L}$-theory (cf. [Lecture Notes, Exercise 2.5.4]).

(c) Let $\mathcal{L}$ be a language and let T be an $\mathcal{L}$-theory. Show that T is complete if and only if, whenever $\mathcal{A}$ and $\mathcal{B}$ are models of T, then $\mathcal{A} \equiv \mathcal{B}$.

(d) For each of the five structures $\omega_{\mathrm{semr}}, \mathbb{Z}_{\mathrm{semr}}, \mathbb{Q}_{\mathrm{semr}}, \mathbb{R}_{\mathrm{semr}}, \mathbb{C}_{\mathrm{semr}}$, write down an $\mathcal{L}_{\mathrm{semr}}$-sentence which is satisfied by that structure but not by any of the other four structures.

(e) Show that the $\mathcal{L}_{\exp}$-theory T_{rcef} of real closed exponential fields is not complete.

Exercise 3.2 (Quantifier-Elimination)

(a) Does the $\mathcal{L}_{\mathrm{or}}$-theory T_{of} of ordered fields admit quantifier-elimination? Justify your answer.
Hint: Consider an $\mathcal{L}_{\mathrm{or}}$-sentence φ such that $T_{\mathrm{of}} \not\models \varphi$.

(b) Does the $\mathcal{L}_{\mathrm{mag}}$-theory $\mathrm{Th}(\mathbb{Z}, \cdot)$ admit quantifier-elimination? Justify your answer.
Hint: Show that for any quantifier-free $\mathcal{L}_{\mathrm{mag}}$-formula $\varphi(\underline{x}, y)$ and any $\underline{a} \in \mathbb{Z}$ the set $\varphi(\underline{a}, (\mathbb{Z}, \cdot))$ is finite or cofinite in $\mathbb{Z}$.[1]

Exercise 3.3 (Axiomatisation and Elementary Equivalence)
Let $\mathcal{L}$ be a language and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures such that M is finite and $\mathcal{M} \equiv \mathcal{N}$. The aim of this exercise is to prove $\mathcal{M} \cong \mathcal{N}$ (cf. [Lecture Notes, Exercise 2.5.10]).

[1] A subset $C \subseteq \mathbb{Z}$ is called **cofinite** in $\mathbb{Z}$ if its complement $\mathbb{Z} \setminus C$ is finite.

(i) Denote by $\mathcal{L}_=$ the **empty language** containing no relation, function, or constant symbols. For any $n \in \mathbb{N}$, specify an $\mathcal{L}_=$-sentence $\varphi_{\geq n}$ axiomatising the class of sets that contain at least n elements and an $\mathcal{L}_=$-sentence $\varphi_{=n}$ axiomatising the class of sets that contain exactly n elements, i.e. the $\mathcal{L}_=$-sentences shall fulfill

$$\mathcal{A} \models \varphi_{\geq n} \text{ if and only if } |A| \geq n, \text{ and}$$
$$\mathcal{A} \models \varphi_{=n} \text{ if and only if } |A| = n, \text{ respectively,}$$

for any $\mathcal{L}_=$-structure $\mathcal{A} = (A)$.

(ii) Given $n \in \mathbb{N}$, show that the $\mathcal{L}_=$-sentence $\varphi_{=n}$ has exactly one model up to $\mathcal{L}_=$-isomorphism. Deduce that $|M| = |N|$.

(iii) Write $M = \{a_1, \ldots, a_n\}$ for some $n \in \mathbb{N}$ and let $f\colon M \to N$ be a bijection that is not an $\mathcal{L}$-isomorphism. Find a quantifier-free $\mathcal{L}$-formula $\varphi_f(x_1, \ldots, x_n)$ such that

$$\mathcal{M} \models \varphi_f(a_1, \ldots, a_n) \text{ but } \mathcal{N} \not\models \varphi_f(f(a_1), \ldots, f(a_n)).$$

(iv) Prove that $\mathcal{M} \cong \mathcal{N}$.

Bonus Exercise (Axiomatisation of ReCloFIP)

Consider the language $\mathcal{L}_{\text{orip}} = \mathcal{L}_{\text{or}} \cup \{Z\}$, where Z denotes a unary relation symbol, and the class

$$\mathcal{C}_{\text{rakuga}} := \Big\{\mathcal{K} = \big(K, +, -, \cdot, 0, 1, <, Z^{\mathcal{K}}\big) \;\Big|\; K_{\text{or}} \text{ is a real closed field and } Z(\mathcal{K}) \text{ is an integer part of } K\Big\}$$

of $\mathcal{L}_{\text{orip}}$-structures. Find an axiomatisation for $\mathcal{C}_{\text{rakuga}}$.

Definition E.3.1. An ordered ring $(R, +, -, \cdot, 0, 1, <)$ is called **discretely ordered** if 1 is the least positive element.

Definition E.3.2. Let K be an ordered field. An $\mathcal{L}_{\text{or}}$-substructure Z of K is an **integer part** of K if it is a discretely ordered ring and for any $a \in K$ there exists $z \in Z$ such that $z \leq a < z + 1$.

Please hand in your solutions by **Thursday, 5 May 2022, 11:45 (postbox 18 in F4).**