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Tutorials for 'Real Closed Fields and Integer Parts'

Exercise Sheet 4: Peano Arithmetic

General Note: All statements must always be proven. The bonus exercise is voluntary and will be awarded extra points.

Exercise 4.1 (Definability of the Non-Standard Part)
Consider the $\mathcal{L}_{\mathrm{semr}}$-substructure of the polynomial ring $\mathbb{Z}[X]$ with domain

$$\omega[X] = \left\{ p = \sum_{i=0}^{n} a_i X^i \in \mathbb{Z}[X] \;\middle|\; n \in \omega \text{ and } a_i \in \omega \text{ for all } i \in \{0, \ldots, n\} \right\}.$$

(i) Define a linear ordering $<$ on $\omega[X]$ such that ω is an initial segment of $(\omega[X], <)$ and the resulting $\mathcal{L}_{\mathrm{PA}}$-structure $\omega[X]_{\mathrm{PA}}$ satisfies the axioms (ii), (iii), (vii) and (xiii) from Definition 3.1.1.

(ii) Show that the $\mathcal{L}_{\mathrm{PA}}$-structure $\omega[X]_{\mathrm{PA}}$, equipped with the ordering from (i), is not a model of PA^- by proving that it does not satisfy axiom (viii) from Definition 3.1.1.

(iii) Show that the set $\omega[X] \setminus \omega$ of infinite elements is $\mathcal{L}_{\mathrm{PA}}$-definable in $\omega[X]_{\mathrm{PA}}$.

Exercise 4.2 (Cuts, Least Number Principle, Underspill and Overspill)

(a) Let $\mathcal{M} \models \mathrm{PA}^-$. Prove the following statements of Remark 3.1.7.(ii):

(i) The standard part of $\mathcal{M}$ is a cut of $\mathcal{M}$, which is proper if and only if $\mathcal{M}$ is a non-standard model.

(ii) The standard part of $\mathcal{M}$ is the smallest cut of $\mathcal{M}$.

(b) Let $\mathcal{M} \models \mathrm{PA}$ be non-standard.

(i) Let $\varphi(x)$ be an $\mathcal{L}_{\mathrm{PA}}$-formula. Suppose that for any non-standard element $a \in M$ there exists a non-standard element $b \in M$ such that $\mathcal{M} \models b < a \wedge \varphi(b)$. Show that there exists a standard element $c \in M$ such that $\mathcal{M} \models \varphi(c)$.

(ii) Let $\varphi(x)$ be an $\mathcal{L}_{\mathrm{PA}}$-formula. Suppose that for any standard element $a \in M$ there exists a standard element $b \in M$ such that $\mathcal{M} \models a \leq b \wedge \varphi(b)$. Show that for any non-standard element $c \in M$ there exists a non-standard element $d \in M$ such that $\mathcal{M} \models d < c \wedge \varphi(d)$.

(iii) Show that neither the standard part nor the non-standard part of $\mathcal{M}$ are $\mathcal{L}_{\mathrm{PA}}$-definable in $\mathcal{M}$ (cf. [Lecture Notes, Exercise 3.1.11]).

Exercise 4.3 (Models of PA^- and Discretely Ordered Rings)

(a) Let $\mathcal{M} \models \mathrm{PA}^-$. Define an $\mathcal{L}_{\mathrm{or}}$-structure $\mathcal{Z}_M = (Z_M, +, -, \cdot, 0, 1, <)$ fulfilling the following conditions:

- $\mathcal{Z}_M \models T_{\mathrm{dor}}$ (see Definition E.4.1),
- $\mathcal{M} \subseteq (Z_M, +, \cdot, 0, 1, <)$,
- $M = Z_M^{\geq 0} = \{z \in Z_M \mid z \geq 0\}$.

(b) Let $\mathcal{Z} \models T_{\mathrm{dor}}$. Define an $\mathcal{L}_{\mathrm{PA}}$-structure $\mathcal{M}_Z = (M_Z, +, \cdot, 0, 1, <)$ fulfilling the following conditions:

- $\mathcal{M}_Z \models \mathrm{PA}^-$,
- $\mathcal{M}_Z \subseteq Z_{\mathrm{PA}}$,
- for any $z \in Z$ there exists $m \in M_Z$ such that $m = z$ or $-m = z$.

(c) Show that the maps

$$\Phi\colon \ \mathrm{Mod}(\mathrm{PA}^-) \to \mathrm{Mod}(T_{\mathrm{dor}}), \ \mathcal{M} \mapsto \mathcal{Z}_M,$$
$$\Psi\colon \ \mathrm{Mod}(T_{\mathrm{dor}}) \to \mathrm{Mod}(\mathrm{PA}^-), \ \mathcal{Z} \mapsto \mathcal{M}_Z$$

are inverses of each other.[1] This correspondence justifies calling PA^- the theory of non-negative parts of discretely ordered rings.

Definition E.4.1. The $\mathcal{L}_{\mathrm{or}}$-theory T_{dor} of **discretely ordered rings** is axiomatised by the extension of T_{or} by the following axiom:

$\forall (x > 0)\ 1 \leq x.$

Bonus Exercise (Primes and Irreducibles)

(a) Show that $\mathrm{PA} \models \forall (x > 1) \exists y\ (\mathrm{irr}(y) \wedge y \mid x)$, i.e. PA implies that every $x > 1$ has an irreducible divisor. Deduce the statement of [Lecture Notes, Exercise 3.1.17].

(b) Consider the ring $R = \mathbb{Z}[X, Y, Z]/\langle XZ - Y^2 \rangle$, and for any $p \in \mathbb{Z}[X, Y, Z]$ denote by $\overline{p}$ the element $p + \langle XZ - Y^2 \rangle$ of R.

   (i) Show that the ring R can be discretely ordered with $\overline{0} < \overline{X} < \overline{Y} < \overline{Z}$.

   (ii) Deduce that the formulas $\mathrm{pr}(x)$ and $\mathrm{irr}(x)$ are not equivalent over PA^-, i.e. show that $\mathrm{PA}^- \not\models \forall x\ (\mathrm{pr}(x) \leftrightarrow \mathrm{irr}(x))$.

Please hand in your solutions by **Thursday, 12 May 2022, 11:45 (postbox 18 in F4).**

[1] Recall that, given a language $\mathcal{L}$ and a set Σ of $\mathcal{L}$-sentences, $\mathrm{Mod}(\Sigma)$ denotes the class of $\mathcal{L}$-structures axiomatised by Σ.