Prof. Dr. Salma Kuhlmann,
Dr. Lothar Sebastian Krapp,
Laura Wirth, Moritz Schick

Universität Konstanz
Fachbereich Mathematik und Statistik
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Tutorials for 'Real Closed Fields and Integer Parts'

Exercise Sheet 5: Peano Arithmetic, Open Induction and Real Algebra

General Note: All statements must always be proven. Exercises with references to the lecture notes may only be solved by using results that have been established in the lecture notes *prior* to the respective exercise in order to avoid circular arguments. The bonus exercise is voluntary and will be awarded extra points.

Exercise 5.1 (Irrationality of $\sqrt{2}$)

(a) Show that $\mathrm{PA} \models \neg\exists m \exists (n \neq 0)\ m^2 = 2n^2$ (cf. [Lecture Notes, Exercise 3.1.16]).

(b) Consider the ring $R = \mathbb{Z}[X,Y]/\langle X^2 - 2Y^2\rangle$. Show that R can be discretely ordered. Deduce that $\mathrm{PA}^- \not\models \neg\exists m \exists (n \neq 0)\ m^2 = 2n^2$.

Exercise 5.2 (Non-Standard Models of PA^-)

Consider the polynomial ring $\mathbb{Z}[X]$ and the binary relation $<$ on $\mathbb{Z}[X]$ defined by

$$p < q \;:\Leftrightarrow\; \mathrm{lcf}(q-p) > 0$$

for any $p, q \in \mathbb{Z}[X]$.[1]

(a) Show that $(\mathbb{Z}[X], <) \models T_{\mathrm{lo}}$.

(b) Consider the $\mathcal{L}_{\mathrm{PA}}$-structure $\mathbb{Z}[X]_{\mathrm{PA}}$, equipped with the usual operations on polynomials and the ordering defined above, and its $\mathcal{L}_{\mathrm{PA}}$-substructure $\mathcal{M}$ with domain

$$\mathbb{Z}[X]^{\geq 0} := \{p \in \mathbb{Z}[X] \mid p \geq 0\}.$$

(i) Show that $\mathcal{M} \models \mathrm{PA}^-$.
(ii) Show that $\mathcal{M} \not\models \mathrm{IOpen}$.
(iii) Deduce that $\mathcal{M}$ is a non-standard model of PA^-.
(iv) Show that the $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(x)$ given by

$$\exists (y \leq x)\ \neg(2 \mid y \vee 2 \mid y+1)$$

defines the non-standard part of $\mathcal{M}$ in $\mathcal{M}$.

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[1] Given a polynomial $p \in \mathbb{Z}[X]$, its leading coefficient is denoted by $\mathrm{lcf}(p)$, and we set $\mathrm{lcf}(0) = -\infty < 0$.

Exercise 5.3 (Real Fields)

(a) Let K be an algebraically closed field. Show that K is not real.

(b) Let $(K, <)$ be an ordered field, i.e. $(K, <) \models T_{\text{of}}$. Show that K is real (cf. [Lecture Notes, Exercise 4.1.3]).

(c) Use Zorn's Lemma to prove that any real field has a real closure (cf. [Lecture Notes, Exercise 4.1.14]).

(d) Let K be a real field and let $a \in K$ with $-a \notin \sum K^2$. Show that there is a binary relation $<$ on K such that $(K, <) \models T_{\text{of}}$ and $a > 0$ (cf. [Lecture Notes, Exercise 4.1.14]). (Hint/Note: You may use the statement of [Lecture Notes, Exercise 4.1.11] without proving it.)

Bonus Exercise (Ordered Fields and Positive Cones)

(a) Let K be a field.

(i) Let $<$ be a binary relation on K such that $(K, <) \models T_{\text{of}}$. Show that

$$P_< := \{x \in K \mid x > 0\}$$

is a positive cone of K.

(ii) Let P be a positive cone of K. Show that the binary relation $<_P$ on K, defined by

$$a <_P b \ :\Leftrightarrow \ b - a \in P$$

for any $a, b \in K$, fulfills $(K, <_P) \models T_{\text{of}}$.

(iii) Deduce that there is a bijective correspondence between the set of positive cones of K and the set of binary relations $<$ on K making $(K, <)$ an ordered field.

(b) Find a proper field extension of $\mathbb{R}$ that is real (cf. [Lecture Notes, Exercise 4.1.9]).

Definition E.5.1. Let K be a field. A subset $P \subseteq K^\times$ is called a **positive cone** of K if it fulfills the following conditions:

- $P + P \subseteq P$, where $P + P = \{a + b \mid a, b \in P\}$,
- $P \cdot P \subseteq P$, where $P \cdot P = \{a \cdot b \mid a, b \in P\}$,
- $a^2 \in P$ for any $a \in K^\times$,
- $-1 \notin P$,
- $-P \cup P = K^\times$, where $-P = \{-a \mid a \in P\}$.

Please hand in your solutions by **Thursday, 19 May 2022, 11:45 (postbox 18 in F4).**