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Tutorials for 'Real Closed Fields and Integer Parts'

Exercise Sheet 6: Real Closed Fields and Roots of Polynomials

General Note: All statements must always be proven. Exercises with references to the lecture notes may only be solved by using results that have been established in the lecture notes *prior* to the respective exercise in order to avoid circular arguments. The bonus exercise is voluntary and will be awarded extra points.

Exercise 6.1 (cf. [Lecture Notes, Exercise 4.1.11])
Let K be a real closed field and denote by $<$ the binary relation on K from Definition 4.1.10.

(a) Let $a \in K^\times$. Show that either $a \in K^2$ or $-a \in K^2$, where $K^2 := \{c^2 \mid c \in K\} \subseteq K$, and deduce that $K^2 = \sum K^2$.

(b) Show that $(K, <) \models T_{\mathrm{of}}$, i.e. $(K, <)$ is an ordered field.

(c) Let $\prec$ be a binary relation on K that does not coincide with $<$. Show that $(K, \prec) \not\models T_{\mathrm{of}}$. *(Hence, $<$ is the unique binary relation on K making $(K, <)$ an ordered field.)*

Exercise 6.2 (Bounding Roots & Sign Determination)
Let $(K, <)$ be an ordered field and for some $n \in \mathbb{N}$ let

$$p = \sum_{i=0}^{n} a_i X^i \in K[X]$$

with $a_n \neq 0$.

(a) Prove the statement of [Lecture Notes, Exercise 4.1.20], i.e. show that for

$$B(p) := \sum_{i=0}^{n} \left| \frac{a_i}{a_n} \right| + 1$$

no zero of p in K lies outside the interval $(-B(p), B(p))_K$.

(b) Assume that $(K, <)$ is a real closed field.

(i) Show that for any $c \in [B(p), \infty)_K$, we have

$$\mathrm{sign}(p(c)) = \mathrm{sign}(\mathrm{lm}(p)(1)) = \mathrm{sign}(a_n).$$

(This identity justifies the definition of $\mathrm{Var}(\underline{f}; \infty)$ in [Lecture Notes, Definition 4.1.31 (ii)].)

(ii) Show that for any $c \in (-\infty, -B(p)]_K$, we have

$$\mathrm{sign}(p(c)) = \mathrm{sign}(\mathrm{lm}(p)(-1)) = (-1)^n \mathrm{sign}(a_n).$$

(This identity justifies the definition of $\mathrm{Var}(\underline{f}; -\infty)$ in [Lecture Notes, Definition 4.1.31 (ii)].)

(c) Assume that $(K, <)$ has the intermediate value property and that $\deg(p) = n$ is odd. Deduce that p has a root in the interval $(-B(p), B(p))_K$.

Exercise 6.3 (Cauchy Index, Tarski Query and Signed Remainder Sequence)

(a) Let $(R, <)$ be a real closed field and consider the polynomials

$$q = (X^2 + 5X + 4)(X^2 - 4)(X^2 - 4X - 5) \in R[X] \text{ and}$$
$$p = (X^2 - 10X + 25)(X - 1)^3(X^2 - 16)(X + 3)(X + 7)^4 \in R[X].$$

Determine $\mathrm{CInd}\left(\frac{q}{p}; -\infty, 0\right), \mathrm{CInd}\left(\frac{q}{p}; 0, \infty\right)$ and $\mathrm{CInd}\left(\frac{q}{p}\right)$.

(b) Let $(R, <)$ be a real closed field and let $p, q \in R[X]$ with $p \neq 0$. Verify the following identities:

(i) $|\{c \in R \mid p(c) = 0 \text{ and } q(c) = 0\}| = \mathrm{TaQ}(1, p) - \mathrm{TaQ}(q^2, p)$,

(ii) $|\{c \in R \mid p(c) = 0 \text{ and } q(c) > 0\}| = \frac{1}{2}\left(\mathrm{TaQ}(q^2, p) + \mathrm{TaQ}(q, p)\right)$,

(iii) $|\{c \in R \mid p(c) = 0 \text{ and } q(c) < 0\}| = \frac{1}{2}\left(\mathrm{TaQ}(q^2, p) - \mathrm{TaQ}(q, p)\right)$.

(c) Consider the ordered field $(K, <)$, where $K = \mathbb{Q}(\mathrm{e}, \pi) \subseteq \mathbb{R}$ and $<$ is the ordering induced by the standard ordering on $\mathbb{R}$, and consider the polynomials $p = \mathrm{e}X^2 - 1 \in K[X]$ and $q = (\pi + 1)X + 2 \in K[X]$. Compute $\mathrm{SRS}(p, p'q)$.

Bonus Exercise (cf. [Lecture Notes, Exercise 4.1.22])

Let $(K, <_K)$ be an ordered field. Show that there exists a real closure R of K such that $(K, <_K) \subseteq (R, <)$, where $<$ denotes the unique binary relation on R making $(R, <)$ an ordered field (cf. Exercise 6.1).

Please hand in your solutions by **Friday, 27 May 2022, 11:45 (postbox 18 in F4).**