Prof. Dr. Salma Kuhlmann,
Dr. Lothar Sebastian Krapp,
Laura Wirth, Moritz Schick

Universität Konstanz
Fachbereich Mathematik und Statistik
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Tutorials for 'Real Closed Fields and Integer Parts'

Exercise Sheet 7: Counting Roots, Quantifier Elimination and Integer Parts

General Note: All statements must always be proven. Exercises with references to the lecture notes may only be solved by using results that have been established in the lecture notes *prior* to the respective exercise in order to avoid circular arguments. The bonus exercise is voluntary and will be awarded extra points.

Exercise 7.1 (Counting Roots)
Let $(R, <)$ be a real closed field.

(a) Let $p = X^3 - b \in R[X]$ with $b \neq 0$. Determine the number of roots of p in R.

(b) Consider the polynomial $p = X^4 - 5X^2 + 4 \in R[X]$.

 (i) Compute the Sturm sequence $\mathrm{SRS}(p, p')$ of p.

 (ii) Determine the number of roots of p in $(-\infty, 0]_R$ and in $[0, \infty)_R$.

Exercise 7.2 (Quantifier Elimination)
Consider the $\mathcal{L}_{\mathrm{or}}$-formula $\varphi(a, b)$ given by $\exists x\ x^2 + b = -ax$, where a and b are free variables of φ. Find a quantifier-free $\mathcal{L}_{\mathrm{or}}$-formula $\psi(a, b)$ such that

$$T_{\mathrm{rcf}} \models \forall a \forall b\ (\varphi(a, b) \leftrightarrow \psi(a, b))$$

by performing the steps of the quantifier elimination algorithm described in [Lecture Notes, Section 4.2.1].
(Hint: While performing the steps of the algorithm, you may omit false subformulas of disjunctions and true subformulas of conjunctions in order to obtain simplified but equivalent formulas.)

Exercise 7.3 (Properties of Integer Parts)
Let $(K, <)$ be an ordered field and let Z be an integer part of $(K, <)$.

(a) Show that for any $a \in K$ there exists a *unique* $z_a \in Z$ such that $z_a \leq a < z_a + 1$.

(b) Show that the fraction field of Z is dense in $(K, <)$, i.e. show that for any $a, b \in K$ with $a < b$ there exist $u \in Z$ and $v \in Z \setminus \{0\}$ such that $av < u < bv$.

Bonus Exercise (Integer Parts and Archimedean Fields)

Let $(K, <)$ be an ordered field. Show that $(K, <)$ is archimedean if and only if $\mathbb{Z}$ is its unique integer part (see Definition E.7.1).

Definition E.7.1. Let $(K, <)$ be an ordered field. We regard $\mathbb{Z}_{\mathrm{or}}$ as an $\mathcal{L}_{\mathrm{or}}$-substructure of $(K, <)$ via the unique $\mathcal{L}_{\mathrm{or}}$-embedding $\iota\colon \mathbb{Z}_{\mathrm{or}} \hookrightarrow (K, <)$ extending the map

$$\omega \to K,\ n \mapsto \underbrace{1 + \ldots + 1}_{n \text{ times}}.$$

In particular, we consider ω as subset of K. The ordered field $(K, <)$ is called **archimedean** if for any $a \in K$ there exists $n \in \omega$ such that $a < n$.

Please hand in your solutions by **Thursday, 2 June 2022, 11:45 (postbox 18 in F4).**