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Tutorials for 'Real Closed Fields and Integer Parts'

Exercise Sheet 8: Shepherdson's Theorem

General Note: All statements must always be proven. Exercises with references to the lecture notes may only be solved by using results that have been established in the lecture notes *prior* to the respective exercise in order to avoid circular arguments. The bonus exercise is voluntary and will be awarded extra points.

Exercise 8.1 (Properties of Integer Parts)
Let $(K,<)$ be an ordered field.

(a) Let Z be an integer part of $(K,<)$. Show that $\mathbb{Z}$ is a convex subring of Z, i.e. show that for any $z_1, z_2 \in \mathbb{Z}$ and any $z \in Z$ with $z_1 < z < z_2$ we have $z \in \mathbb{Z}$.

(b) Let Z_1 and Z_2 be integer parts of $(K,<)$. Show that Z_1 and Z_2 are $\mathcal{L}_<$-isomorphic.

Exercise 8.2 (Integer Parts and Density)
Let $(K,<)$ be an ordered field and let L be a subfield of K.

(a) Show that if L is dense in $(K,<)$, then every integer part of $(L,<)$ is also an integer part of $(K,<)$.

(b) Show that if there exists an integer part of $(L,<)$ that is also an integer part of $(K,<)$, then L is dense in $(K,<)$.

Definition E.8.1. Let $(A,<) \models T_{\mathrm{lo}}$. A subset $B \subseteq A$ is called **dense** in $(A,<)$ if for any $a, a' \in A$ with $a < a'$ there exists $b \in B$ such that $a < b < a'$.

Exercise 8.3 (Integer Parts and IOpen')
Consider the ordered ring $(\mathbb{R}[X],<)$, where the ordering $<$ is defined by

$$p < q \;:\Leftrightarrow\; \mathrm{lcf}(q-p) > 0$$

for any $p, q \in \mathbb{R}[X]$. Moreover, consider its subrings

$$Z_1 = \{p \in \mathbb{R}[X] \mid p(0) \in \mathbb{Z}\}$$

and

$$Z_2 = \left\{ p = \sum_{i=0}^{n} a_i X^i \in \mathbb{R}[X] \;\middle|\; a_0 \in \mathbb{Z}, a_1 = 0 \text{ and } n \in \omega \right\},$$

which are discretely ordered by $<$.

(a) Show that for any $n, m \in \mathbb{N}$ the polynomial $X^n - m$ has a root in Q_{Z_1} and a root in Q_{Z_2}. (In particular, $\sqrt{2}$ is rational over Z_1 and Z_2.)

(b) Prove that Z_1 is not an integer part of the real closure of Q_{Z_1}.[1]

(c) Prove that Z_2 is not an integer part of Q_{Z_2}.[2]

(d) Deduce that $Z_1 \not\models \mathrm{IOpen}'$ and $Z_2 \not\models \mathrm{IOpen}'$.

Bonus Exercise (Least Number Principle for IOpen)
Show that for any $\mathcal{M} \models \mathrm{PA}^-$ the following statements are equivalent:

(i) $\mathcal{M} \models \mathrm{IOpen}$, i.e. $\mathcal{M}$ satisfies the induction axiom

$$\forall \underline{y}\ ((\varphi(0, \underline{y}) \land \forall n\ (\varphi(n, \underline{y}) \to \varphi(n+1, \underline{y}))) \to \forall n\ \varphi(n, \underline{y}))$$

for any quantifier-free $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(x, \underline{y})$.

(ii) Given any quantifier-free $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(x, \underline{y})$ and any $\underline{a} \in M$ such that $\mathcal{M} \models \varphi(m, \underline{a})$ for some $m \in M$, the set $\varphi(\mathcal{M}, \underline{a})$ contains a least element. In other words, $\mathcal{M}$ satisfies

$$\forall \underline{y}\ (\exists m\ \varphi(m, \underline{y}) \to \exists n\ (\varphi(n, \underline{y}) \land \forall (n' < n)\ \neg\varphi(n', \underline{y})))$$

for any quantifier-free $\mathcal{L}_{\mathrm{PA}}$-formula $\varphi(x, \underline{y})$.

Please hand in your solutions by **Thursday, 9 June 2022, 11:45 (postbox 18 in F4).**

[1] We endow the real closure of Q_{Z_1} with the unique ordering $<$ making it an ordered field (cf. [Lecture Notes, Definition 4.1.10 and Exercise 4.1.11]), and thus obtain $(Z_1, <) \subseteq (Q_{Z_1}, <)$

[2] We endow the field Q_{Z_2} with the ordering $<$ defined in [Lecture Notes, Remark 5.1.7] to obtain $(Z_2, <) \subseteq (Q_{Z_2}, <)$.