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Tutorials for 'Real Closed Fields and Integer Parts'
Exercise Sheet 9: Well-Orderings, Ordinals and Hahn Series

General Note: All statements must always be proven. Exercises with references to the lecture notes may only be solved by using results that have been established in the lecture notes *prior* to the respective exercise in order to avoid circular arguments. The bonus exercise is voluntary and will be awarded extra points.

Exercise 9.1 (Well-Orderings)

(a) Let $(I, <) \models T_{\mathrm{lo}}$ and let $J \subseteq I$. Show that J is well-ordered if and only if it contains no infinite strictly decreasing sequence, i.e. for any sequence $(a_i)_{i \in \omega}$ in J it is not possible that $a_0 > a_1 > a_2 > \ldots$ (cf. [Lecture Notes, Exercise 6.1.3]).

(b) Let $(I, <)$ be a well-ordering and let $f\colon (I, <) \to (I, <)$. Prove the statements of [Lecture Notes, Exercise 6.1.4]:

- (i) Show that for any $i \in I$ we have $f(i) \geq i$.
- (ii) Suppose that $f\colon (I, <) \cong (I, <)$. Show that $f = \mathrm{id}_I$.
- (iii) Let $(J, <)$ be an $\mathcal{L}_<$-structure with $(I, <) \cong (J, <)$. Show that there is a unique $\mathcal{L}_<$-isomorphism from I to J.

Exercise 9.2 (Successor Ordinals and Limit Ordinals)
Let $\alpha \in \mathbf{On}$.

(a) Show that $\alpha + 1 = \alpha \cup \{\alpha\}$ is the smallest ordinal greater than α.
(Therefore, $\alpha + 1$ is referred to as *successor* of α.)

(b) Show that α is a successor ordinal if and only if α has a greatest element.

(c) Suppose that $\alpha > 0$. Show that there exist a limit ordinal γ and an ordinal $n \in \omega$ such that $\alpha = \gamma + n$.

Definition E.9.1. Given $\alpha \in \mathbf{On}$ and $n \in \omega$, we define $\alpha + n$ recursively by $\alpha + 0 := \alpha$ and $\alpha + (n + 1) := (\alpha + n) + 1$ for any $n \in \omega$.[1]

[1] By Exercise 9.2 (a), $\alpha + n$ is again an ordinal.

Exercise 9.3 (Laurent Series)

(a) Consider the ordered abelian group $\mathbb{Z}$ and let $S \subseteq \mathbb{Z}$ be non-empty. Show that S is well-ordered if and only if S has a least element.
(Hence, given a field k, for any $s \in k((\mathbb{Z}))$ there exists $z \in \mathbb{Z}$ such that

$$\operatorname{supp}(s) \subseteq [z, \infty)_{\mathbb{Z}} = \{n \in \mathbb{Z} \mid z \leq n\},$$

and therefore we denote s by

$$s = \sum_{n=z}^{\infty} s_n t^n.$$

Usually, $k((\mathbb{Z}))$ is denoted by $k((t))$ and its elements are referred to as **Laurent series**.)

(b) Let k be a field and consider the Laurent series $s = 1 + 3t^2 \in k((t))$.

(i) Show that any Laurent series $r \in k((t))$ with $s \cdot r = 1$ fulfills $\operatorname{supp}(r) \subseteq \mathbb{Z}^{\geq 0}$.

(ii) Find a Laurent series $r \in k((\mathbb{Z}))$ such that $s \cdot r = 1$ and express it both in the standard notation and in the order type notation.

Bonus Exercise

(a) Let $(x, <)$ be an ordinal number. Show that the binary relation $<$ on x coincides with the binary relation $\in$ on x (cf. [Lecture Notes, Exercise 6.1.12]).

(b) Let $\alpha, \beta \in \mathbf{On}$. Show that $\alpha \in \beta$ if and only if $\alpha \subsetneq \beta$ (cf. [Lecture Notes, Exercise 6.1.15]).

(c) Show that $(\mathbf{On}, <)$ is a well-ordering (cf. [Lecture Notes, Exercise 6.1.17]).
(Hint: You may apply the statement of [Lecture Notes, Exercise 6.1.16] without proving it.)

Please hand in your solutions by **Thursday, 23 June 2022, 11:45 (postbox 18 in F4).**