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**Tutorials for 'Real Closed Fields and Integer Parts'**

**Exercise Sheet 10: Hahn Fields and Rayner Fields**

**General Note:** All statements must always be proven. Exercises with references to the lecture notes may only be solved by using results that have been established in the lecture notes *prior* to the respective exercise in order to avoid circular arguments. The bonus exercise is voluntary and will be awarded extra points.

**Exercise 10.1** (Sum of Well-Ordered Sets)
Let $k$ be a field, let $G$ be an ordered abelian group and let $A, B \subseteq G$ be non-empty and well-ordered subsets. Show that the subset

$$A + B := \{a + b \mid a \in A, b \in B\}$$

is again well-ordered.

**Exercise 10.2** (Hahn Fields)
Let $k$ be a field, let $G$ be an ordered abelian group and recall that

$$k[G] = \{s \in k(\!(G)\!) \mid \mathrm{supp}(s) \text{ is finite}\}$$

is an integral domain (cf. [Lecture Notes, Exercise 6.2.8 (i)]).

(a) Show that $k(G) = \mathrm{ff}(k[G])$ is the smallest subfield of $k(\!(G)\!)$ containing all monomials, i.e. the set

$$\{at^g \mid a \in k, g \in G\} \subseteq k(\!(G)\!)$$

(cf. [Lecture Notes, Exercise 6.2.8 (ii)]).

(b) Show that every Rayner field is a Hahn field, i.e. show that for any field $k$, any ordered abelian group $G$ and any $\mathcal{F} \subseteq \mathrm{wo}(G)$, if $k(\!(\mathcal{F})\!)$ is a Rayner field, then

$$k(G) \subseteq k(\!(\mathcal{F})\!) \subseteq k(\!(G)\!)$$

(cf. [Lecture Notes, Exercise 6.2.10]).

**Exercise 10.3** (Puiseux Series)
Let $k$ be a field. Consider the ordered abelian group $\mathbb{Q}$ and the set

$$\mathcal{F} = \left\{ \left\{ \frac{m}{n} \;\middle|\; m \in A \right\} \;\middle|\; A \in \mathrm{wo}(\mathbb{Z}), n \in \mathbb{N} \right\} \subseteq \mathrm{wo}(\mathbb{Q}).$$

Show that $k(\!(\mathcal{F})\!)$ is a Rayner field.

(By Exercise 9.3 (a), for any series $a \in k(\!(\mathcal{F})\!)$ there exist $n \in \mathbb{N}$ and $z \in \mathbb{Z}$ such that

$$\operatorname{supp}(a) \subseteq \left\{ \frac{m}{n} \;\middle|\; m \in [z, \infty)_{\mathbb{Z}} \right\},$$

and therefore we write

$$a = \sum_{m=z}^{\infty} a_m t^{\frac{m}{n}},$$

where $a_m = a(\frac{m}{n})$. Usually, $k(\!(\mathcal{F})\!)$ is denoted by $k\langle\!\langle t \rangle\!\rangle$ and its elements are referred to as **Puiseux series**.)

**Bonus Exercise**

Let $k$ be a field and let $G$ be an ordered abelian group. Show that $k(\!(G)\!)$ is a Rayner field (cf. [Lecture Notes, Exercise 6.2.6]).
(Hint: You may apply Neumann's Lemma.)

Please hand in your solutions by **Thursday, 30 June 2022, 11:45 (postbox 18 in F4).**