Prof. Dr. Salma Kuhlmann,
Dr. Lothar Sebastian Krapp,
Laura Wirth, Moritz Schick

Universität Konstanz
Fachbereich Mathematik und Statistik
Sommersemester 2022

Tutorials for 'Real Closed Fields and Integer Parts'

Exercise Sheet 11: Valuation Theory

General Note: All statements must always be proven. Exercises with references to the lecture notes may only be solved by using results that have been established in the lecture notes *prior* to the respective exercise in order to avoid circular arguments. The bonus exercise is voluntary and will be awarded extra points.

Exercise 11.1 (Properties of Valued Fields)
Let (K, v) be a valued field.

(a) Verify the following properties:

  (i) $v(0) = \infty$ and $v(1) = v(-1) = 0$.
  (ii) $v(a^{-1}) = -v(a)$ for any $a \in K^{\times}$.
  (iii) $v(-a) = v(a)$ for any $a \in K$.
  (iv) For any $a, b \in K$, the following hold:
    - $v(a) \neq v(b) \;\Rightarrow\; v(a+b) = \min\{v(a), v(b)\}$,
    - $v(a+b) > v(a) \;\Rightarrow\; v(a) = v(b)$.

(b) Verify that $\mathcal{O}_v$ is a subring of K and that $\mathcal{I}_v$ is a maximal ideal of $\mathcal{O}_v$. Moreover, show that $\mathcal{I}_v$ is the *unique* maximal ideal of $\mathcal{O}_v$ (cf. [Lecture Notes, Exercise 6.3.2]).

Exercise 11.2 (Natural Valuation)
Let $(K, <)$ be an ordered field.

(a) Show that

$$a \sim b \;\Leftrightarrow\; \text{both } n \cdot |a| \geq |b| \text{ and } n \cdot |b| \geq |a| \text{ for some } n \in \omega$$

for any $a, b \in K^{\times}$.

(b) Show that

$$[a] < [b] \;\Leftrightarrow\; \text{the inequality } n \cdot |b| < |a| \text{ holds for any } n \in \omega$$

for any $a, b \in K^{\times}$.

(c) Show that (K, v_{nat}) is a valued field (cf. [Lecture Notes, Exercise 6.3.8]).
(Note: You may assume that $G = v_{\mathrm{nat}}K$ is an ordered abelian group.)

(d) Show that the convex hull of $\mathbb{Z}$ in K coincides with the valuation ring of (K, v_{nat}), i.e. we have

$$\{a \in K \mid c \leq a \leq d \text{ for some } c, d \in \mathbb{Z}\} = \mathcal{O}_{v_{\mathrm{nat}}} = \{a \in K \mid v_{\mathrm{nat}}(a) \geq 0\}.$$

Exercise 11.3 (Equivalence of $v_{\min}$ and v_{nat})
Let $(k, <)$ be an archimedean ordered field and let G be an ordered abelian group. Denote by v_{nat} the natural valuation on $(K, <) = (k((G)), <)$. Prove the statements of [Lecture Notes, Exercise 6.3.9]:

(i) $\mathcal{O}_{v_{\min}} = \mathcal{O}_{v_{\mathrm{nat}}}$.

(ii) The map

$$\varphi \colon G \to v_{\mathrm{nat}}K,\ g \mapsto v_{\mathrm{nat}}(t^g)$$

defines an $\mathcal{L}_{\mathrm{og}}$-isomorphism.

(iii) For any $a \in k((G))^\times$ we have

$$v_{\mathrm{nat}}(a) = \varphi(v_{\min}(a)).$$

Bonus Exercise
Let $(K, <)$ be an ordered field and denote by v its natural valuation.

(a) Show that the relation $<$ on the residue field $\overline{K}$ from [Lecture Notes, Definition 6.3.11] is well-defined, i.e. that $\overline{a} < \overline{b}$ implies $\overline{a+c} < \overline{b+d}$ for any $a, b \in \mathcal{O}_v$ and any $c, d \in \mathcal{I}_v$.

(b) Show that $(\overline{K}, <)$ is an archimedean ordered field (cf. [Lecture Notes, Exercise 6.3.11]).
(Note: You may assume that $(\overline{K}, <)$ is a linear order.)

Please hand in your solutions by **Thursday, 7 July 2022, 11:45 (postbox 18 in F4).**