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Sommersemester 2022

Tutorials for 'Real Closed Fields and Integer Parts'

Exercise Sheet 12: Embeddings and Integer Parts of Hahn Fields

General Note: All statements must always be proven. Exercises with references to the lecture notes may only be solved by using results that have been established in the lecture notes *prior* to the respective exercise in order to avoid circular arguments. The bonus exercise is voluntary and will be awarded extra points.

Exercise 12.1 (Embedding the Value Group, cf. [Lecture Notes, Exercise 7.1.9 (ii)])
Let K be a real closed field and let $G = vK$. Consider both G and $K^{>0}$ as $\mathbb{Q}$-vector spaces as in [Lecture Notes, Exercise 7.1.8]. Deduce from [Lecture Notes, Exercise 7.1.9 (i)] that there exists an embedding

$$\varphi\colon (G, +, 0) \hookrightarrow (K^{>0}, \cdot, 1)$$

with $v(\varphi(g)) = g$ for any $g \in G$ and $\varphi(g) > \varphi(h)$ for any $g, h \in G$ with $g < h$.

Exercise 12.2 (cf. [Lecture Notes, Hint for Exercise 7.1.10])
Let K be a real closed field. Set $k = \overline{K}$ and $G = vK$. Show that there exists an $\mathcal{L}_{\mathrm{or}}$-embedding

$$\psi\colon (k[G], <) \hookrightarrow (K, <).$$

(One then obtains an $\mathcal{L}_{\mathrm{or}}$-embedding $\Psi\colon (k(G), <) \hookrightarrow (K, <)$ by setting

$$\Psi\left(\frac{p}{q}\right) = \frac{\psi(p)}{\psi(q)}$$

for any $p, q \in k[G]$ with $q \neq 0$.)

Exercise 12.3 (Integer Parts of Hahn Fields)

(a) Let $(k, <)$ be an archimedean ordered field and let G be an ordered abelian group. Consider a truncation closed Hahn field K over k and G and its subring

$$Z = \left\{ s \in K \mid \mathrm{supp}(s) \subseteq G^{\leq 0} \text{ and } s_0 \in \mathbb{Z} \right\}.$$

Show that Z is an integer part of K.

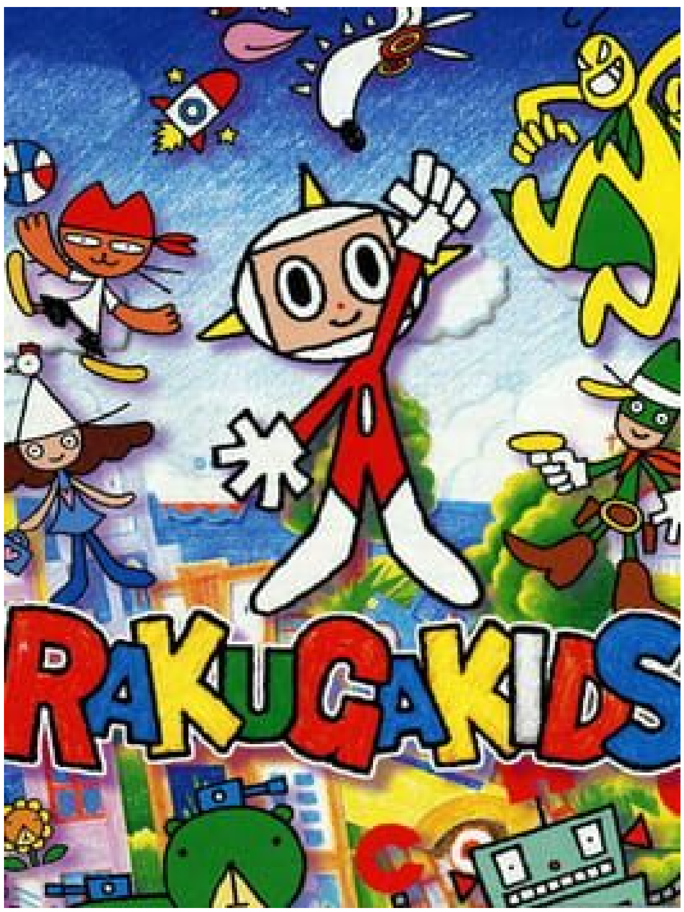
(b) Show that $\mathrm{IOpen} \not\models \neg\exists(x \neq 0)\exists(y \neq 0)\exists(z \neq 0)\ x^n + y^n = z^n$ for any $n \in \mathbb{N}$.
(Note: You may apply [Lecture Notes, Theorem 7.1.11].)

Bonus Exercise (cf. [Lecture Notes, Exercise 7.1.8 (ii)])

Let K be a real closed field. Show that $(K^{>0}, \boxplus, \boxdot)$ is a $\mathbb{Q}$-vector space where the addition operation on $K^{>0}$ is given by $a \boxplus b = ab$ for any $a, b \in K^{>0}$ (i.e. the standard multiplication) and the scalar multiplication is given by

$$\frac{m}{n} \boxdot a = \sqrt[n]{a^m}$$

for any $m \in \mathbb{Z}, n \in \mathbb{N}$ and $a \in K^{>0}$. Here, for any $b \in K^{>0}$, we denote by $\sqrt[n]{b}$ the unique positive element c of K with $c^n = b$.



YOU MADE IT!

Please hand in your solutions by **Thursday, 14 July 2022, 11:45 (postbox 18 in F4).**